FUNCTIONS OF OPEN FLOW-CHART PROVING IN INTRODUCTORY LESSONS OF FORMAL PROVING[1]

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Amongst important and under-researched questions are how introductory lessons can be designed for teaching initial proofs to junior high school students, and how such lessons enrich students' understanding of proofs. With a view to improving the learning situation in the classroom, in this paper we report on the various functions of introductory flow-chart proofs that use 'open problems' that have multiple possible solutions. Through an analysis of a teaching experiment in Grade 8, and by using a model of levels of understanding of proof structure, we identify the functions as enhancing the transition towards a relational understanding of the structure of formal proof, and encouraging forms of forward/backward thinking interactively that accompany such a relational understanding of the structure of proofs in mathematics.

INTRODUCTION

With proving and reasoning universally recognized as key competencies of mathematics education, it remains the case that students at the lower secondary school level can experience difficulties in understanding formal proofs (eg: Hanna & de Villiers, 2012; Mariotti, 2006). In order to enhance the capabilities of junior high school students with formal proving (from around the age of 14), it is important to have a clear framework to inform the design of introductory proof lessons. This is because such lessons aim to initiate inexperienced students into understanding the meaning of formal proofs fruitfully so that they can develop the competencies to construct proofs for themselves. We have previously reported that students who have experienced such introductory lessons can score around 10% better than expected on a question that involved choosing reasons to deduce a conclusion (see Miyazaki, Fujita and Jones, 2012). In this paper we report a further qualitative analysis that focuses on why the students did well in such mathematical proofs. Our research questions are as follows: how can introductory lessons for formal proofs be designed, and how do such lessons enrich students' understanding of proofs?

In order to enrich the introductory lessons of formal proving, our research study focuses on the students learning to use flow-chart proofs in 'open problem' situations where they can construct multiple solutions for congruent triangle tasks by deciding the assumptions and intermediate propositions necessary to deduce a given conclusion in a flow-chart format. Such proofs involve using the conditions for triangle

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congruency as these are often used to introduce formal proofs in geometry in Japanese lower secondary schools (Jones & Fujita, 2013), and our discussions and analyses are related to this topic. The aim of this paper is to evaluate the introductory lessons designed on the basis of our theoretical framework by identifying their pedagogical functions and implications.

THEORETICAL FRAMEWORK: UNDERSTANDING PROOF STRUCTURE

We take as our starting point that a formal proof generally consists of deductive reasoning between assumptions and conclusions. Within this reasoning process at least two types of deductive reasoning are employed: universal instantiation (which deduces a singular proposition from a universal proposition) and hypothetical syllogism (where the conclusion necessarily results from the premises).

In order to understand the structure of proof, students need to pay attention to the elements of the proof and their inter-relationships. Research studies by Heinze and Reiss (2004) and by McCrone and Martin (2009) have identified that an appreciation of proof structure is an important component of learner competence with proof. In this paper we use the following levels of learner's understanding of proof structure initially elaborated by Miyazaki and Fujita (2010): Pre-, Partial- and Holistic structural levels. These levels are described in Table 1 and the overall framework illustrated in Figure 1.

Level	Description
Pre-structural	The basic status in terms of an understanding of proof structure where learners regard proof as a kind of 'cluster' of possibly symbolic objects.
Partial-structural	Once learners have begun paying attention to each element, then we consider they are at the *Partial-structural Elemental* sub-level. To reach the next level, learners need to recognize some relationships between these elements (such as universal instantiations and syllogism). If learners have started paying attention to each relationship, then we consider them to be at the *Partial-structural Relational* sub-level, with this sub-level being further sub-divided into a) universal instantiation and b) syllogism (see Figure 1).
Holistic-structural	At this level, learners understand the relationships between singular and universal propositions, and see a proof as 'whole' in which premises and conclusions are logically connected through universal instantiations and hypothetical syllogism.

Table 1: Levels of learner understanding of proof structure

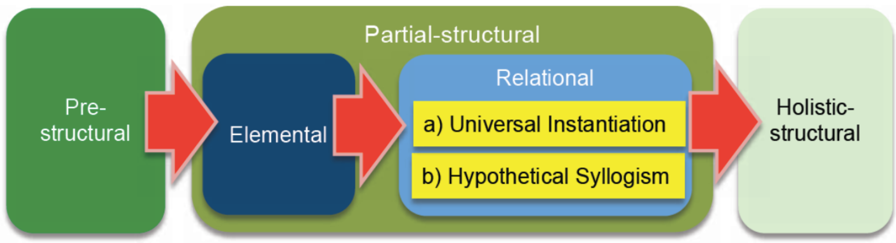

Figure 1: Framework of learner understanding of the structure of proof

To date we have utilized this framework to demonstrate students' explorative activity to overcome logical circularity in a proof problem (Fujita, Jones, & Miyazaki, 2011), and considered how a hypothetical learning trajectory for introductory lessons of formal proving could be designed so that students can be helped to develop their understanding of the structure of proof (Miyazaki, Fujita, & Jones, 2012). In this paper we focus on the design of introductory lessons of formal proofs.

