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DEVELOPING A CURRICULUM FOR EXPLORATIVE PROVING IN LOWER SECONDARY SCHOOL GEOMETRY

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Proving is explorative in nature. It means that proving involves producing statements, producing proofs, looking back (examining, improving and advancing) these productions, and their interactions among these aspects. We aim to echo the explorative nature of proving in curriculum development by mainly focusing on the planning aspects and constructing aspects in producing proofs. As the result we found two kinds of learning progressions as a framework, developed a curriculum of geometry for junior high school by corresponding the progressions with the units of "Course of Study" in Japan. We further refined the provisional curriculum by implementing lessons by expert teachers and reflecting on these lessons with them.

INTRODUCTION

The teaching and learning of proof is recognised internationally as a key component of mathematics curricula. Yet it remains the case that students at the junior high school level (and beyond) experience difficulties in learning proofs in mathematics in general. Hanna and de Villiers (2012, p. 3) explain, "a narrow view of proof [as solely a formal derivation] neither reflects mathematical practice nor offers the greatest opportunities for promoting mathematical understanding". In school geometry, proofs are often presented in an arrangement generally called the 'two-column format'. Yet, as Herbst and Brach (2006) show, such an approach does not necessarily support students to build up reasoned arguments for themselves creatively. In contrast to a rigid view of proof, we regard proving as a flexible, dynamic and productive activity in nature. In order to realize teaching of proofs based on this view, we are currently undertaking a study based on a design experiment (Cobb, et al, 2003) for a development of curriculum for explorative proving in lower secondary schools (G7-8). This paper reports our first cycle of the experiment around the following questions; a) What can the idea of 'Explorative Proving' be conceptualized?; b) How can we realize the idea to a curriculum?, c) How can we develop the provisional curriculum?; d) How can we establish the curriculum by realizing and reflecting on the lessons?

THEORETICAL UNDERPINNINGS: EXPLORATIVE PROVING

Concerning Question a), based on the work of Fawcett (1938), Waerden (1967), and Lakatos (1976), we argue that proving activities in mathematics are not limited only to writing a proof, but also involve producing statements inductively/deductively/analogically, planning and constructing proofs, looking back over proving processes and overcoming global/local counter-examples or errors, and utilizing already-proved statements in the context of working on further proofs (see Figure 1) in order to reflect the nature of proving as an activity in mathematics (Freudenthal, 1971).

By considering insights from the above, we define explorative proving as having the following three components and their relationships: producing propositions, producing proofs (planning and construction) and looking back (examining, improving and advancing) (Miyazaki & Fujita, 2015). By making these aspects and their relationships more explicit in the curriculum of proving, we expect that students would produce mathematical statements, produce proofs, and examine/improve/advance statements and proofs by themselves.

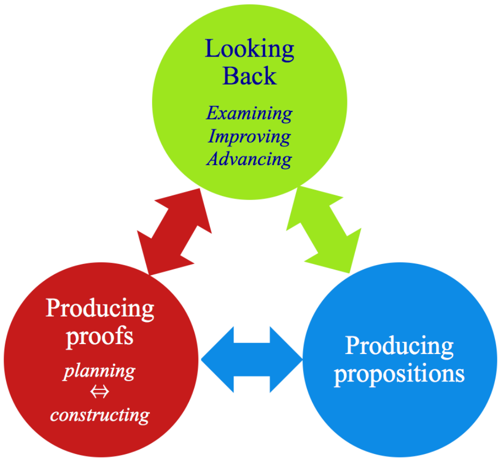


Fig. 1: Explorative proving

FRAMEWORK TO DEVELOP CURRICULUM OF EXPLORATIVE PROVING

Focusing on "Planning a proof" and "Constructing a proof"

For developing our curriculum of explorative proving, it is useful to set up some theoretical learning levels as the scope of curriculum, which enables us to consider achievable learning progression for students. Especially, due to the low achievement to produce proofs of junior high school students repeatedly reported by the national survey, we particularly focus on how we support the following two aspects that enables active production of proofs: 'Planning a proof' and 'Constructing a proof'. Provided students need to reach the elemental sub-level of partial structural level of understanding a proof (Miyazaki & Fujita, 2010) to accomplish our curriculum.

Levels of 'Planning a proof'

'Planning a proof' means students' activity to seek ways how to connect premises and conclusions by deductive reasoning (Tsujiyama, 2011). In this activity necessary conditions are deduced in the direction from premises to conclusion by thinking forward. On the contrary, sufficient conditions are deduced in the opposite direction by thinking backward. The former proposes the network of propositions that can be deduced from premises, and the latter proposes the other network of propositions that can be deduced from conclusions. In planning a proof, it needs to be sought the common propositions as joint points of the two networks while these networks expand respectively.

