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More on the Continuity of Real Functions[1]

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Summary. In this article we demonstrate basic properties of the continuous functions from $\mathbb{R}$ to $\mathcal{R}^n$ which correspond to state space equations in control engineering.

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The terminology and notation used here have been introduced in the following articles: [3], [7], [17], [2], [4], [12], [13], [14], [16], [1], [5], [9], [15], [18], [10], [8], [20], [21], [19], [11], [22], and [6].

For simplicity, we use the following convention: n, i denote elements of $\mathbb{N}$, X, X_1 denote sets, r, p, s, x_0, x_1, x_2 denote real numbers, f, f_1, f_2 denote partial functions from $\mathbb{R}$ to $\mathcal{R}^n$, and h denotes a partial function from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle$.

Let us consider n, f, x_0. We say that f is continuous in x_0 if and only if:

(Def. 1) There exists a partial function g from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ such that $f = g$ and g is continuous in x_0.

We now state four propositions:

(1) If $h = f$, then f is continuous in x_0 iff h is continuous in x_0.

(2) If $x_0 \in X$ and f is continuous in x_0, then $f{\restriction}X$ is continuous in x_0.

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(3) f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in \operatorname{dom} f$ and $|x_1 - x_0| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(4) Let r be a real number, z be an element of $\mathcal{R}^n$, and w be a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $z = w$. Then $\{y \in \mathcal{R}^n \colon |y - z| < r\} = \{y; y$ ranges over points of $\langle \mathcal{E}^n, \|\cdot\| \rangle \colon \|y - w\| < r\}$.

Let n be an element of $\mathbb{N}$, let Z be a set, and let f be a partial function from Z to $\mathcal{R}^n$. The functor $|f|$ yielding a partial function from Z to $\mathbb{R}$ is defined by:

(Def. 2) $\operatorname{dom} |f| = \operatorname{dom} f$ and for every set x such that $x \in \operatorname{dom} |f|$ holds $|f|_x = |f_x|$.

Let n be an element of $\mathbb{N}$, let Z be a non empty set, and let f be a partial function from Z to $\mathcal{R}^n$. The functor $-f$ yields a partial function from Z to $\mathcal{R}^n$ and is defined by:

(Def. 3) $\operatorname{dom}(-f) = \operatorname{dom} f$ and for every set c such that $c \in \operatorname{dom}(-f)$ holds $(-f)_c = -f_c$.

One can prove the following propositions:

(5) Let f_1, f_2 be partial functions from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and g_1, g_2 be partial functions from $\mathbb{R}$ to $\mathcal{R}^n$. If $f_1 = g_1$ and $f_2 = g_2$, then $f_1 + f_2 = g_1 + g_2$.

(6) Let f_1 be a partial function from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, g_1 be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$, and a be a real number. If $f_1 = g_1$, then $a \cdot f_1 = a \cdot g_1$.

(7) For every partial function f_1 from $\mathbb{R}$ to $\mathcal{R}^n$ holds $(-1) \cdot f_1 = -f_1$.

(8) Let f_1 be a partial function from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and g_1 be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. If $f_1 = g_1$, then $-f_1 = -g_1$.

(9) Let f_1 be a partial function from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and g_1 be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. If $f_1 = g_1$, then $\|f_1\| = |g_1|$.

(10) Let f_1, f_2 be partial functions from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and g_1, g_2 be partial functions from $\mathbb{R}$ to $\mathcal{R}^n$. If $f_1 = g_1$ and $f_2 = g_2$, then $f_1 - f_2 = g_1 - g_2$.

(11) f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every subset N_1 of $\mathcal{R}^n$ such that there exists a real number r such that $0 < r$ and $\{y \in \mathcal{R}^n \colon |y - f_{x_0}| < r\} = N_1$ there exists a neighbourhood N of x_0 such that for every x_1 such that $x_1 \in \operatorname{dom} f$ and $x_1 \in N$ holds $f_{x_1} \in N_1$.

(12) f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every subset N_1 of $\mathcal{R}^n$ such that there exists a real number r such that $0 < r$ and $\{y \in \mathcal{R}^n: |y - f_{x_0}| < r\} = N_1$ there exists a neighbourhood N of x_0 such that $f^\circ N \subseteq N_1$.

(13) If there exists a neighbourhood N of x_0 such that $\operatorname{dom} f \cap N = \{x_0\}$, then f is continuous in x_0.

(14) If $x_0 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and f_1 is continuous in x_0 and f_2 is continuous in x_0, then $f_1 + f_2$ is continuous in x_0.

(15) If $x_0 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and f_1 is continuous in x_0 and f_2 is continuous in x_0, then $f_1 - f_2$ is continuous in x_0.

(16) If f is continuous in x_0, then $r \cdot f$ is continuous in x_0.

(17) If $x_0 \in \operatorname{dom} f$ and f is continuous in x_0, then $|f|$ is continuous in x_0.

(18) If $x_0 \in \operatorname{dom} f$ and f is continuous in x_0, then $-f$ is continuous in x_0.

