

# More on Continuous Functions on Normed Linear Spaces

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**Summary.** In this article we formalize the definition and some facts about continuous functions from  $\mathbb{R}$  into normed linear spaces [14].

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The terminology and notation used in this paper have been introduced in the following papers: [2], [12], [3], [4], [10], [11], [1], [5], [13], [7], [17], [18], [15], [9], [8], [16], [19], and [6].

## 1. PRELIMINARIES

For simplicity, we adopt the following rules:  $n$  denotes an element of  $\mathbb{N}$ ,  $X$ ,  $X_1$  denote sets,  $r$ ,  $p$  denote real numbers,  $s$ ,  $x_0$ ,  $x_1$ ,  $x_2$  denote real numbers,  $S$ ,  $T$  denote real normed spaces,  $f$ ,  $f_1$ ,  $f_2$  denote partial functions from  $\mathbb{R}$  to the carrier of  $S$ ,  $s_1$  denotes a sequence of real numbers, and  $Y$  denotes a subset of  $\mathbb{R}$ .

The following propositions are true:

- (1) Let  $s_2$  be a sequence of real numbers and  $h$  be a partial function from  $\mathbb{R}$  to the carrier of  $S$ . If  $\text{rng } s_2 \subseteq \text{dom } h$ , then  $s_2(n) \in \text{dom } h$ .
- (2) Let  $h_1$ ,  $h_2$  be partial functions from  $\mathbb{R}$  to the carrier of  $S$  and  $s_2$  be a sequence of real numbers. If  $\text{rng } s_2 \subseteq \text{dom } h_1 \cap \text{dom } h_2$ , then  $(h_1 + h_2)_* s_2 = (h_1 *_s s_2) + (h_2 *_s s_2)$  and  $(h_1 - h_2)_* s_2 = (h_1 *_s s_2) - (h_2 *_s s_2)$ .

- (3) For every sequence  $h$  of  $S$  and for every real number  $r$  holds  $rh = r \cdot h$ .
- (4) Let  $h$  be a partial function from  $\mathbb{R}$  to the carrier of  $S$ ,  $s_2$  be a sequence of real numbers, and  $r$  be a real number. If  $\text{rng } s_2 \subseteq \text{dom } h$ , then  $rh_*s_2 = r \cdot (h_*s_2)$ .
- (5) Let  $h$  be a partial function from  $\mathbb{R}$  to the carrier of  $S$  and  $s_2$  be a sequence of real numbers. If  $\text{rng } s_2 \subseteq \text{dom } h$ , then  $\|h_*s_2\| = \|h\|_*s_2$  and  $-(h_*s_2) = -h_*s_2$ .

## 2. CONTINUOUS REAL FUNCTIONS INTO NORMED LINEAR SPACES

Let us consider  $S, f, x_0$ . We say that  $f$  is continuous in  $x_0$  if and only if:

- (Def. 1)  $x_0 \in \text{dom } f$  and for every  $s_1$  such that  $\text{rng } s_1 \subseteq \text{dom } f$  and  $s_1$  is convergent and  $\lim s_1 = x_0$  holds  $f_*s_1$  is convergent and  $f_{x_0} = \lim(f_*s_1)$ .

Next we state a number of propositions:

- (6) If  $x_0 \in X$  and  $f$  is continuous in  $x_0$ , then  $f|X$  is continuous in  $x_0$ .
- (7)  $f$  is continuous in  $x_0$  if and only if the following conditions are satisfied:
  - (i)  $x_0 \in \text{dom } f$ , and
  - (ii) for every  $s_1$  such that  $\text{rng } s_1 \subseteq \text{dom } f$  and  $s_1$  is convergent and  $\lim s_1 = x_0$  and for every  $n$  holds  $s_1(n) \neq x_0$  holds  $f_*s_1$  is convergent and  $f_{x_0} = \lim(f_*s_1)$ .
- (8)  $f$  is continuous in  $x_0$  if and only if the following conditions are satisfied:
  - (i)  $x_0 \in \text{dom } f$ , and
  - (ii) for every  $r$  such that  $0 < r$  there exists  $s$  such that  $0 < s$  and for every  $x_1$  such that  $x_1 \in \text{dom } f$  and  $|x_1 - x_0| < s$  holds  $\|f_{x_1} - f_{x_0}\| < r$ .
- (9) Let given  $S, f, x_0$ . Then  $f$  is continuous in  $x_0$  if and only if the following conditions are satisfied:
  - (i)  $x_0 \in \text{dom } f$ , and
  - (ii) for every neighbourhood  $N_1$  of  $f_{x_0}$  there exists a neighbourhood  $N$  of  $x_0$  such that for every  $x_1$  such that  $x_1 \in \text{dom } f$  and  $x_1 \in N$  holds  $f_{x_1} \in N_1$ .
- (10) Let given  $S, f, x_0$ . Then  $f$  is continuous in  $x_0$  if and only if the following conditions are satisfied:
  - (i)  $x_0 \in \text{dom } f$ , and
  - (ii) for every neighbourhood  $N_1$  of  $f_{x_0}$  there exists a neighbourhood  $N$  of  $x_0$  such that  $f^\circ N \subseteq N_1$ .
- (11) If there exists a neighbourhood  $N$  of  $x_0$  such that  $\text{dom } f \cap N = \{x_0\}$ , then  $f$  is continuous in  $x_0$ .
- (12) If  $x_0 \in \text{dom } f_1 \cap \text{dom } f_2$  and  $f_1$  is continuous in  $x_0$  and  $f_2$  is continuous in  $x_0$ , then  $f_1 + f_2$  is continuous in  $x_0$  and  $f_1 - f_2$  is continuous in  $x_0$ .
- (13) If  $f$  is continuous in  $x_0$ , then  $rf$  is continuous in  $x_0$ .

