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Partial Differentiation, Differentiation and Continuity on n-Dimensional Real Normed Linear Spaces

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Summary. In this article, we aim to prove the characterization of differentiation by means of partial differentiation for vector-valued functions on n-dimensional real normed linear spaces (refer to [15] and [16]).

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The notation and terminology used in this paper have been introduced in the following papers: [2], [7], [1], [3], [4], [5], [17], [11], [13], [6], [9], [14], [10], [8], [12], and [18].

One can prove the following propositions:

(1) Let n, i be elements of $\mathbb{N}$, q be an element of $\mathcal{R}^n$, and p be a point of $\mathcal{E}^n_{\mathrm{T}}$. If $i \in \operatorname{Seg} n$ and $q = p$, then $|p_i| \leq |q|$.

(2) For every real number x and for every element v_1 of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ such that $v_1 = \langle x \rangle$ holds $\|v_1\| = |x|$.

(3) Let n be a non empty element of $\mathbb{N}$, x be a point of $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and i be an element of $\mathbb{N}$. If $1 \leq i \leq n$, then $\|(\mathrm{Proj}(i, n))(x)\| \leq \|x\|$.

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(4) For every non empty element n of $\mathbb{N}$ and for every element x of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and for every element i of $\mathbb{N}$ holds $\|(\mathrm{Proj}(i,n))(x)\| = |(\mathrm{proj}(i,n))(x)|$.

(5) Let n be a non empty element of $\mathbb{N}$, x be an element of $\mathcal{R}^n$, and i be an element of $\mathbb{N}$. If $1 \leq i \leq n$, then $|(\mathrm{proj}(i,n))(x)| \leq |x|$.

(6) Let m, n be non empty elements of $\mathbb{N}$, s be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and i be an element of $\mathbb{N}$. Suppose $1 \leq i \leq n$. Then $\mathrm{Proj}(i,n)$ is a bounded linear operator from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and $(\mathrm{BdLinOpsNorm}(\langle \mathcal{E}^n, \|\cdot\| \rangle, \langle \mathcal{E}^1, \|\cdot\| \rangle))(\mathrm{Proj}(i,n)) \leq 1$.

(7) Let m, n be non empty elements of $\mathbb{N}$, s be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and i be an element of $\mathbb{N}$. Suppose $1 \leq i \leq n$. Then

(i) $\mathrm{Proj}(i,n) \cdot s$ is a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$, and

(ii) $(\mathrm{BdLinOpsNorm}(\langle \mathcal{E}^m, \|\cdot\| \rangle, \langle \mathcal{E}^1, \|\cdot\| \rangle))(\mathrm{Proj}(i,n) \cdot s) \leq (\mathrm{BdLinOpsNorm}(\langle \mathcal{E}^n, \|\cdot\| \rangle, \langle \mathcal{E}^1, \|\cdot\| \rangle))(\mathrm{Proj}(i,n)) \cdot (\mathrm{BdLinOpsNorm}(\langle \mathcal{E}^m, \|\cdot\| \rangle, \langle \mathcal{E}^n, \|\cdot\| \rangle))(s)$.

(8) For every non empty element n of $\mathbb{N}$ and for every element i of $\mathbb{N}$ holds $\mathrm{Proj}(i,n)$ is homogeneous.

(9) Let n be a non empty element of $\mathbb{N}$, x be an element of $\mathcal{R}^n$, r be a real number, and i be an element of $\mathbb{N}$. Then $(\mathrm{proj}(i,n))(r \cdot x) = r \cdot (\mathrm{proj}(i,n))(x)$.

(10) Let n be a non empty element of $\mathbb{N}$, x, y be elements of $\mathcal{R}^n$, and i be an element of $\mathbb{N}$. Then $(\mathrm{proj}(i,n))(x+y) = (\mathrm{proj}(i,n))(x) + (\mathrm{proj}(i,n))(y)$.

(11) Let n be a non empty element of $\mathbb{N}$, x, y be points of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and i be an element of $\mathbb{N}$. Then $(\mathrm{Proj}(i,n))(x-y) = (\mathrm{Proj}(i,n))(x) - (\mathrm{Proj}(i,n))(y)$.

