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LAKATOS' HEURISTIC RULES AS A FRAMEWORK FOR PROOFS AND REFUTATIONS IN MATHEMATICAL LEARNING: LOCAL-COUNTEREXAMPLE AND MODIFICATION OF PROOF

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Proof and proving should be the center of students' mathematical experiences at all grades. It is important to construct frameworks for capturing students' proving activities, because such frameworks enable us to deepen our understanding about students' behaviors and to support their learning. Among various activities related to proving, this study focuses on a process of proofs and refutations, and constructs a framework for the process in mathematical learning by utilizing Lakatos' heuristic rules. In particular, this paper illustrates the framework through analyzing a lesson in which ninth graders faced a local counterexample of their proof and modified it. Based on the analysis, this paper also proposes a hypothesis about important actions in modification of proofs.

INTRODUCTION

Proof and proving are at the core of mathematical activities, and thus should be the center of students' mathematical experiences at all grades. In order to enrich students' learning, it is essential to construct frameworks for capturing students' proving activities, because such frameworks enable us to deepen our understanding about students' mathematical thought processes and to support their learning. Mathematics educators have so far constructed frameworks or utilized existing ones for analyzing students' behaviors, teachers' instructions and so on (for example, Inglis, Mejia-Ramos & Simpson, 2007; Pedemonte, 2007; A. Stylianides, 2007; G. Stylianides, 2008, 2009; Weber and Alcock, 2009).

Among various activities related to proof and proving, this study focuses on processes of proofs and refutations. According to Lakatos (1976), "informal, quasi-empirical, mathematics … grow(s) through … the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations" (p. 5). Thus, providing students with an opportunity in which they can make conjectures, prove them, and refine their conjectures and proofs by refutations should lead to authentic mathematical learning (Lampert, 1990). Through such mathematical activities, students could also enhance their critical thinking and attitude which are essential for literate citizens.

This study utilizes Lakatos' research in order to construct a framework for capturing students' behaviors. Lakatos (1976) attempted to describe a process of mathematical development through reconstructing rationally a history about Descartes-Euler conjecture on polyhedra. Davis and Hersh (1981) appraised Lakatos' endeavor, stating that Lakatos could show a process in which mathematical theory matured gradually with problems, conjectures, proofs and refutations. Hence, through constructing a framework based on Lakatos' research and analyzing students' behaviors through the framework, it would become possible to derive important actions for achieving mathematical learning which mirrors mathematical research. These actions may also provide us with some insights and ideas as to what steps teachers might take to encourage students to think and act like a process of proofs and refutations.

This study addresses the following research questions:

- What framework can be constructed for capturing proofs and refutations in mathematical learning?
- Which actions by students are important for achieving proofs and refutations?

THEORETICAL FRAMEWORK: HEURISTIC RULES IN PROOFS AND REFUTATIONS

Some mathematics educators have already made frameworks or analyzed students' behaviors by utilizing Lakatos' research (e.g. Balacheff, 1991; Davis & Hersh, 1981; Larsen & Zandieh, 2008; Reid, 2002). However, they seemed to refer to his research only partially, for example, monster-barring, exception-barring and proof-analysis. Whereas their studies successfully analyzed students' mathematical processes, Lakatos (1976) described various activities other than the above three methods. Furthermore, the interactions between imaginary students and their teacher in *Proofs and Refutations* (Lakatos, 1976) seemed to attach more weight to certain methods among the activities.

According to Silver and Herbst (2007), there are four distinct ways in which a theory plays the role of mediator between research and practice, and two of them are *understanding* (description or explanation) and *prescription*. From the viewpoint of both roles, this study focuses on Lakatos' *heuristic rules* in order to construct a theoretical framework for proofs and refutations in mathematical learning.

First, Lakatos formulated five heuristic rules, and the second rule referred to proof-analysis that mathematics educators have utilized (Larsen & Zandieh, 2008). In other words, there are other heuristic rules which include contents educational researches have not considered. Thus, by using these rules, it would become possible to construct a more descriptive framework which arrows us to understand students' behaviors more broadly.