INTRODUCTORY LESSONS USING OPEN FLOW-CHART PROVING

To design introductory proof lessons we used the following two pedagogical ideas: flow-chart proof format and 'open problem' tasks. A flow-chart proof shows a 'story line' of the proof. McMurray (1978) and others have provided accounts of the value of using flow-chart proofs prior to the use of formats such as the 'two column proof'.

Given the evidence that flow-chart proofs can help students to visualize the structure of proofs, in our research we are investigating how the power of flow-chart proofs might be enhanced at the introductory stage of proof learning by using 'open problem' situations where students can construct multiple solutions by deciding the assumptions and intermediate propositions necessary to deduce a given conclusion.

For example, the problem in Figure 2 is intentionally designed so that students can freely choose which assumptions they use to show the conclusion that ∠B=∠C. After drawing a line AO, for instance, students might decide ΔABO and ΔACO should be congruent to show ∠B=∠C by using the theorems "If two figures are congruent, then corresponding angles are equal." Based on AO=AO as a same line, ΔABO≡ΔACO can be shown by assuming AB=AC and ∠BAO=∠CAO using the SAS condition. However, other solutions are also possible. One approach might be to use the fact that ΔABO≡ΔACO can be shown by assuming AO=AO, AB=AC and BO=CO, using the SSS condition. As students can construct more than one suitable proof, we refer to this type of problem situation as 'open'.

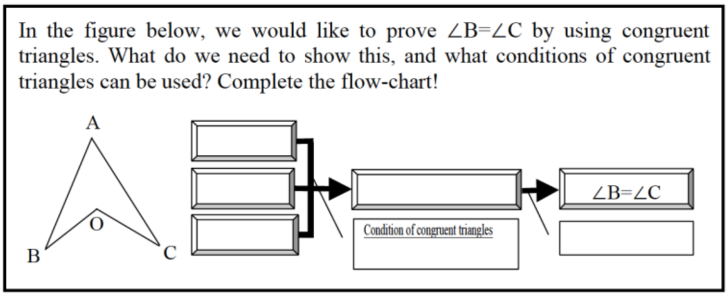

Figure 2: An example of flow-chart proving in an 'open-problem' situation

In accordance with our theoretical framework, in the introductory proof lessons it is particularly important to support transitions from the Partial-Structural to the Holistic-Structural level. The flow-chart format aims to help students to visualize that a formal proof consists of two kinds of propositional layers, one of which contains universal propositions (theorems) and the other contains the chain of singular propositions. Also, the flow-chart format can show clearly that a singular proposition is deduced by the universal instantiation of universal proposition, and that the chain of singular propositions between assumptions and conclusions would be established by hypothetical syllogism. Moreover, in order to show a given conclusion in the 'open problem' situation, students would be encouraged to seek out the necessary assumptions and intermediate propositions diversely. Then, they have a chance to originate alternative proofs by replacing the used theorems into others, and so on.

METHODOLOGY

To investigate the functions of open flow-chart proving in the introductory lessons of formal proving in Grade 8 (aged 14), we developed nine lessons based on the learning progression with three phases as follows (Miyazaki, Fujita, & Jones, 2012).

- Constructing flow-chart proofs in an 'open problem' situation (four lessons)
- Constructing a formal proof by reference to a flow-chart proof in a 'closed problem' situation (two lessons)
- Refining formal proofs by placing them into flow-chart proof format in a 'closed problem' situation (three lessons).

During the first phase of lessons, students constructed flow-chart proofs in 'open problem' situations. Through these tasks, the students were expected to learn how to think forward/backward between assumptions/conclusions and how to organize their thinking in order to connect assumptions and conclusions. Thus this phase aimed at supporting them to understand how to 'assemble' a proof as a structural entity. Note that they study proof in 'closed-problem' situations after the first phase.

Our main data are taken from one of our lesson implementations in which a teacher with 18 years of teaching experience conducted the set of the nine Grade 8 lessons in a junior high school in Japan during October 2013. The lessons were video-recorded and then transcribed. In the next section we report selected scenes from the fourth lesson in which students undertook the problem in Figure 2. By this data analysis, we identify the functions of open flow-chart proving during the introductory lessons designed using our theoretical framework of the understanding of structure of proof.