(12) Let n be a non empty element of $\mathbb{N}$, x, y be elements of $\mathcal{R}^n$, and i be an element of $\mathbb{N}$. Then $(\mathrm{proj}(i,n))(x-y) = (\mathrm{proj}(i,n))(x) - (\mathrm{proj}(i,n))(y)$.

(13) Let m, n be non empty elements of $\mathbb{N}$, s be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$, i be an element of $\mathbb{N}$, and s_1 be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$. If $s_1 = \mathrm{Proj}(i,n) \cdot s$ and $1 \leq i \leq n$, then $\|s_1\| \leq \|s\|$.

(14) Let m, n be non empty elements of $\mathbb{N}$, s, t be points of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$, s_1, t_1 be points of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$, and i be an element of $\mathbb{N}$. If $s_1 = \mathrm{Proj}(i,n) \cdot s$ and $t_1 = \mathrm{Proj}(i,n) \cdot t$ and $1 \leq i \leq n$, then $\|s_1 - t_1\| \leq \|s - t\|$.

(15) Let K be a real number, n be an element of $\mathbb{N}$, and s be an element of $\mathcal{R}^n$. Suppose that for every element i of $\mathbb{N}$ such that $1 \leq i \leq n$ holds

$|s(i)| \leq K$. Then $|s| \leq n \cdot K$.

(16) Let K be a real number, n be a non empty element of $\mathbb{N}$, and s be an element of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Suppose that for every element i of $\mathbb{N}$ such that $1 \leq i \leq n$ holds $\|(\mathrm{Proj}(i, n))(s)\| \leq K$. Then $\|s\| \leq n \cdot K$.

(17) Let K be a real number, n be a non empty element of $\mathbb{N}$, and s be an element of $\mathcal{R}^n$. Suppose that for every element i of $\mathbb{N}$ such that $1 \leq i \leq n$ holds $|(\mathrm{proj}(i, n))(s)| \leq K$. Then $|s| \leq n \cdot K$.

(18) Let m, n be non empty elements of $\mathbb{N}$, s be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and K be a real number. Suppose that for every element i of $\mathbb{N}$ and for every point s_1 of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$ such that $s_1 = \mathrm{Proj}(i, n) \cdot s$ and $1 \leq i \leq n$ holds $\|s_1\| \leq K$. Then $\|s\| \leq n \cdot K$.

(19) Let m, n be non empty elements of $\mathbb{N}$, s, t be points of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and K be a real number. Suppose that for every element i of $\mathbb{N}$ and for all points s_1, t_1 of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$ such that $s_1 = \mathrm{Proj}(i, n) \cdot s$ and $t_1 = \mathrm{Proj}(i, n) \cdot t$ and $1 \leq i \leq n$ holds $\|s_1 - t_1\| \leq K$. Then $\|s - t\| \leq n \cdot K$.

(20) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$, X be a subset of $\langle \mathcal{E}^m, \| \cdot \| \rangle$, and i be an element of $\mathbb{N}$. Suppose $1 \leq i \leq m$ and X is open. Then the following statements are equivalent

(i) f is partially differentiable on X w.r.t. i and $f{\restriction}^i X$ is continuous on X,

(ii) for every element j of $\mathbb{N}$ such that $1 \leq j \leq n$ holds $\mathrm{Proj}(j, n) \cdot f$ is partially differentiable on X w.r.t. i and $\mathrm{Proj}(j, n) \cdot f{\restriction}^i X$ is continuous on X.

(21) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and X be a subset of $\langle \mathcal{E}^m, \| \cdot \| \rangle$. Suppose X is open. Then f is differentiable on X and $f'_{\restriction X}$ is continuous on X if and only if for every element j of $\mathbb{N}$ such that $1 \leq j \leq n$ holds $\mathrm{Proj}(j, n) \cdot f$ is differentiable on X and $(\mathrm{Proj}(j, n) \cdot f)'_{\restriction X}$ is continuous on X.

(22) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and X be a subset of $\langle \mathcal{E}^m, \| \cdot \| \rangle$. Suppose X is open. Then for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ holds f is partially differentiable on X w.r.t. i and $f{\restriction}^i X$ is continuous on X if and only if f is differentiable on X and $f'_{\restriction X}$ is continuous on X.

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