Second, Polya (1957) stated the aims of heuristic as "(t)he aim of heuristic is to study the methods and rules of discovery and invention" (p. 112) and "(m)odern heuristic endeavors to understand the process of solving problems, especially the *mental operations typically useful* in this process" (pp. 129-130, emphasis is original). Since Lakatos was largely influenced by Polya's mathematical heuristic (Davis & Hersh, 1981; Lakatos, 1976), he would also attribute a prescriptive meaning to the word "heuristic". That is, he seemed to formulate heuristic rules as a set of methods which were productive and valuable in mathematical research. Lakatos' heuristic rules therefore would enable us to construct a framework which guides students to desirable actions for advances of mathematical activities.

Lakatos (1976) formulated the following five heuristic rules:

Rule 1. If you have a conjecture, set out to prove it and to refute it. Inspect the proof carefully to prepare a list of non-trivial lemmas (proof-analysis); find counterexamples both to the conjecture (global counterexamples) and to the suspect lemmas (local counterexamples).

Rule 2. If you have a global counterexample discard your conjecture, add to your proof-analysis a suitable lemma that will be refuted by the counterexample, and replace the discarded conjecture by an improved one that incorporates that lemma as a condition. Do not allow a refutation to be dismissed as a monster. Try to make all 'hidden lemmas' explicit.

Rule 3. If you have a local counterexample, check to see whether it is not also a global counterexample. If it is, you can easily apply Rule 2. (Lakatos, 1976, p. 50)

Rule 4. If you have a counterexample which is local but not global, try to improve your proof-analysis by replacing the refuted lemma by an unfalsified one. (*ibid.*, p. 58)

Rule 5. If you have counterexamples of any type, try to find, by deductive guessing, a deeper theorem to which they are counterexamples no longer. (*ibid.*, p. 76)

Rule 1 refers to a conjecture, proof and refutation. There are two kinds of counterexamples, that is, "global counterexample" that refutes a conjecture itself, and "local counterexample" that refutes a proof.

Rules 2 and 5 indicate how one should act when facing global counterexamples. Suppose that one proves a conjecture and then faces its counterexamples. Rule 2, which Lakatos called "the method of lemma-incorporation", requires that one should analyze the proof first, and then discover a certain part of the proof refuted by the counterexamples (this part was called guilty-lemma). After that, one needs to restrict the domain of the conjecture by incorporating this guilty-lemma to a condition of the conjecture. In contrast to Rule 2, Rule 5 indicates to invent a more general conjecture which also holds true in the case of the counterexamples; Lakatos named it as "increasing content by deductive guessing".

Rules 3 and 4 imply how one should cope with local counterexamples of proofs. Rule 3 requires that we check whether the local counterexamples are also global or not. If they are global counterexamples, one should just apply Rule 2 or 5. If they are not global, one has to modify the proofs so that the local counterexamples can no longer refute the modified proofs (Rule 4). This modification of proof can be interpreted in two ways. The first is to keep the overall structure of the proof and to modify only a part of the proof which was refuted by the local counterexamples. The second, which is more radical than the first, is to invent a deeper proof that is completely different from the original proof (Lakatos, 1976, pp. 58-59).

Figure 1 is a summary of Lakatos' heuristic rules, and this study adopts it as a theoretical framework for capturing students' processes of proofs and refutations. Though Fig. 1 is slightly monotonous, actual activities by both mathematicians and students would become cyclic processes; each element in Fig. 1 would be involved mutually in a more complex way.

This framework is more descriptive in two aspects than those which mathematics educators have previously constructed or utilized. First, the framework takes into account "increasing content by deductive guessing" that has not been sufficiently deliberated in mathematics education research (for example, Balacheff, 1991; Larsen & Zandieh, 2008; Reid, 2002). Second, mathematics educators utilizing Lakatos' research have tended to focus on only global counterexamples that refute conjectures (Balacheff, 1991; Komatsu, 2011; Larsen & Zandieh, 2008; Reid, 2002). In contrast, the framework of this study enables us to deal with cases in which students face local counterexamples of their proofs.

As stated earlier, this framework also indicates desirable behaviors in mathematical activities. Indeed, Lakatos formulated the heuristic rules as the form "if …, try to …". For example, if students face global counterexamples, they are expected not to ignore the counterexamples as monsters or exceptions, but to restrict their conjectures by lemma-incorporation or to invent more general conjectures which are true for the counterexamples.

