On Weyl modules of cyclotomic q-Schur algebras

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Abstract. We study Weyl modules of cyclotomic q-Schur algebras. In particular, we give a character formula of the Weyl modules by using the Kostka numbers and some numbers obtained from a generalization of Littlewood-Richardson rule. We also study corresponding symmetric functions. Finally, we give some simple applications to modular representations of the cyclotomic q-Schur algebras.

0. Introduction

Let ${}_R\mathscr{H}_{n,r}$ be the Ariki-Koike algebra over a commutative ring R with parameters $q, Q_1, \cdots, Q_r \in R$ associated to the complex reflection group $\mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$, and let ${}_R\mathscr{S}_{n,r}$ be the cyclotomic q-Schur algebra associated to ${}_R\mathscr{H}_{n,r}$ introduced by Dipper, James and Mathas in [**DJM**]. Put $\mathcal{A} = \mathbb{Z}[q, q^{-1}, Q_1, \cdots, Q_r]$, where $q, Q_1, \cdots, Q_r$ are indeterminate, and $\mathcal{K} = \mathbb{Q}(q, Q_1, \cdots, Q_r)$ is the quotient field of $\mathcal{A}$.

In the case where $r = 1$, ${}_R\mathscr{H}_{n,1}$ is the Iwahori-Hecke algebra associated to the symmetric group $\mathfrak{S}_n$, and ${}_R\mathscr{S}_{n,1}$ is the q-Schur algebra associated to ${}_R\mathscr{H}_{n,1}$. In this case, the q-Schur algebra ${}_R\mathscr{S}_{n,1}$ comes from the Schur-Weyl duality as follows. Let $\mathfrak{gl}_m$ be the general linear Lie algebra, and $U_q(\mathfrak{gl}_m)$ be the corresponding quantum group over $\mathcal{K}$. We consider the vector representation V of $U_q(\mathfrak{gl}_m)$, then $U_q(\mathfrak{gl}_m)$ acts on the tensor space $V^{\otimes n}$ via the coproduct. ${}_\mathcal{K}\mathscr{H}_{n,1}$ also acts on the tensor space $V^{\otimes n}$ by a q-analogue of the permutations of factors of the tensor product. Then the Schur-Weyl duality between $U_q(\mathfrak{gl}_m)$ and ${}_\mathcal{K}\mathscr{H}_{n,1}$ holds via this tensor space $V^{\otimes n}$ as shown in [**J**]. Moreover, the Schur-Weyl duality between ${}_\mathcal{A}U_q(\mathfrak{gl}_m)$ and ${}_\mathcal{A}\mathscr{H}_{n,1}$ also holds via the tensor space $V^{\otimes n}$, where ${}_\mathcal{A}U_q(\mathfrak{gl}_m)$ is the Lusztig's integral form of $U_q(\mathfrak{gl}_m)$ (see [**Du**]). Hence, we can specialize this Shur-Weyl duality to any ring R with a parameter $q \in R^\times$. Then the q-Schur algebra ${}_R\mathscr{S}_{n,1}$ coincides with the image of ${}_RU_q(\mathfrak{gl}_m) \to \mathrm{End}(V^{\otimes n})$ which comes from the action of ${}_RU_q(\mathfrak{gl}_m)$ on $V^{\otimes n}$.

On the other hand, in the case where $r \geq 2$, the Schur-Weyl duality is also known in [**SakS**]. Let $\mathfrak{g} = \mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}$ be a Levi subalgebra of $\mathfrak{gl}_m$, and $U_q(\mathfrak{g})$ be the corresponding quantum group over $\mathcal{K}$. $U_q(\mathfrak{g})$ acts on $V^{\otimes n}$ by the restriction of the action of $U_q(\mathfrak{gl}_m)$. We can also define an action of ${}_\mathcal{K}\mathscr{H}_{n,r}$ on $V^{\otimes n}$