For planning a proof, it should be considered carefully that what can be used to connect premises and conclusions (Object), and how it can be used (Method). The learning to plan a proof requests to differentiate the objects and methods to plan a proof, and then to make use of them in order to connect premises and conclusions. This is the first learning level of planning a proof (P1).

As described above, planning a proof needs to expand the two networks of propositions respectively and to seek the common propositions within the two networks. Therefore, the advanced learning of planning a proof requests to differentiate the method of P1 into thinking forward from premises to conclusions and thinking backward in the opposite direction, and then to make use of them in order to connect premises and conclusions. This is the second learning level of planning a proof (P2).

Now, we can establish the following learning level related to 'Planning a proof'.

P1: Clarify what and how can be used to connect premises and conclusion.

P2: Consider how to think backward from conclusion and think forward from premises, and how to connect them.

Levels of 'Constructing a proof'

'Constructing a poof' consists of finding the common propositions in two relational networks and expressing the deductive connection between premises and conclusions, which are suggested by planning. This connection can be mainly realized by two kinds of deductive reasoning (universal instantiation and hypothetical syllogism). Finally, constructing a proof can be achieved by expressing the realized connection with language, diagram, etc.

Especially in a geometrical proof, premises and conclusions can be connected mainly by hypothetical syllogism based on singular propositions peculiar to the diagram. Therefore, the learning of "constructing a proof" firstly requests to express the part of connection based on hypothetical syllogism. This is the first learning level of constructing a proof (C1).

Considering a proof more strictly, each singular proposition would be deduced with universal proposition (ex. theorems). This deduction can be realized by universal instantiation. Then, the leaning to construct a proof needs to differentiate universal instantiation and hypothetical syllogism from deductive reasoning, and to express singular propositions and universal propositions with clear distinction. This is the second learning level of constructing a proof (C2).

Now, we can establish the following learning level related to constructing a proof.

C1: Form and express the deductive connection between premises and conclusions.

C2: Form and express the deductive connection between premises and conclusions with differentiating universal instantiation and hypothetical syllogism from deductive reasoning.

Setting the Imaginary Learning Levels

'Planning a proof' and 'Constructing a proof' are interrelated essentially to realize explorative proving in mathematics. On the other hand, in order to develop school mathematics curriculum, assuming that 'Planning a proof' and 'Constructing a proof' are independent each other, Combining two levels of 'Planning a proof' and two levels of 'Constructing a proof' can produce the four learning levels: (P1, C1), (P2, C1), (P1, C2), (P2, C2).

The learning level would be called '0' as the starting point of learning progression of explorative proving. At this level, even there is no differentiation between planning a proof and constructing a proof, producing a proof can be completed. Similarly, there are learning levels that focus on either planning a proof or constructing a proof despite the differentiation of them. These learning levels are called P1, P2 and C1, C2. Now, we can set the nine imaginary learning levels as follows.

Two Kinds of Shifts of Learning Levels and their Processes

In lower secondary school geometry of Japan our curriculum of explorative proving should start the shift from Level 0 to Level (P2, C2) due to the correspondence with current national curriculum. The shift needs to pass through Level (P1, C1) in order to enhance planning a proof and constructing a proof reciprocally. Therefore, the shift can be divided into the former shift [0 ⇒ (P1, C1)] and the latter shift [(P1, C1) ⇒ (P2, C2)].

The transition process of the former shift passes through either Level C1 or Level P1. In the case passing though Level C1 this level attains to make students connect assumptions and conclusions by hypothetical syllogism. Then, at the next level (P1, C1) the learning of P1, that is, clarifying what and how can be used to connect premises and conclusion can be realized. On the other hand, in the case passing through Level P1 the learning of P1 cannot be realized because in order to learn clarifying what and how can be used to connect premises and conclusion (P1) it is necessary to have a chance to form and express the deductive connection between premises and conclusions (C1)

The transition process of latter shift passes through either Level (P1, C2) or (P2, C1). In the case passing through (P1, C2) this level attains to form and express the connection between premises and conclusions with differentiating universal instantiation and hypothetical syllogism from deductive reasoning. Due to carrying out deductive reasoning based on universal instantiation, at the next Level (P2, C2) thinking backward from conclusions and thinking forward from assumptions can be differentiated and carried out together. On the other hand, in the case passing through Level (P2, C1) the learning of P2 cannot be realized because in order to learn considering how to think backward from conclusion and think forward from premises, and how to connect them (P2) it is necessary to have a chance to form and express the deductive connection based on universal instantiation that makes possible to distinguish thinking forward and thinking backward.

