# Elementary Properties of Infinitely Divisible Probability Distributions for $\tau_T$-Semigroups

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## § 1. Introduction.

Let $T$ be a binary operation on $[0, 1]$ and $\Delta^+$ be a class of probability distributions. For any $F$, $G$ in $\Delta^+$ and for any real $x$, Moynihan [2] defined

$$\tau_T(F, G) = \sup\{T(F(u), G(v))\ ;\ u+v=x\} \qquad (1.1)$$

and obtained a commutative semigroup $(\Delta^+, \tau_T)$, called the $\tau_T$- semigroup. Moynihan [4], [5] defined the conjugate transform $C_T F$ and obtained the similar properties of characteristic functions in Lukacs [1].

As pointed out by Schweizer and Sklar in [6], several open problems on the arithmetic for $\tau_T$-semigroups are left. One of the interesting problems is a characterization of the class of infinitely divisible elements in $(\Delta^+, \tau_T)$.

In this paper we study two elementary properties for this class (**Theorem3. 1** and **Theorem 3. 2**). These are well-known for the semigroup $(\Delta^+, *)$, the semigroup of probability distributions under the convolution $*$. But $(\Delta^+, \tau_T)$ is not isomorphic to $(\Delta^+, *)$ (Moynihan [3] Theorem 1. 5). Furthermore, $\tau_T$ is not derivable from any function on random variables ([6] p113 Theorem 7. 6. 5).

Therefore our results are not trivial ones.

## § 2. $\tau_T$-Semigroups.

**Definition 2. 1.** A *t-norm* is any two-place function $T : [0, 1] \to [0, 1]$ satisfying

(a) $T(a, 1) = a$

(b) $T(c. d) \geqq T(a, b)$ for $c \geqq a$, $d \geqq b$

(c) $T(a, b) = T(b, a)$

(d) $T(T(a, b), c) = T(a, T(b, c))$.

A t-norm $T$ is *strict* if $T$ is continuous on $[0, 1] \times [0, 1]$ and strictly increasing in each place on $[0, 1] \times [0, 1]$.

A strict t-norm has the following representation ([6] p68 Theorem 5. 5. 5. and Theorem 5. 5. 7).

**Theorem 2. 1.** (1) *A t-norm $T$ is strict if and only if it admits the representation*

$$T(x, y) = k^{-1}(k(x)\ k(y)) \qquad \textit{for all } x, y \textit{ in } [0, 1] \qquad (2.1)$$

*where $k$ is a continuous and strictly increasing function from* $[0, 1]$ *onto* $[0, 1]$, *so that*

$k(0)=0$ *and* $k(1)=1$.

(We say that $k\ (\cdot)$ is the *multiplicative generator* of $T$.)

(2) *If* $k_1$ *and* $k_2$ *are multiplicative generators of a strict t-norm* $T$ *then there is an* $r>0$ *such that*

$$k_2(x)=(k_1(x))^r \qquad \text{for all } x \text{ in } [0, 1] \tag{2. 2}$$

**Examples.**

**1.** $T(x, y)= xy$ is a strict t-norm with the multiplicative generator $k(x)= x$.

**2.**

$$T_p(x, y)=\begin{cases} (\mathrm{Max}\,(x^p+y^p-1,\ 0))^{1/p} & p\neq 0 \\ xy & p=0 \end{cases} \tag{2. 3}$$

$T_p$ is a strict t-norm if and only if $p\leqq 0$, and has the multiplicative generator

$$k_P(x)=\begin{cases} \exp\{(x^P-1)/p\} & p<0 \\ x & p=0 \end{cases} \tag{2. 4}$$

**Definition 2. 2.** $\Delta^+=\{F:\mathbf{R}\to[0,1]\ ;\ F$ is left-cotinuous nondecreasing and $F(0)=0\}$.

For $F$, $G$ in $\Delta^+$, $x$ in $\mathbf{R}$ and for a strict t-norm $T$,

$\tau_T(F, G)(x)$ is defined by (1. 1). Then we have ([2])

**Theorem 2. 2.** (1) *$(\Delta^+, \tau_T)$ is a commutative semigroup, that is*

(a) *$\tau_T(F, G)$ is in $\Delta^+$ for any $F$, $G$ in $\Delta^+$.*

(b) *$\tau_T(F, \epsilon_0)=F$, where $\epsilon_0(x)=1$ for $x>0$ and $=0$ for $x\leqq 0$.*