- (14) If  $x_0 \in \text{dom } f$  and  $f$  is continuous in  $x_0$ , then  $\|f\|$  is continuous in  $x_0$  and  $-f$  is continuous in  $x_0$ .
- (15) Let  $f_1$  be a partial function from  $\mathbb{R}$  to the carrier of  $S$  and  $f_2$  be a partial function from the carrier of  $S$  to the carrier of  $T$ . Suppose  $x_0 \in \text{dom}(f_2 \cdot f_1)$  and  $f_1$  is continuous in  $x_0$  and  $f_2$  is continuous in  $(f_1)_{x_0}$ . Then  $f_2 \cdot f_1$  is continuous in  $x_0$ .

Let us consider  $S, f$ . We say that  $f$  is continuous if and only if:

(Def. 2) For every  $x_0$  such that  $x_0 \in \text{dom } f$  holds  $f$  is continuous in  $x_0$ .

Next we state two propositions:

- (16) Let given  $X, f$ . Suppose  $X \subseteq \text{dom } f$ . Then  $f \upharpoonright X$  is continuous if and only if for every  $s_1$  such that  $\text{rng } s_1 \subseteq X$  and  $s_1$  is convergent and  $\lim s_1 \in X$  holds  $f_*s_1$  is convergent and  $f_{\lim s_1} = \lim(f_*s_1)$ .
- (17) Suppose  $X \subseteq \text{dom } f$ . Then  $f \upharpoonright X$  is continuous if and only if for all  $x_0, r$  such that  $x_0 \in X$  and  $0 < r$  there exists  $s$  such that  $0 < s$  and for every  $x_1$  such that  $x_1 \in X$  and  $|x_1 - x_0| < s$  holds  $\|f_{x_1} - f_{x_0}\| < r$ .

Let us consider  $S$ . One can check that every partial function from  $\mathbb{R}$  to the carrier of  $S$  which is constant is also continuous.

Let us consider  $S$ . Note that there exists a partial function from  $\mathbb{R}$  to the carrier of  $S$  which is continuous.

Let us consider  $S$ , let  $f$  be a continuous partial function from  $\mathbb{R}$  to the carrier of  $S$ , and let  $X$  be a set. Observe that  $f \upharpoonright X$  is continuous.

Next we state the proposition

- (18) If  $f \upharpoonright X$  is continuous and  $X_1 \subseteq X$ , then  $f \upharpoonright X_1$  is continuous.

Let us consider  $S$ . Observe that every partial function from  $\mathbb{R}$  to the carrier of  $S$  which is empty is also continuous.

Let us consider  $S, f$  and let  $X$  be a trivial set. Observe that  $f \upharpoonright X$  is continuous.

Let us consider  $S$  and let  $f_1, f_2$  be continuous partial functions from  $\mathbb{R}$  to the carrier of  $S$ . Observe that  $f_1 + f_2$  is continuous and  $f_1 - f_2$  is continuous.

The following two propositions are true:

- (19) Let given  $X, f_1, f_2$ . Suppose  $X \subseteq \text{dom } f_1 \cap \text{dom } f_2$  and  $f_1 \upharpoonright X$  is continuous and  $f_2 \upharpoonright X$  is continuous. Then  $(f_1 + f_2) \upharpoonright X$  is continuous and  $(f_1 - f_2) \upharpoonright X$  is continuous.
- (20) Let given  $X, X_1, f_1, f_2$ . Suppose  $X \subseteq \text{dom } f_1$  and  $X_1 \subseteq \text{dom } f_2$  and  $f_1 \upharpoonright X$  is continuous and  $f_2 \upharpoonright X_1$  is continuous. Then  $(f_1 + f_2) \upharpoonright (X \cap X_1)$  is continuous and  $(f_1 - f_2) \upharpoonright (X \cap X_1)$  is continuous.

Let us consider  $S$ , let  $f$  be a continuous partial function from  $\mathbb{R}$  to the carrier of  $S$ , and let us consider  $r$ . One can check that  $r f$  is continuous.