Establishing Theoretical Framework to Develop Curriculum of Explorative Proving

Concerning Question b) ‘How can we echo the idea to a curriculum?’, we could specify the six learning levels and establish the two transition processes as learning progressions. The former one is from Level 0 to (P1, C1) via C1, and the latter one is from Level (P1, C1) to (P2, C2) via (P1, C2). For each level the component ‘Looking back (<u>E</u>xamining, <u>I</u>mproving and <u>A</u>dvancing, EIA)’ (Fig. 1) can be expected and encouraged as explorative proving. Depending on the contents it should be decided whether ‘Looking back’ (EIA) actually should be intended or not. Therefore, by regarding these transition as learning progressions we can establish thef ramework to develop curriculum of explorative proving (Fig. 2)

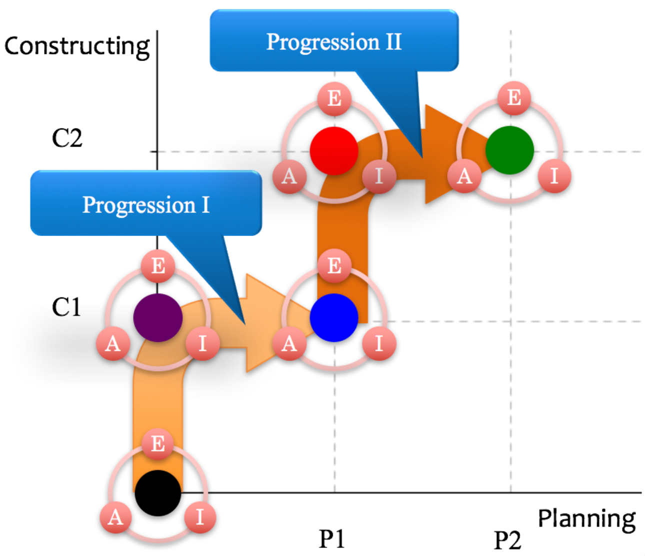


Fig. 2: Learning progressions in proving

MAKING CORRESPONDENCE TABLES OF UNITS WITH LEARNING PROGRESSIONS

To answer Question c) ‘How can we develop the provisional curriculum?’, we firstly examine the existing intended curriculum and lessons according to “Course of Study” in Japan, and show how we can make them more explorative - based on our theoretical frameworks described above.

In lower high school of Japan, "Course of Study" demands to learn various properties of plane and space figures mainly based on congruency and similarity, and to learn meaning of proofs and how to prove formally. Although "Course of Study" encourages to introduce formal proofs gradually until the end of Grade 8, there shown no gradual way clearly to realize planning a proof and constructing a proof. In order to improve this, we first consider the correspondence of the intended units in |"Course of Study" with the local progressions of two learning progressions in our theoretical framework, illustrated in the table 2.

For example, in Grade 8 'Geometry', "Course of Study" requires to learn the following units: properties of parallel lines and angles, properties of angles of polygons, meaning of congruent and conditions of triangle congruent, meaning of proof and how to prove, and properties of triangles and quadrilaterals. Assuming that Level (P1, C1) is attained by the end of Grade 7, the transition process from (P1, C1) to (P2, C2) +EIA can be subdivided into the five local progressions as follows:（P1, C1）→（P1, C2）, (P1, C2) ↑ (P1,C2)+EIA, （P1, C2）→（P2, C2）, (P2, C2) ↑ (P2, C2) +EIA. By combing the intended units with the local progressions with considering the characteristics of units which local progression can be accomplished, we can make the correspondence table of the intended units with our local progressions (Table 1).

Units in "Course of Study"	Local Progressions
Properties of parallel lines and angles	（P1, C1）→（P1, C2）
Properties of angles of polygons	(P1, C2)↑(P1,C2)+EIA
Meaning of congruent and conditions of triangle congruent	
Meaning of formal proofs and how to prove formally	（P1, C2）→（P2, C2）
Properties of triangles and quadrilaterals	(P2, C2)↑(P2, C2) +EIA

Table 1: Correspondence of intended units with local progressions in Grade 8 geometry

REALIZING CLASSROOM LESSONS BASED ON THE CORRESPONDENCE TABLES

To answer Question d) 'How can we establish the curriculum by realizing and reflecting on the realized lessons?', we design and implement the lessons in junior high schools under our correspondence tables, derived our theoretical examinations described above. We take the method of lesson study (Lewis, Perry and Murata, 2006) and a design experiment (Cobb, et al, 2003) to plan, implement and reflect on these lessons. Especially we take the extra care of cooperation between researchers and expert teachers to produce the desirable activities for our learning progressions. We are currently at the first cycle of the experiment.