(c) *$\tau_T(F, G)=\tau_T(G, F)$.*

(d) *$\tau_T(\tau_T(F, G), H)=\tau_T(F, \tau_T(G, H))$.*

(2) *If we introduce the modified Lévy metric $L$ in $\Delta^+$, then $(\Delta^+, \tau_T, L)$ is a topological semigroup, where*

$$L(F, G)=\inf\{\delta\ ;\ F(x)\leqq G(x+\delta)+\delta,$$
$$G(x)\leqq F(x+\delta)+\delta,\ 0<x<1/\delta\}.$$

*Following [2], we say that $(\Delta^+, \tau_T)$ is the $\tau_T$-semigroup.* In [4], [5] Moynihan introduced the *conjugate transform* in $(\Delta^+, \tau_T)$, which plays the similar role as the characteristic function in usual probability theory.

**Definition 2. 3.** Let $T$ be a strict t-norm and $k(\cdot)$ be the multiplicative generator of $T$. For any $F$ in $\Delta^+$ and for $z\geqq 0$ the conjugate transform of $T$ is defined by

$$C_T\ F(z)=\sup\{e^{-xz}\ kF(x)\ ;\ x\geqq 0\} \tag{2. 5}$$

where $kF(x)=k(F(x))$ for $x>0$ and $=0$ for $x\leqq 0$.

**Definition 2. 4.** For any strict t-norm $T$

( i ) Let $A_T=\{\phi:[0,\infty]\to[0,1]\ ;\ \phi$ is positive, non-increasing, log-convex and continuous$\}\ \cup\ \{\theta_\infty\}$, where $\theta_\infty(z)=0$ for all $z\geqq 0$.

( ii ) For any $\phi$ in $A_T$

$$C_T^* \cdot \phi(x) = k^{-1}(\inf\{e^{xz}\ \phi(z)\ ;\ z \geqq 0\}). \qquad (2.\ 6)$$

(iii) For $F$ in $\Delta^+$, $b_F = \sup\{x\ ;\ F(x) = 0\}$.

The following fundamental properties of conjugate transforms are in Moynihan [4], [5].

**Theorem 2. 3.** (1) $C_T(\tau_T(F,\ G))(z) = C_TF(z) \cdot C_TG(z)$.

(2) $A_T = \{C_TF\ ;\ F \in \Delta^+\}$.

(3) $C_T : \Delta^+{}_T \to A_T$ *is a bijection with inverse* $C_T^*$, *where* $\Delta_T{}^+ = \{F \in \Delta^+\ ;\ kF$ *is log-concave*$\}$ .

(4) $C_T^*(C_TF \cdot C_TG) = \tau_T(F_T,\ G_T)$, *where* $F_T$ *and* $G_T$ *are the log-concave envelope of* $F$ *and* $G$ *respectively.*

(5) *Let* $\{G_n\}$ *be a sequence in* $\Delta^+$, *which coverges weakly to some* $G$ *in* $\Delta^+$ *then* $C_T\ G_n\ (z)$ *converges to* $C_T\ G(z)$ *for any* $z > 0$.

(6) *Let* $\{\phi_n\}$ *be a sequence in* $A_T \diagdown \{\theta_\infty\}$ *and* $\lim_{n \to \infty} \phi_n(z) = \phi(z)$ *for any* $z \geqq 0$ *and if* $\phi\ (z)$ *is cotinuous at* 0 *then there exists a* $F$ *in* $\Delta^+ \diagdown \{\epsilon_\infty\}$ *such that* $\phi(z) = C_TF(z)$ *for any* $z \geqq 0$.

(6) is an analogy of the Lḗvy's continuity theorem for chracteristic functions.

**Definition 2. 5.** $F$ in $\Delta^+$ is *infinitely divisible under* $\tau_T$ if for any positive integer $n$ there exists a $G$ in $\Delta^+$ such that

$G^n = F$, where $G^n$ is recursively defined by $G^1 = G$, $G^n = \tau_T(G^{n-1},\ G)$ for $n \geqq 2$.

Let $\mathfrak{B}_T = \{F \in \Delta^+\ ;\ \ kF$ is log-concave on $(b_F,\ \infty)\}$, then following result is in Moynihan [5] Theorem 4. 2.

**Theorem 2. 4.** *Every* $F$ *in* $\mathfrak{B}_T \diagdown \{\epsilon_\infty\}$ *is infinitely divisible under* $\tau_T$.

## § 3. Main results.

**Theorem 3. 1.** *Let* $T$ *be a strict t-norm. Then the* $\tau_T$*-product of a finite number of infinitely divisible probability distributions in* $\Delta^+$ *is infinitely divisible under* $\tau_T$.