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which is a generalization of the action of ${}_{\mathcal{K}}\mathscr{H}_{n,1}$. Then $U_q(\mathfrak{g})$ and ${}_{\mathcal{K}}\mathscr{H}_{n,r}$ satisfy the Schur-Weyl duality via the tensor space $V^{\otimes n}$ as shown in [**SakS**]. Unfortunately, this Schur-Weyl duality does not hold over $\mathcal{A}$ since the action of ${}_{\mathcal{K}}\mathscr{H}_{n,r}$ on $V^{\otimes n}$ is not defined over $\mathcal{A}$. However, we can replace ${}_{\mathcal{K}}\mathscr{H}_{n,r}$ with the modified Ariki-Koike algebra ${}_{R}\mathscr{H}^0_{n,r}$ over R with parameters $q, Q_1, \cdots, Q_r$ such that $Q_i - Q_j$ $(i \neq j)$ is invertible in R which was introduced in [**Sho**]. Then, the Schur-Weyl duality between ${}_{R}U_q(\mathfrak{g})$ and ${}_{R}\mathscr{H}^0_{n,r}$ holds via the tensor space $V^{\otimes n}$ (see [**SawS**]). Let ${}_{R}\mathscr{S}^0_{n,r}$ be the image of ${}_{R}U_q(\mathfrak{g}) \to \mathrm{End}(V^{\otimes n})$ which comes from the action of ${}_{R}U_q(\mathfrak{g})$ on $V^{\otimes n}$. Then some relations between ${}_{R}\mathscr{S}^0_{n,r}$ and ${}_{R}\mathscr{S}_{n,r}$ are studied in [**SawS**] and [**Saw**]. In particular, ${}_{R}\mathscr{S}^0_{n,r}$ is realized as a subquotient algebra of ${}_{R}\mathscr{S}_{n,r}$. Then, some decomposition numbers of ${}_{R}\mathscr{S}_{n,r}$ coincide with the decomposition numbers of ${}_{R}\mathscr{S}^0_{n,r}$ (which are also decomposition numbers for ${}_{R}U_q(\mathfrak{g})$) when R is a field. In [**SW**], we obtained a certain generalization of these results (see also Remark 5.7). Motivated by this generalization together with the Schur-Weyl duality between ${}_{R}U_q(\mathfrak{g})$ and ${}_{R}\mathscr{H}^0_{n,r}$, the author gave two presentations of ${}_{\mathcal{K}}\mathscr{S}_{n,r}$ (also ${}_{\mathcal{A}}\mathscr{S}_{n,r}$) by generators and fundamental relations in [**W**]. By using this presentation, we can define a (not surjective) homomorphism $\Phi_{\mathfrak{g}} : U_q(\mathfrak{g}) \to {}_{\mathcal{K}}\mathscr{S}_{n,r}$. We also have $\Phi_{\mathfrak{g}}|_{\mathcal{A}U_q(\mathfrak{g})} : {}_{\mathcal{A}}U_q(\mathfrak{g}) \to {}_{\mathcal{A}}\mathscr{S}_{n,r}$ by restriction. Thus we can specialize it to any commutative ring R and parameters $q, Q_1. \cdots, Q_r \in R$. In this paper, we study ${}_{R}\mathscr{S}_{n,r}$-modules by restricting the action to ${}_{R}U_q(\mathfrak{g})$ when R is a field.

First, we consider the problem over $\mathcal{K}$. In this case, ${}_{\mathcal{K}}\mathscr{S}_{n,r}$ is semi-simple, and finite dimensional $U_q(\mathfrak{g})$-modules are also semi-simple. Put $\Lambda^+_{n,r} = \{\lambda = (\lambda^{(1)}, \cdots, \lambda^{(r)}) \mid \lambda^{(k)}\text{: partition}, \sum_{k=1}^r |\lambda^{(k)}| = n\}$, the set of r-partitions of size n. Let $W(\lambda)$ be the Weyl module of ${}_{\mathcal{K}}\mathscr{S}_{n,r}$ corresponding to $\lambda \in \Lambda^+_{n,r}$. It is well known that $\{W(\lambda) \mid \lambda \in \Lambda^+_{n,r}\}$ gives a complete set of non-isomorphic simple ${}_{\mathcal{K}}\mathscr{S}_{n,r}$-modules. On the other hand, let $W(\lambda^{(k)})$ be the Weyl module of $U_q(\mathfrak{gl}_{m_k})$ with the highest weight $\lambda^{(k)}$. By investigating the appearing weights, we see that $\{W(\lambda^{(1)}) \boxtimes \cdots \boxtimes W(\lambda^{(r)}) \mid \lambda \in \Lambda^+_{n,r}\}$ gives a complete set of non-isomorphic simple $U_q(\mathfrak{g})$-modules which appear as $U_q(\mathfrak{g})$-submodules of ${}_{\mathcal{K}}\mathscr{S}_{n,r}$-modules through the homomorphism $\Phi_{\mathfrak{g}}$. Then we can consider the irreducible decomposition of the Weyl module $W(\lambda)$ of $\mathscr{S}_{n,r}$ as $U_q(\mathfrak{g})$-modules through the homomorphism $\Phi_{\mathfrak{g}}$ as follows:

$$W(\lambda) \cong \bigoplus_{\mu \in \Lambda^+_{n,r}} \Big(W(\mu^{(1)}) \boxtimes \cdots \boxtimes W(\mu^{(r)}) \Big)^{\oplus \beta_{\lambda\mu}} \text{ as } U_q(\mathfrak{g})\text{-modules.} \tag{0.1}$$

In order to compute the multiplicity $\beta_{\lambda\mu}$ in this decomposition, we describe the $U_q(\mathfrak{g})$-crystal structure on $W(\lambda)$ by using a generalization of admissible reading for $U_q(\mathfrak{gl}_m)$-crystal given in [**KN**] (Theorem 2.15). As a consequence, we can compute the multiplicity $\beta_{\lambda\mu}$ by the combinatorial way which can be regarded as a generalization of the Littlewood-Richardson rule (Corollary 3.8. See also Remark 3.9).