Localizing Correspondence Table according to Leaning Contents

Every unit in "Course of Study" includes many contents. To realize lessons based on the correspondence table (table 1) of units with local progression it is necessary to localize each pair of them into each content included in the units.

For example, Unit 'Properties of parallel lines and angles' for Grade 8 includes seven contents content. Six of them are essential properties and the remained is a standard activity in Japan to apply the learned properties. According to the correspondence table (Table 2), learning this unit intend to achieve the following local progression: (P1, C1)→(P1, C2). Because aiming the progression in every content is not realistic nor effective we should produce a "gentle" progress for students along with contents included in a unit (Table 2). In preparing the series of lessons expert mathematics teachers consider further how to design the local progressions, and we designed the following local progressions by getting valuable suggestion from these teachers.

No.	Learning Contents	Local Progressions
1	Vertical angles are equal.	(P1, C1)
2	Corresponding angles are equal. ⇔ Two lines are parallel.	(P1, C1) →(P1, C2)
3	Alternate interior angles are equal. ⇔Two lines are parallel.	(P1, C1) →(P1, C2)
4	Sum of inner angles at the same side is 180°.	(P1, C1) →(P1, C2)
5	Sum of inner angles of a triangle	(P1, C2)
6	Relation between inner angles and an exterior angle in a triangle	(P1, C2)
7	Angles in slanting L-shaped lines between two parallel lines	(P1, C2)

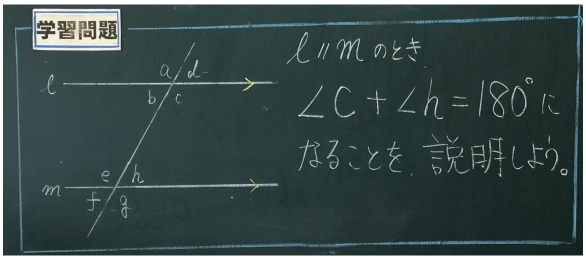
Table 2: Local progressions in Unit "Properties of parallel lines and angles"

Planning and Implementing Lessons: Unit "Properties of parallel lines and angles"

The lessons based on the local progressions in Unit 'Properties of parallel lines and angles' were implemented to a class (16 boys and 19 girls) of Grade 8 during the period from December 10 (2014) to January 20 (2015) including winter vacation (about two weeks) in a public junior high school of Nagano-Shi (Nagano Pref.). This school is located in the centre of the city and has six classes in Grade 8. At that time the teacher had an experience of fifteen years as mathematics teacher in junior high school.

On account of limited space we introduce the implementation of 4th lesson corresponding the local progression (P1, C1) →(P1, C2), i.e. aiming at shifting students' understanding of 'Construction of proof' from C1 to C2. Before the 4th lesson the six properties of angles in lines were found inductively and proved by simple deductive reasoning, and then shared as theorems. Each of these theorems was written on the piece of papers individually with adding different numbers from "1" to "6", and these theorems were listed accumulatively on the worksheet for students and the blackboard every lesson: *① Straight angle is 180°*; *② Vertical angles are equal*; *③ Corresponding angles are equal if the two lines are parallel; ④ The two lines are parallel if the corresponding angles are equal; ⑤ Alternate interior angles are equal if the two lines are parallel;* ⑥ The two lines are parallel if the alternate interior angles are equal.

In 4th lesson a problem for students to explain the reason why the sum of interior angles on the same side is 180° was proposed with the following diagram (Fig. 3).

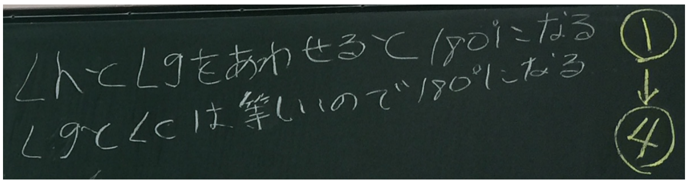


Given *l* // *m*.

Explain why ∠c + ∠h =180°.