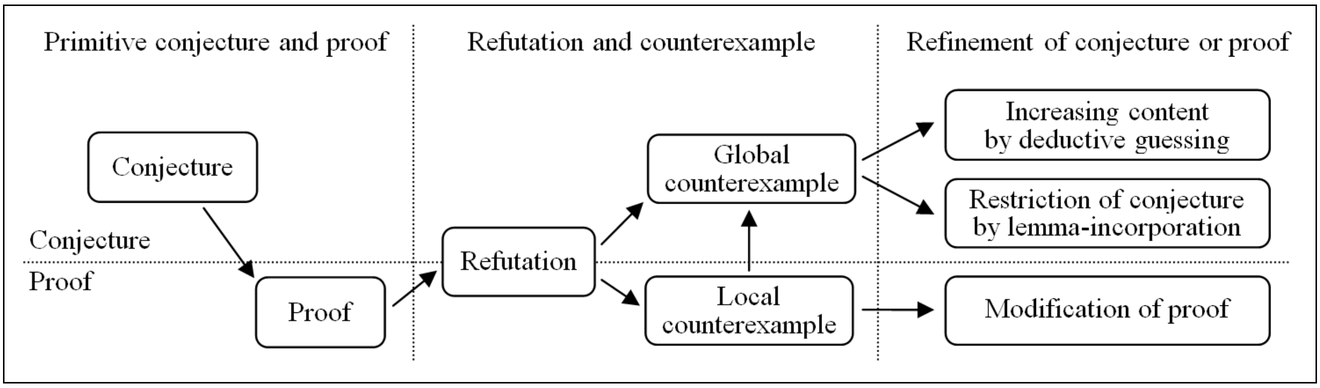

Fig. 1: Heuristic rules in *Proofs and Refutations* (Lakatos, 1976)

In the following, this paper first deals with an episode and illustrates that the framework of this study has a descriptive nature which allows us to capture students' behaviors. After that, by analyzing the episode according to the framework, this paper proposes a hypothesis about important actions in proofs and refutations.

BACKGROUND OF EPISODE

This paper analyzes a Japanese lesson in the ninth grade (14-15 years old) that dealt with the inscribed angle theorem and its proof. In this lesson, students faced a local counterexample that refuted their previous proof, and modified the proof. As explained above, existing studies have only focused on students' responses to global counterexamples (Balacheff, 1991; Komatsu, 2010; Larsen & Zandieh, 2008; Reid, 2002), and this paper attempts to add more insights into this areas of research by focusing on students' actions with local counterexamples in their proving and refuting processes.

The author observed the lesson as a non-participant observer. The school is a junior high school which is attached to a national university, and the students' standards in the school seem to be comparatively above average. In Japan, students start to learn geometric proof from the eighth grade (13-14 years old). Japanese eighth and ninth graders usually learn to prove geometric statements related to various properties of triangles, quadrilaterals and circles, using conditions for congruent triangles or similar triangles.

The lesson was recorded and transcribed, and the transcripts were analyzed with the focus on cognitive aspects of proofs and refutations, in particular how the students coped with a local counterexample. In addition, what they wrote on worksheets was referred to obtain further insight into their understanding of proof in geometry. The students' names are pseudonyms.

EPISODE: LOCAL COUNTEREXAMPLE AND MODIFICATION OF PROOF

The following episode is from one of the lessons of the unit which was about the inscribed angle theorem 'the measurement of the central angle is equal to twice the measurement of the inscribed angle subtended by the same arc'. Before this lesson, the students conjectured the theorem by examining some examples empirically; in each example, the center of the circle was located in the inside of the inscribed angle. After that, they proved the theorem as Fig. 2. In this lesson, they attempted to prove it in the other case (Fig. 3)

We draw line PO,
and name the intersection point of line PO and circle O as point K.
Because segment OA, OB, and OP are radiuses of circle O (hypothesis),
OA = OB = OP.
Hence, $\triangle$OAP and $\triangle$OBP are isosceles triangles.
Because two base angles of isosceles triangle are equal,
$\angle OAP = \angle OPA$, $\angle OBP = \angle OPB$ · · · ①
Due to ① and property of external angle of triangle,
$\angle AOK = \angle OAP + \angle OPA = 2\angle OPA$
$\angle BOK = \angle OBP + \angle OPB = 2\angle OPB$
And, $\angle AOB = \angle AOK + \angle BOK$
$= 2\angle OPA + 2\angle OPB$
$= 2(\angle OPA + \angle OPB)$
$= 2\angle APB$
Therefore, $\angle AOB = 2\angle APB$