DATA ANALYSIS AND DISCUSSION

In reporting our findings from the fourth lesson, first we show the students' levels of thinking at this stage; in particular their incomplete understanding of universal instantiations. Then, we show how learning with 'open problem' proof tasks helped them to start to see proofs from a more structural point of view.

Enhancing the structural understanding of formal proof: universal instantiations

While prior to the lesson the students had used a one-step flow-chart proof to prove that two given triangles are congruent, during this lesson they tackled the problem in Figure 2. This has two steps; first deducing the congruence of triangles, and second, concluding the equivalence of angles. As one purpose of the lesson was to make students aware of the importance of universal instantiation (which deduces a singular proposition from a universal proposition), the teacher oriented the students to confirm the necessity of supplementary line AC to deduce $\angle B=\angle C$ by using the congruency of ΔABO and ΔACO, and wrote "$\Delta ABO\equiv\Delta ACO$" into the flow-chart on the board.

Thereafter, the students started to complete the flow-chart proof by themselves. After a suitable time the teacher asked student SA to say what he would put in the flow-chart box for the properties of congruent figures. SA answered “because of ΔABO≡ΔACO” (see line 6 SA in the transcript below); the teacher wrote this answer on the blackboard. Next, the teacher directed two other students to show their answer. One of them said, “Due to congruent triangles, angles are congruent”, and another said, “In congruent triangles the corresponding angles are equivalent.” The teacher also wrote these answers on the blackboard. At this time the teacher compared these three answers, and asked SA to explain more; their dialogue is shown as follows.

1	T:	SA, can you tell us why you wrote this?
2	SA:	Umm, I considered why the angles are equal; then I found an arrow is drawn.
3	T:	OK, because the arrow can be drawn (pointing the corresponding part of flow-chart on the blackboard).
4	SA:	I put ‘it’.
5	T:	What is ‘it’?
6	SA:	ΔABO and ΔACO are congruent.
7	T:	OK, if we can say these two are congruent, then we can use the arrow. So, SA, if two triangles are congruent, what can we show?
8	SA:	Angles are also equal.
9	T:	Good, angles are also equal? Anything else?
10	SA:	Sides are equal, too.
11	T:	Yes, sides are equal too. So, umm, in this case our conclusion is to say the angles are equal, so it is OK. But in general if two triangles are congruent, it can be angles but also sides as well, so we should add information generally about angles such as ‘because angles are congruent or equal’.

Given that prior to this lesson the students could find the appropriate conditions of triangle congruency, and write them into the theorem box (universal proposition) given in the one-step flow-chart proof. It was expected that they would reach the *partial-structural elemental* sub-level (by paying attention to elements of proofs) during this lesson. Beyond this, some students might start reaching the *relational* sub-level (by understanding both universal instantiation and hypothetical syllogism) through examining the properties of congruent figures.

Nevertheless, during the early parts of this lesson it was evident that only a small proportion of the students could reach the *relational* sub-level. In fact, about half the students could not correctly write two boxes of flow-chart, each of which requested the condition of congruent triangles and the properties of congruent figures. Others just wrote a singular proposition “because of ΔABO≡ΔACO” into the theorem box (like student SA said). This singular proposition is not precise enough from a universal instantiation point of view. It is clear that such students remained at the *elemental* sub-level, and could not reach the *relational* one. In particular, the students who wrote the singular proposition could not understand that a singular proposition should be deduced by the universal instantiation of a universal proposition.

In order to resolve the student' lack of understanding, the teacher compared SA's answer with others answer in which universal propositions were correctly used (the relational sub-level), and pointed out that it was necessary to express the property of congruent figures generally because it was being used to deduce the equivalence of angles in this case (although it could be used to deduce the equivalence of both angles and sides). This resolution managed by the teacher might have supported the students to enhance their understanding of the universal instantiation that deduces a singular proposition with a universal proposition. This, in turn, could promote the transition from the elemental sub-level to the relational one.

From the above we can identify as the functions of 'open problem' flow-chart proving that it can enhance the transition towards a relational understanding of the structure of formal proof by helping student to visualize the connection of singular proposition to hypothetical syllogism and the connection with universal instantiation between a singular proposition and the necessary universal proposition. This 'open problem' flow-chart format can help visualize not only the connection of singular propositions by hypothetical syllogism but also the connections of a singular proposition with a universal one by universal instantiation. With this visualized format, students could be supported effectively to focus on the characteristics of the two kinds of deductive reasoning, by checking the expression of theorems and confirming their meaning and/or roles.

Encouraging thinking forward/backward interactively by using open proof situations

After most of the students made their own flow-chart proofs, the teacher picked up three answers, each of which used different conditions of congruent triangles (this was possible because of the 'open problem' situation). The teacher checked with the class if three pairs of angle/sides were necessary to deduce $\Delta ABO \equiv \Delta ACO$ with each congruent condition, and then also checked the reason why they chose these pairs on the basis of the words written in the box below each of the three pairs.