Fig. 3 Problem teacher proposed

Confirming that the six theorems can be used to solve the problem, the teacher rewrote the problem as follows: "By using the learnt theorems, write the reason why ∠c + ∠h =180°". After individual and collaborative solving the teacher picked up Student K's explanation, and asked to him "Which theorems did you use and how?". Through classroom discussion the teacher confirmed that the theorems in the order of "①→④"was used and wrote it on the blackboard (Fig. 4).

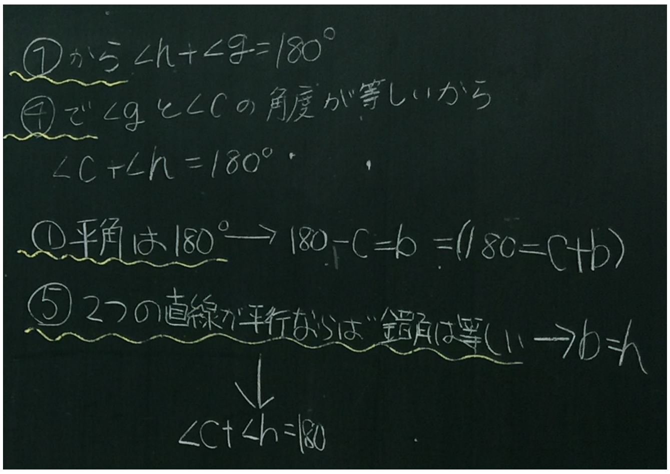


Combining ∠h with ∠g makes 180°.

∠g is equal to ∠c, then 180°.

Fig. 4 Student K's explanation and the order of the used theorems

Next, the teacher picked up the two other explanations as follows, and admired that the number of theorems (e.g. ⑤) were embedded correctly with drawing wavy lines in yellow (Fig .5).



Due to ① ∠h + ∠g = 180°.
By ④ ∠g is equal to ∠c.
Then, ∠c + ∠h = 180°

① "Straight angle is 180°."→180-c = b = (180=c+b)
⑤ "Alternate interior angles are equal if the two lines are parallel." → b = h

↓

∠c + ∠h = 180°

Fig. 5 Two explanations and Teacher's suggestions by yellow lines

Finally, the teacher named this relation of two angles '同側内角' (interior angles on the same side), and wrote the theorem with the number "7" on a piece of drawing paper as follows: "⑦ The sum of interior angles on the same side is 180° if two line are parallel"

According to the local progressions in Unit 'Properties of parallel lines and angles' (Table. 3) the lessons from the 1st to 4th aimed to achieve the shift of learning levels from (P1, C1) to (P1, C2). Concerning P1 in every lesson the teacher always asked students how to solve the problem from the point of what could be used, and how to use them. On the other hand, the shift from C1 to C2 needs to differentiate universal instantiation and hypothetical syllogism from deductive reasoning. Especially in 4th lesson, the teacher clarified not only the implicit theorems but also the order to use them, and recommended the explanation expressing theorems adequately. These suggestions made it clear for students that the how theorems used in universal instantiation should be embedded in their explanation to construct a proof in Level C2.

REFINEMENT OF CURRICULUM BY REFRECTING ON PRACTICING LESSONS

After implementing the seven lessons we found a positive possibility to realize Level (P1, C2) + EIA in 6th and 7th lessons. For example, in 6th lesson students started focusing on the relation between inner angles and an exterior angle in a triangle and deducting reasons why based on the proof constructed in the previous 5th lesson on 'Sum of inner angles of a triangle'. These episodes made us modify the original local progression (P1, C2) corresponding to 6th and 7th lesson to (P1, C2)+EIA. By this change the 6th lesson would more strongly focus on how to discover the relation between inner angles and an exterior angle from the proof constructed in 5th lesson, and we will discuss the necessity of 7th lesson from the point of our learning progression.

TOWARD DEVELOPING ROBUST CURRICULUM OF EXPLORATIVE PROVING

We have already made the correspondence tables of units with our learning progressions for Grade 7-9 on geometry along with Japanese course of study. Under the fruitful collaboration with expert teachers these tables are going to be subdivided into local progressions according to the learning contents included in a units, and then planning & implementing lessons are also going to proceed. We will conduct another cycle of our design experiment and by reflecting on the lessons implemented in the first cycle, and it is expected by the end of the second one we will be able to develop a more implementable curriculum. On the other hand, we should evaluate our curriculum from the point of learning of explorative proving, e.g. How can students achieve explorative proving with our curriculum? – this is one of the most crucial problems. In order to evaluate this ability, we should develop the standards for evaluation.

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