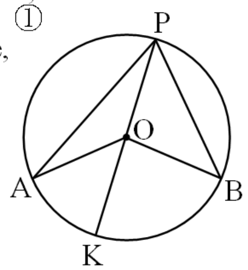

Fig. 2: Proof of the inscribed angle theorem

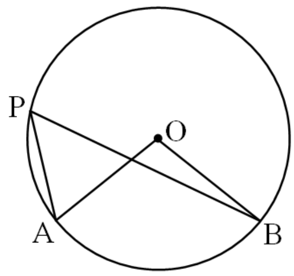

Fig. 3: Local counterexample

Awareness of Local Counterexample

At the beginning of the lesson, the teacher and students reviewed the inscribed angle theorem and their proof (Fig. 2). In particular, they recalled that they had used two facts in their proof; one was that triangle OAP and OBP are isosceles triangles, and the other was that an exterior angle is equal to the sum of the two interior angles which are not adjacent to the exterior angle. After that, they briefly thought about whether their proof was applicable to Fig. 3 or not:

15. Teacher: Next, … we fix these (point A and B) and move point P. … So, we get this figure, this figure (the teacher put Fig. 3 on the blackboard). Can we also prove this case by using our proof (Fig. 2)? Let's think about it.

17. Takumi: Like the last time, PO ….

18. Teacher: If we do so, the same proof seems to be right?

19. Takumi: It's kind of ….

20. Teacher: It's kind of …. You have not much seen yet. … The similar proof to this (Fig. 2) seems to be OK? How about? Riku has a little different idea?

21. Riku: Well, same. PO, line PO, it's a diameter. If we draw the line, I think we can do it.

22. Teacher: Can you do it if we draw this (the teacher drew the diameter PO on the blackboard)? How about Kenta?

23. Kenta: Well, my guess is similar to theirs. PO, if we use the auxiliary line, I think that we will manage to prove it.

Objectively, it is not possible to apply directly the proof (Fig. 2) to the case where the center of the circle is not inside the inscribed angle (Fig. 3), because angle AOB is equal not to the sum of angle AOK and BOK, but to their difference. Fig. 3 therefore becomes a local counterexample that refutes the proof. However, the students did not inspect the details of their proof at that point, and they examined only the outline of their proof such as "drawing line PO". Hence, they did not regard Fig. 3 as a local counterexample, and thought that it would be possible to apply their proof to Fig. 3.

Riku then proposed to check their proof, and the teacher required the students to check individually whether their proof was really applicable to Fig. 3. After the individual check, Riku said that "the same proof is right halfway, but after the middle, it seems to be a little different, I think we can't use the completely same proof". Based on Riku's comment, the teacher asked the students to make small groups and to talk about 'strange points' or 'flawed points' of their proof.

After the students exchanged their ideas within their small groups, the teacher and Hayato started to discuss flawed points of their proof:

33. Teacher: Well, can you find some strange points of the proof? How about? … Hayato.

36. Hayato: In our proof, after "And" (the fifth line from the bottom in Fig. 2).

37. Teacher: After "And", yeah.

38. Hayato: In the proof, $\angle AOB = \angle AOK + \angle BOK$. But, since angle AOK is already bigger than angle AOB, and since this is not true, the proof is strange after this point.

Thus, the students noticed that $\angle AOB = \angle AOK + \angle BOK$ in their proof (Fig. 2) did not hold in the case of Fig. 3. In fact, some students wrote 'same' above the "And", and 'different' or 'impossible' below it in their worksheets. Other students described more concretely the reason why $\angle AOB = \angle AOK + \angle BOK$ was not true, as Hayato's remark (38. Hayato); for example, they wrote that "(the sum of angle AOK and BOK) becomes larger than angle AOB" or "it's not true, it's strange because angle AOB is smaller than angle AOK". They therefore started recognizing Fig. 3 as a local counterexample that refuted their proof.