We now state several propositions:

- (21) If  $X \subseteq \text{dom } f$  and  $f \upharpoonright X$  is continuous, then  $(r f) \upharpoonright X$  is continuous.

- (22) If  $X \subseteq \text{dom } f$  and  $f \upharpoonright X$  is continuous, then  $\|f\| \upharpoonright X$  is continuous and  $(-f) \upharpoonright X$  is continuous.
- (23) If  $f$  is total and for all  $x_1, x_2$  holds  $f_{x_1+x_2} = f_{x_1} + f_{x_2}$  and there exists  $x_0$  such that  $f$  is continuous in  $x_0$ , then  $f \upharpoonright \mathbb{R}$  is continuous.
- (24) If  $\text{dom } f$  is compact and  $f \upharpoonright \text{dom } f$  is continuous, then  $\text{rng } f$  is compact.
- (25) If  $Y \subseteq \text{dom } f$  and  $Y$  is compact and  $f \upharpoonright Y$  is continuous, then  $f^\circ Y$  is compact.

### 3. LIPSCHITZ CONTINUITY

Let us consider  $S, f$ . We say that  $f$  is Lipschitzian if and only if:

- (Def. 3) There exists a real number  $r$  such that  $0 < r$  and for all  $x_1, x_2$  such that  $x_1, x_2 \in \text{dom } f$  holds  $\|f_{x_1} - f_{x_2}\| \leq r \cdot |x_1 - x_2|$ .

The following proposition is true

- (26)  $f \upharpoonright X$  is Lipschitzian if and only if there exists a real number  $r$  such that  $0 < r$  and for all  $x_1, x_2$  such that  $x_1, x_2 \in \text{dom}(f \upharpoonright X)$  holds  $\|f_{x_1} - f_{x_2}\| \leq r \cdot |x_1 - x_2|$ .

Let us consider  $S$ . Observe that every partial function from  $\mathbb{R}$  to the carrier of  $S$  which is empty is also Lipschitzian.

Let us consider  $S$ . One can verify that there exists a partial function from  $\mathbb{R}$  to the carrier of  $S$  which is empty.

Let us consider  $S$ , let  $f$  be a Lipschitzian partial function from  $\mathbb{R}$  to the carrier of  $S$ , and let  $X$  be a set. One can check that  $f \upharpoonright X$  is Lipschitzian.

The following proposition is true

- (27) If  $f \upharpoonright X$  is Lipschitzian and  $X_1 \subseteq X$ , then  $f \upharpoonright X_1$  is Lipschitzian.

Let us consider  $S$  and let  $f_1, f_2$  be Lipschitzian partial functions from  $\mathbb{R}$  to the carrier of  $S$ . One can check that  $f_1 + f_2$  is Lipschitzian and  $f_1 - f_2$  is Lipschitzian.

One can prove the following propositions:

- (28) If  $f_1 \upharpoonright X$  is Lipschitzian and  $f_2 \upharpoonright X_1$  is Lipschitzian, then  $(f_1 + f_2) \upharpoonright (X \cap X_1)$  is Lipschitzian.
- (29) If  $f_1 \upharpoonright X$  is Lipschitzian and  $f_2 \upharpoonright X_1$  is Lipschitzian, then  $(f_1 - f_2) \upharpoonright (X \cap X_1)$  is Lipschitzian.

Let us consider  $S$ , let  $f$  be a Lipschitzian partial function from  $\mathbb{R}$  to the carrier of  $S$ , and let us consider  $p$ . Note that  $p f$  is Lipschitzian.

Next we state the proposition

- (30) If  $f \upharpoonright X$  is Lipschitzian and  $X \subseteq \text{dom } f$ , then  $(p f) \upharpoonright X$  is Lipschitzian.

Let us consider  $S$  and let  $f$  be a Lipschitzian partial function from  $\mathbb{R}$  to the carrier of  $S$ . Note that  $\|f\|$  is Lipschitzian.

One can prove the following proposition

- (31) If  $f|X$  is Lipschitzian, then  $-f|X$  is Lipschitzian and  $(-f)|X$  is Lipschitzian and  $\|f\||X$  is Lipschitzian.

Let us consider  $S$ . One can verify that every partial function from  $\mathbb{R}$  to the carrier of  $S$  which is constant is also Lipschitzian.

Let us consider  $S$ . Observe that every partial function from  $\mathbb{R}$  to the carrier of  $S$  which is Lipschitzian is also continuous.

Next we state two propositions:

- (32) If there exists a point  $r$  of  $S$  such that  $\text{rng } f = \{r\}$ , then  $f$  is continuous.  
 (33) For all points  $r, p$  of  $S$  such that for every  $x_0$  such that  $x_0 \in X$  holds  $f_{x_0} = x_0 \cdot r + p$  holds  $f|X$  is continuous.

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