For example, student KA used the ASA condition and the teacher asked him why he chose the followings; '$AO=AO$', '$\angle BAO=\angle CAO$', '$\angle AOB=\angle AOC$'.

The student's explanation was as follows:

1	KA:	Because we can see AO=AO from the given figure.
2	T:	Can see it from the given figure?
3	KA:	And it is an assumption. I assumed by myself $\angle BAO=\angle CAO$, and also $\angle AOB=\angle AOC$ as well. And then we can show $\Delta AOB \equiv \Delta AOC$, and the condition is 'Two pairs of corresponding angles are equal and the included sides equal'. Due to congruent triangles, corresponding angles are equal and therefore $\angle B=\angle C$.

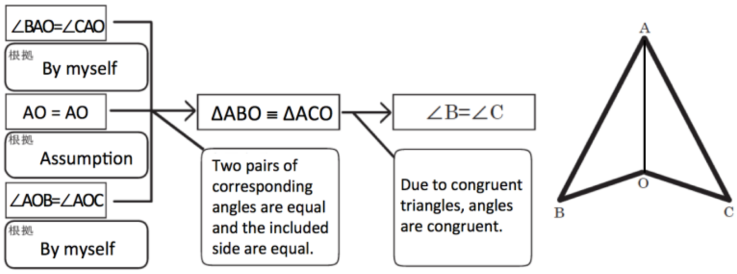

Figure 3: One of the flow-chart proofs by KA on the blackboard

As can be seen from the dialogue and the flow-chart proofs by KA shown in Figure 3, for the reason why "AO=AO", KA wrote "Assumption" in the box and explained that this equivalence was apparent by means of the given figure (see line 3 KA). In contrast, for the reasons why "∠BAO=∠CAO" and "∠AOB=∠AOC" KA wrote "By myself" and explained that they were decided by himself (see line 3 KA). In this thinking process, there were the two ways of approach. One way is thinking forward, i.e. in order to find the conditions for ΔABO≡ΔACO, KA focused on the corresponding angles/sides of these triangles and judged that "AO=AO" could be one of the conditions. A second way is thinking backward, i.e. KA chose ASA as a condition and then looked for the other conditions (in this case "∠BAO=∠CAO" and "∠AOB=∠AOC") which were necessary to satisfy this condition. It is the 'open problem' situation that made it possible for KA to use these two ways of thinking interactively. Furthermore, KA actually wrote in his worksheet two types of flow-chart proof. Each of these used different conditions: SSS and SAS. To complete these proofs he similarly determined the assumptions that were necessary to deduce the congruent triangles. Likewise, most other students in the class constructed three different proofs using similar thinking processes.

From the above we can identify as the functions of 'open problem' flow-chart proving that it can encourage thinking forward/backward interactively, accompanied by relational understanding of the structure of proof. The amplification of thinking backward, in particular, can be triggered by the 'open problem' situation. Moreover, the flow-chart proof format can support students to associate two modes of forward/backward thinking visually. This systematic learning with thinking forward/backward interactively is useful for the planning of formal proof that usually precedes its construction (Tsujiyama, 2012). Thus the learning of 'open problem' flow-chart proving in the first phase of introductory lessons of formal proving can be preparatory to the planning of formal proof in a 'closed problem' situation.

CONCLUSIONS

Within our focus on students understanding of the structure of proof, we can identify two functions of 'open problem' flow-chart proving. One is that it can enhance the transition towards the relational understanding of the structure of formal proof by visualizing both the connection of singular proposition by hypothetical syllogism and the connection with universal instantiation between a singular proposition and the necessary universal proposition. A second function is that 'open problem' flow-chart proving can encourage thinking forward/backward interactively, accompanied by relational understanding of the structure of proof. In particular, this study illustrates that 'open problem' flow-chart proving can give students a chance to find necessary conditions and combine them in order to connect assumptions with conclusions. This systematic learning with thinking forward/backward interactively is required to make the planning of formal proofs. We suggest that it is these functions that contribute to developing students' understanding of proofs, and that is why the students who experienced our introductory lessons scored 10% better than the national average of proof problems in general (Miyazaki, Fujita & Jones, 2012).

Due to page limitation we cannot show that some students, after finishing solving the assigned task, attempted to 'expand' and/or 'break' the given flow-chart proof format so that they could show their own way of proving. This further illustrates that the innovative use of 'open problem' flow-chart proving, as in our project, can cultivate students' productive thinking about formal proofs even in introductory proof lessons.

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