(19) Let S be a real normed space, z be a point of $\langle \mathcal{E}^n, \| \cdot \| \rangle$, f_1 be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$, and f_2 be a partial function from the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ to the carrier of S. Suppose $x_0 \in \operatorname{dom}(f_2 \cdot f_1)$ and f_1 is continuous in x_0 and $z = (f_1)_{x_0}$ and f_2 is continuous in z. Then $f_2 \cdot f_1$ is continuous in x_0.

(20) Let S be a real normed space, f_1 be a partial function from $\mathbb{R}$ to the carrier of S, and f_2 be a partial function from the carrier of S to $\mathbb{R}$. Suppose $x_0 \in \operatorname{dom}(f_2 \cdot f_1)$ and f_1 is continuous in x_0 and f_2 is continuous in $(f_1)_{x_0}$. Then $f_2 \cdot f_1$ is continuous in x_0.

Let us consider n, let f be a partial function from $\mathcal{R}^n$ to $\mathbb{R}$, and let x_0 be an element of $\mathcal{R}^n$. We say that f is continuous in x_0 if and only if the condition (Def. 4) is satisfied.

(Def. 4) There exists a point y_0 of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ and there exists a partial function g from the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ to $\mathbb{R}$ such that $x_0 = y_0$ and $f = g$ and g is continuous in y_0.

One can prove the following two propositions:

(21) Let f be a partial function from $\mathcal{R}^n$ to $\mathbb{R}$, h be a partial function from the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ to $\mathbb{R}$, x_0 be an element of $\mathcal{R}^n$, and y_0 be a point of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Suppose $f = h$ and $x_0 = y_0$. Then f is continuous in x_0 if and only if h is continuous in y_0.

(22) Let f_1 be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$ and f_2 be a partial function from $\mathcal{R}^n$ to $\mathbb{R}$. Suppose $x_0 \in \operatorname{dom}(f_2 \cdot f_1)$ and f_1 is continuous in x_0 and f_2 is continuous in $(f_1)_{x_0}$. Then $f_2 \cdot f_1$ is continuous in x_0.

Let us consider n, f. We say that f is continuous if and only if:

(Def. 5) For every x_0 such that $x_0 \in \operatorname{dom} f$ holds f is continuous in x_0.

One can prove the following propositions:

(23) Let g be a partial function from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. If $g = f$, then g is continuous iff f is continuous.

(24) Suppose $X \subseteq \operatorname{dom} f$. Then $f \restriction X$ is continuous if and only if for all x_0, r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in X$ and $|x_1 - x_0| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

Let us consider n. Observe that every partial function from $\mathbb{R}$ to $\mathcal{R}^n$ which is constant is also continuous.

Let us consider n. Observe that there exists a partial function from $\mathbb{R}$ to $\mathcal{R}^n$ which is continuous.

Let us consider n, let f be a continuous partial function from $\mathbb{R}$ to $\mathcal{R}^n$, and let X be a set. One can verify that $f \restriction X$ is continuous.

One can prove the following proposition

(25) If $f \restriction X$ is continuous and $X_1 \subseteq X$, then $f \restriction X_1$ is continuous.

Let us consider n. Note that every partial function from $\mathbb{R}$ to $\mathcal{R}^n$ which is empty is also continuous.

Let us consider n, f and let X be a trivial set. One can verify that $f \restriction X$ is continuous.

Let us consider n and let f_1, f_2 be continuous partial functions from $\mathbb{R}$ to $\mathcal{R}^n$. One can check that $f_1 + f_2$ is continuous.

The following propositions are true:

(26) If $X \subseteq \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and $f_1 \restriction X$ is continuous and $f_2 \restriction X$ is continuous, then $(f_1 + f_2) \restriction X$ is continuous and $(f_1 - f_2) \restriction X$ is continuous.

(27) If $X \subseteq \operatorname{dom} f_1$ and $X_1 \subseteq \operatorname{dom} f_2$ and $f_1 \restriction X$ is continuous and $f_2 \restriction X_1$ is continuous, then $(f_1 + f_2) \restriction (X \cap X_1)$ is continuous and $(f_1 - f_2) \restriction (X \cap X_1)$ is continuous.

Let us consider n, let f be a continuous partial function from $\mathbb{R}$ to $\mathcal{R}^n$, and let us consider r. Observe that $r \cdot f$ is continuous.

The following propositions are true:

(28) If $X \subseteq \operatorname{dom} f$ and $f \restriction X$ is continuous, then $(r \cdot f) \restriction X$ is continuous.

(29) If $X \subseteq \operatorname{dom} f$ and $f \restriction X$ is continuous, then $|f| \restriction X$ is continuous and $(-f) \restriction X$ is continuous.

(30) If f is total and for all x_1, x_2 holds $f_{x_1 + x_2} = f_{x_1} + f_{x_2}$ and there exists x_0 such that f is continuous in x_0, then $f \restriction \mathbb{R}$ is continuous.

(31) For every subset Y of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ such that $\operatorname{dom} f$ is compact and $f \restriction \operatorname{dom} f$ is continuous and $Y = \operatorname{rng} f$ holds Y is compact.

(32) Let Y be a subset of $\mathbb{R}$ and Z be a subset of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $Y \subseteq \operatorname{dom} f$ and $Z = f^\circ Y$ and Y is compact and $f \restriction Y$ is continuous. Then Z is compact.