Thanks to the decomposition (0.1), we obtain the character formula of $W(\lambda)$ by using Kostka numbers and multiplicities $\beta_{\lambda\mu}$ $(\lambda, \mu \in \Lambda^+_{n,r})$ (Note that the weight space as the $\mathscr{S}_{n,r}$-module coincides with the weight space as the $U_q(\mathfrak{g})$-module from the homomorphism $\Phi_{\mathfrak{g}}$). We also describe the character of $W(\lambda)$ as a linear combination of products of the Schur polynomials with coefficients $\beta_{\lambda\mu}$ $(\lambda, \mu \in \Lambda^+_{n,r})$. Moreover, we see that the set of characters of the Weyl modules for all

r-partitions gives a new basis of the ring of symmetric polynomials (Theorem 4.3). Then we also study on some properties for such symmetric functions.

Second, as an application of the decomposition (0.1), we have a certain factorization of decomposition matrix of ${}_R\mathscr{S}_{n,r}$ when R is a field (Theorem 5.5), and we give an alternative proof of the product formula for decomposition numbers of ${}_R\mathscr{S}_{n,r}$ given in [**Saw**] (Corollary 5.6, See also Remark 5.7.) For the special parameters ($Q_1 = \cdots = Q_r = 0$ or $q = 1$, $Q_1 = \cdots = Q_r$), we determine the decomposition matrix of ${}_R\mathscr{S}_{n,r}$ from the factorization of decomposition matrix (Corollary 5.8).

Finally, we realize the Ariki-Koike algebra ${}_R\mathscr{H}_{n,r}$ as a subalgebra of ${}_R\mathscr{S}_{n,r}$ by using the generators of ${}_R\mathscr{S}_{n,r}$ (Proposition 6.3). As a corollary of Corollary 5.8, we give an alternative proof for the classification of simple ${}_R\mathscr{H}_{n,r}$-modules for the special parameters ($Q_1 = \cdots = Q_r = 0$ or $q = 1$, $Q_1 = \cdots = Q_r$) which was already obtained by Ariki and Mathas in [**AM**] and [**M1**] (Corollary 6.5).

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1. Review on cyclotomic q-Schur algebras

In this section, we recall the definition of the cyclotomic q-Schur algebra $\mathscr{S}_{n,r}$ introduced in [**DJM**], and we review presentations of $\mathscr{S}_{n,r}$ by generators and fundamental relations given in [**W**].

1.1. Let R be a commutative ring, and we take parameters $q, Q_1, \cdots, Q_r \in R$ such that q is invertible in R. The Ariki-Koike algebra ${}_R\mathscr{H}_{n,r}$ associated to the complex reflection group $\mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$ is the associative algebra with 1 over R generated by $T_0, T_1, \cdots, T_{n-1}$ with the following defining relations:

$$\begin{aligned}
&(T_0 - Q_1)(T_0 - Q_2)\cdots(T_0 - Q_r) = 0, && \\
&(T_i - q)(T_i + q^{-1}) = 0 && (1 \le i \le n-1), \\
&T_0T_1T_0T_1 = T_1T_0T_1T_0, && \\
&T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} && (1 \le i \le n-2), \\
&T_iT_j = T_jT_i && (|i-j| \ge 2).
\end{aligned}$$

The subalgebra of ${}_R\mathscr{H}_{n,r}$ generated by $T_1, \cdots, T_{n-1}$ is isomorphic to the Iwahori-Hecke algebra ${}_R\mathscr{H}_n$ of the symmetric group $\mathfrak{S}_n$ of degree n. For $w \in \mathfrak{S}_n$, we denote by $\ell(w)$ the length of w, and denote by T_w the standard basis of ${}_R\mathscr{H}_n$ corresponding to w.

1.2. Let $\mathbf{m} = (m_1, \cdots, m_r) \in \mathbb{Z}^r_{>0}$ be an r-tuple of positive integers. Put

$$\Lambda_{n,r}(\mathbf{m}) = \left\{ \mu = (\mu^{(1)}, \cdots, \mu^{(r)}) \;\middle|\; \begin{array}{c} \mu^{(k)} = (\mu^{(k)}_1, \cdots, \mu^{(k)}_{m_k}) \in \mathbb{Z}^{m_k}_{\ge 0} \\ \sum_{k=1}^r \sum_{i=1}^{m_k} \mu^{(k)}_i = n \end{array} \right\}.$$