Modification of Proof

The teacher then wrote on the blackboard "Let's modify the sum of the angles in our proof and prove $\angle AOB = 2\angle APB$", and the students tackled this task individually or in small groups. Afterwards, Tsubasa shared his ideas to modify the original proof:

40. Teacher: Well, you seem to have made good progress with your modifications. I want you to tell me your proposals. How about you, Tsubasa?

43. Tsubasa: The point to modify is, a little while ago, below the "And".

44. Teacher: This (the fifth line from the bottom in Fig. 2).

45. Tsubasa: Yes. $\angle AOB = \angle AOK - \angle BOK$, is equal to $2\angle OPA - 2\angle OPB$, is equal to $2(\angle OPA - 2\angle OPB)$, equal to two times angle APB. Oh, there is no modification after that.

When we examine students' worksheets, we can identify that the students modified their original proof (Fig. 2) in mainly two ways. The first was to change only the part in which they had deduced $\angle AOB = 2\angle APB$, and the second was to rewrite the whole proofs which were true in the case of Fig. 3. However, even in the latter case, the students wrote the almost same as they had for their original proof except for the part in which they had deduced $\angle AOB = 2\angle APB$.

As Tsubasa said that "there is no modification after that" (45. Tsubasa), they understood that Fig. 3 did not refute the inscribed angle theorem itself at that point. In fact, some students wrote in their worksheets "no change in the conclusion". Therefore, the students recognized Fig. 3 as a counterexample which was local but not global.

At the end of the lesson, the teacher asked the students to analyze common and different points between their original proof and their modified one:

52. Kaede: The common point between our prior and new proofs is to use isosceles triangles and how to find exterior angles.

53. Teacher: Yes. Using these properties is common. … Other common points unlike this, …, Riku.

54. Riku: Well, at first, we drew the diameter which passed the point P.

55. Teacher: Yeah, an auxiliary line …. Well, next, different, about different points, Ren.

56. Ren: The other day we used the sum (angle AOB is equal to the sum of angle AOK and BOK), but today we've been using the difference (angle AOB is equal to their difference).

59. Teacher: Is there anyone thinking why this time we use the difference? Shota.

60. Shota: Well, because, before the last lesson, the auxiliary line was inside the arc. But, this time, K was outside of the arc, outside. Well, because angle AOK became larger than angle AOB ….

Kaede and Riku indicated the common points between the two proofs. Ren stated their difference, and Shota further analyzed the cause why the two proofs became different from each other. That is, in the case of Fig. 2, a part of the auxiliary line PK used in the proof was inside the central angle AOB, but in the case of Fig. 3, the line PK was outside the angle AOB. Shota pointed out that this cause led to the difference of the two proofs.

DISCUSSION

Description of the Episode

Before this lesson, the students conjectured the inscribed angle theorem, and they proved the theorem in the case where the center of the circle was inside the inscribed angle (Fig. 2). At the beginning of this lesson, the teacher asked questions by which he intended to refute their proof; he showed the case in which the center of the circle was outside the inscribed angle (Fig. 3), and asked whether their proof was still applicable to the case. At first, the students did not regard Fig. 3 as a local counterexample of their proof (for example, 23. Kenta), because they examined only the outline of their proof such as "drawing line PO" at that point.

After that, the students inspected the details of their proof, and noticed that Fig. 3 became the local counterexample of their proof; they found that $\angle AOB = \angle AOK + \angle BOK$ in their proof (Fig. 2) was not true in the case of Fig. 3 (38. Hayato). The students started to modify their proof so that it was tenable for the local counterexample. They first found that angle AOB was equal not to the sum of angle AOK and BOK, but to their difference, and then they deduced $\angle AOB = 2\angle APB$ which was the conclusion of the inscribed angle theorem. In addition, when they modified their proof, they grasped clearly that this local counterexample was not a global one which refuted the inscribed angle theorem itself (45. Tsubasa).

Fig. 4 is a summary of the above description. It is important to mention two different points between Lakatos' heuristic rules and this lesson. The first is the meaning of modification of proof. For Lakatos, the modification of proof was to construct a more general proof which was applicable to local counterexamples (the fourth heuristic rule and two kinds of its interpretation; Lakatos, 1976, pp. 58-59). In contrast, the students in this paper modified their proof by case analysis. That is, when facing a local counterexample (Fig. 3), they constructed the proof for the local counterexample by changing slightly their original proof (Fig. 2), and they separated the two proofs according to the place of point P or K.