Let us consider n, f. We say that f is Lipschitzian if and only if:

(Def. 6) There exists a partial function g from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ such that $g = f$ and g is Lipschitzian.

The following propositions are true:

(33) f is Lipschitzian if and only if there exists a real number r such that $0 < r$ and for all x_1, x_2 such that $x_1, x_2 \in \operatorname{dom} f$ holds $|f_{x_1} - f_{x_2}| \leq r \cdot |x_1 - x_2|$.

(34) If $f = h$, then f is Lipschitzian iff h is Lipschitzian.

(35) $f{\restriction}X$ is Lipschitzian if and only if there exists a real number r such that $0 < r$ and for all x_1, x_2 such that $x_1, x_2 \in \operatorname{dom}(f{\restriction}X)$ holds $|f_{x_1} - f_{x_2}| \leq r \cdot |x_1 - x_2|$.

Let us consider n. Note that every partial function from $\mathbb{R}$ to $\mathcal{R}^n$ which is empty is also Lipschitzian.

Let us consider n. Note that there exists a partial function from $\mathbb{R}$ to $\mathcal{R}^n$ which is empty.

Let us consider n, let f be a Lipschitzian partial function from $\mathbb{R}$ to $\mathcal{R}^n$, and let X be a set. Note that $f{\restriction}X$ is Lipschitzian.

We now state the proposition

(36) If $f{\restriction}X$ is Lipschitzian and $X_1 \subseteq X$, then $f{\restriction}X_1$ is Lipschitzian.

Let us consider n and let f_1, f_2 be Lipschitzian partial functions from $\mathbb{R}$ to $\mathcal{R}^n$. Observe that $f_1 + f_2$ is Lipschitzian and $f_1 - f_2$ is Lipschitzian.

We now state two propositions:

(37) If $f_1{\restriction}X$ is Lipschitzian and $f_2{\restriction}X_1$ is Lipschitzian, then $(f_1+f_2){\restriction}(X \cap X_1)$ is Lipschitzian.

(38) If $f_1{\restriction}X$ is Lipschitzian and $f_2{\restriction}X_1$ is Lipschitzian, then $(f_1-f_2){\restriction}(X \cap X_1)$ is Lipschitzian.

Let us consider n, let f be a Lipschitzian partial function from $\mathbb{R}$ to $\mathcal{R}^n$, and let us consider p. Observe that $p \cdot f$ is Lipschitzian.

Next we state the proposition

(39) If $f{\restriction}X$ is Lipschitzian and $X \subseteq \operatorname{dom} f$, then $(p \cdot f){\restriction}X$ is Lipschitzian.

Let us consider n and let f be a Lipschitzian partial function from $\mathbb{R}$ to $\mathcal{R}^n$. Observe that $|f|$ is Lipschitzian.

Next we state the proposition

(40) If $f{\restriction}X$ is Lipschitzian, then $-f{\restriction}X$ is Lipschitzian and $|f|{\restriction}X$ is Lipschitzian and $(-f){\restriction}X$ is Lipschitzian.

Let us consider n. One can check that every partial function from $\mathbb{R}$ to $\mathcal{R}^n$ which is constant is also Lipschitzian.

Let us consider n. One can verify that every partial function from $\mathbb{R}$ to $\mathcal{R}^n$ which is Lipschitzian is also continuous.

The following propositions are true:

(41) For all elements r, p of $\mathcal{R}^n$ such that for every x_0 such that $x_0 \in X$ holds $f_{x_0} = x_0 \cdot r + p$ holds $f{\restriction}X$ is continuous.

(42) For every element x_0 of $\mathcal{R}^n$ such that $1 \leq i \leq n$ holds $\mathrm{proj}(i, n)$ is continuous in x_0.

(43) Let n be a non empty element of $\mathbb{N}$ and h be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. Then h is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \mathrm{dom}\, h$, and

(ii) for every element i of $\mathbb{N}$ such that $i \in \mathrm{Seg}\, n$ holds $\mathrm{proj}(i, n) \cdot h$ is continuous in x_0.

(44) Let n be a non empty element of $\mathbb{N}$ and h be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. Then h is continuous if and only if for every element i of $\mathbb{N}$ such that $i \in \mathrm{Seg}\, n$ holds $\mathrm{proj}(i, n) \cdot h$ is continuous.

(45) For every point x_0 of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ such that $1 \leq i \leq n$ holds $\mathrm{Proj}(i, n)$ is continuous in x_0.

(46) Let n be a non empty element of $\mathbb{N}$ and h be a partial function from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Then h is continuous in x_0 if and only if for every element i of $\mathbb{N}$ such that $i \in \mathrm{Seg}\, n$ holds $\mathrm{Proj}(i, n) \cdot h$ is continuous in x_0.

(47) Let n be a non empty element of $\mathbb{N}$ and h be a partial function from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Then h is continuous if and only if for every element i of $\mathbb{N}$ such that $i \in \mathrm{Seg}\, n$ holds $\mathrm{Proj}(i, n) \cdot h$ is continuous.

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