**Proof** It is sufficient to prove the theorem for the case of two factors. Suppose that $F$, $G$ in $\Delta^+$ are infinitely divisible under $\tau_T$. Then there exist for any positive integer $n$ two probability distributions $F_n$ and $G_n$ in $\Delta^+$ such that $F = (F_n)^n$, $G = (G_n)^n$, where $(F_n)^n$ and $(G_n)^n$ are defined in Definition 2. 5.

Set $H = \tau_T(F,\ G)$ and $H_n = \tau_T(F_n,\ G_n)$. Then by Theorem 2. 3 (1) we have

$$\begin{aligned} C_T\ H(z) &= C_T\ F(z) \cdot C_T\ G(z) \\ &= (C_T\ F_n)^n(z)(C_T\ G_n)^n(z) \\ &= (C_T\ H_n)^n(z) = (C_T\ H_n{}^n)(z) \end{aligned}$$

Operating $C_T^*$ on both sides, we have by Theorem 2. 3 (3), (4)

$$H = \{(H_n)_T\}^n.$$

Since $(H_n)_T$ is the log-cocave envelope of $H_n$ and so, it is in $\Delta^+$, $H$ is infinitely divisible under $\tau_T$. (Q. E. D.)

**Theorem 3. 2.** *A distribution function in $\Delta^+$, which is the weak limit of a sequence of infinitely divisible distribution functions in $\Delta^+$ is infinitely divisible under $\tau_T$.*

**Proof**. Let $\{F_k\}$ be a sequence of infinitely divisible distribution functions in $\Delta^+$ and suppose that this seqence converges weakly to a probability distribution function $F$ in $\Delta^+$, By Theorem 2. 3 (5) we have

$$(C_T\ F)(z) = \lim_{k\to\infty}(C_T\ F_k)(z) \quad \text{for any } z > 0. \tag{3. 1}$$

Since $\{F_k\}$ is infinitely divisible, there exists for any positive integer $n$ a sequence $\{F_{k,n}\}$ in $\Delta^+$ such that $F_k = (F_{k,n})^n$.

Then, by Theorem 2. 3 (1) we have

$$C_T\ F_k(z) = (C_T\ F_{k,n})^n(z) \tag{3. 2}$$

It follows from (3. 1) and (3. 2) that

$$\begin{aligned}\lim_{k\to\infty}(C_T\ F_{k,n})(z) &= \lim_{k\to\infty}(C_T\ F_k)^{1/n}(z)\\ &= \lim_{k\to\infty}\exp\{(\log\ C_T\ F_k(z))/n\}(z)\\ &= \exp\{\log\ C_T\ F(z))/n\}(z)\\ &= \{(C_T\ F)(z)\}^{1/n},\ (z).\end{aligned}$$

If we define $(C_T\ F)(0)$ by $\lim_{z\downarrow 0}(C_T\ F)(z)$, then $\{(C_T\ F)(z)\}^{1/n}$ is continuous at $z=0$ and by Theorem 2. 3 (6) there exists a $G$ in $\Delta^+\setminus\{\epsilon_\infty\}$ such that

$$(C_T\ G)(z) = \{(C_T\ F)(z)\}^{1/n}.$$

Therefore we have

$$(C_T\ F)(z) = (C_T\ G)(z))^n = (C_T\ G^n)(z) \tag{3. 3}$$

Operating $C_T{}^*$ on both sides, we have by Theorem 2. 3 (3), (4) that $F = (G_T)^n$ and since $G_T$ is in $\Delta^+$, $F$ is infinitely divisible under $\tau_T$. (Q. E. D.)

**Corollary of Theorem 3. 2.** *$F \in \Delta^+$ is infinitely divisible under $\tau_T$ if and only if $(C_T\ F)^r \in A_T$ for any positive $r$, where*

$$(C_T\ F)^r(z) = \exp\{r\log\ C_T\ F(z)\}.$$

**Proof.** Since the only part is trivial, we prove the if part. The general case is proved by Theorem 3. 2, it is sufficient to prove in the case of $r$ is a positive rational number.

If $r = n/m$ then $F^n$ is infinitely divisible by Teorem 3. 1. Then there exists a $G_m$ in $\Delta^+$ such that $F^n = (G_m)^m$.

Therefere we have

$F^r = F^{n/m} = G_m \in \Delta^+$ and by Theorem 2. 3 (3), $(C_T\ F)^r \in A_T$. (Q. E. D.)

## References

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