Secondly, it seemed that Lakatos assumed a case in which one first checked whether local counterexamples were global or not, and then one started to modify proofs if the local counterexamples were not global (the third and fourth heuristic rules). The teacher in this paper also presented the task of the lesson, implying that the local counterexample was not a global one which refuted the inscribed angle theorem. However, it was while the students modified their proof that they recognized clearly the local counterexample was not global (45. Tsubasa). Their recognition seems to be quite natural because, when they conjectured empirically the inscribed angle theorem before this lesson, they dealt with only the case in which the center of the circle was inside the inscribed angle. The bidirectional arrow between local counterexample and modification of proof in Fig. 4 shows this situation.

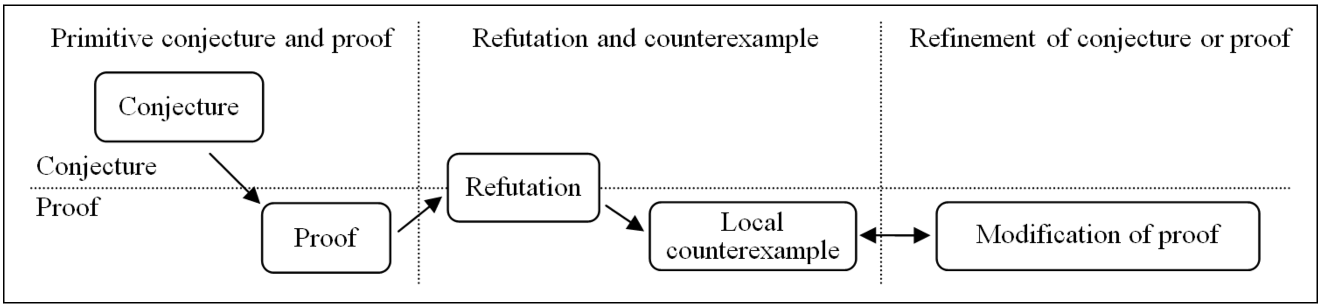

Fig. 4: Description of the episode

As stated earlier, actual activities by both mathematicians and students would become more complex than the theoretical framework of this study itself (Fig. 1). The above two points also show that it is important to utilize the framework more flexibly; for example, it is necessary to interpret the meaning of each element in the framework more broadly and to consider transitions between the elements more flexibly.

Important Action in Modification of Proof

As mentioned previously, the theoretical framework of this study has a somewhat prescriptive nature in the sense that it represents a set of productive and valuable methods for advancing mathematical activities. The framework therefore tells us which parts of incidents we should focus on in order to derive important actions for proofs and refutations.

The important process observed from the above episode is a transition between local counterexample and modification of proof. When the students faced a local counterexample, they not only admitted the falseness of their proof, but also attempted to understand the reason why their proof became false. For example, they grasped why $\angle AOB = \angle AOK + \angle BOK$ became false or why angle AOB was not the sum of angle AOK and BOK but their difference (38. Hayato and 60. Shota).

According to Peled and Zaslavsky (1997), when facing global counterexamples that refute conjectures, it is crucial to examine the conditions of the conjectures that have been modified, because such examination "provides the student with some insight into the mechanism that governs the situation" (p. 5). This statement seemed to be similarly significant in the case of local counterexamples. In fact, in the episode, the students investigated the reason why their original proof collapsed in the case of Fig. 3, and by considering the reason, they could gradually grasp the mechanism of proof by case analysis. Then, they were able to differentiate their proofs according to the place of point P or K.

In summary, it may be an important action for students to understand the reason why their proofs become false when they face local counterexamples of their proofs, because such understanding might become a basis for modification of proofs. However, this is only a hypothesis because the author observed the overall stream of the lesson and did not explore the details of individual students' thinking. It is therefore necessary to scrutinize this hypothesis by dealing with other environments and research methodologies.

From the above discussion, it is also possible to bring some implications for teaching. For example, when students face local counterexamples of their proofs, teachers should lead classroom discussion to focus on the reason why the proofs were false. In addition, teachers should support students' learning so that students can modify their proofs according to their understanding about the reason.

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