We denote by $|\mu^{(k)}| = \sum_{i=1}^{m_k} \mu^{(k)}_i$ (resp. $|\mu| = \sum_{k=1}^r |\mu^{(k)}|$) the size of $\mu^{(k)}$ (resp. the size of μ), and call an element of $\Lambda_{n,r}(\mathbf{m})$ an r-composition of size n. We define the map $\zeta : \Lambda_{n,r}(\mathbf{m}) \to \mathbb{Z}^r_{\ge 0}$ by $\zeta(\mu) = (|\mu^{(1)}|, |\mu^{(2)}|, \cdots, |\mu^{(r)}|)$ for $\mu \in \Lambda_{n,r}(\mathbf{m})$. We also define the partial order "$\succeq$" on $\mathbb{Z}^r_{\ge 0}$ by $(a_1, \cdots, a_r) \succ (a'_1, \cdots, a'_r)$ if $\sum_{j=1}^k a_j \ge \sum_{j=1}^k a'_j$ for any $k = 1, \cdots, r$. Put

$$\Lambda^+_{n,r}(\mathbf{m}) = \{\lambda \in \Lambda_{n,r}(\mathbf{m}) \mid \lambda^{(k)}_1 \ge \lambda^{(k)}_2 \ge \cdots \ge \lambda^{(k)}_{m_k} \text{ for any } k = 1, \cdots, r\}.$$

We also denote by $\Lambda^+_{n,r}$ the set of r-partitions of size n. Then we have $\Lambda^+_{n,r}(\mathbf{m}) = \Lambda^+_{n,r}$ when $m_k \geq n$ for any $k = 1, \cdots, r$.

1.3. For $i = 1, \cdots, n$, put $L_1 = T_0$ and $L_i = T_{i-1}L_{i-1}T_{i-1}$. For $\mu \in \Lambda_{n,r}(\mathbf{m})$, put

$$m_\mu = \Big(\sum_{w \in \mathfrak{S}_\mu} q^{\ell(w)} T_w\Big)\Big(\prod_{k=1}^{r}\prod_{i=1}^{a_k}(L_i - Q_k)\Big), \quad M^\mu = m_\mu \cdot {}_R\mathscr{H}_{n,r},$$

where $\mathfrak{S}_\mu$ is the Young subgroup of $\mathfrak{S}_n$ with respect to μ, and $a_k = \sum_{j=1}^{k-1}|\mu^{(j)}|$ with $a_1 = 0$. The cyclotomic q-Schur algebra ${}_R\mathscr{S}_{n,r}$ associated to ${}_R\mathscr{H}_{n,r}$ is defined by

$${}_R\mathscr{S}_{n,r} = {}_R\mathscr{S}_{n,r}(\Lambda_{n,r}(\mathbf{m})) = \operatorname{End}_{{}_R\mathscr{H}_{n,r}}\Big(\bigoplus_{\mu \in \Lambda_{n,r}(\mathbf{m})} M^\mu\Big).$$

Put $\Gamma(\mathbf{m}) = \{(i,k) \,|\, 1 \leq i \leq m_k,\ 1 \leq k \leq r\}$. For $\mu \in \Lambda_{n,r}(\mathbf{m})$ and $(i,k) \in \Gamma(\mathbf{m})$, we define $\sigma^\mu_{(i,k)} \in {}_R\mathscr{S}_{n,r}$ by

$$\sigma^\mu_{(i,k)}(m_\nu \cdot h) = \delta_{\mu,\nu}\big(m_\mu(L_{N+1} + L_{N+2} + \cdots + L_{N+\mu_i^{(k)}})\big)\cdot h \quad (\nu \in \Lambda_{n,r}(\mathbf{m}),\ h \in {}_R\mathscr{H}_{n,r}),$$

where $N = \sum_{l=1}^{k-1}|\mu^{(l)}| + \sum_{j=1}^{i-1}\mu_j^{(k)}$, and we set $\sigma^\mu_{(i,k)} = 0$ if $\mu_i^{(k)} = 0$. For $(i,k) \in \Gamma(\mathbf{m})$, put $\sigma_{(i,k)} = \sum_{\mu \in \Lambda_{n,r}(\mathbf{m})} \sigma^\mu_{(i,k)}$, then $\sigma_{(i,k)}$ is a Jucys-Murphy element of ${}_R\mathscr{S}_{n,r}$ (See [**M2**] for properties of Jucys-Murphy elements).

1.4. Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}, Q_1, \cdots, Q_r]$, where $q, Q_1, \cdots, Q_r$ are indeterminate over $\mathbb{Z}$, and $\mathcal{K} = \mathbb{Q}(q, Q_1, \cdots, Q_r)$ be the quotient field of $\mathcal{A}$. In order to describe presentations of ${}_\mathcal{K}\mathscr{S}_{n,r}$ (resp. ${}_\mathcal{A}\mathscr{S}_{n,r}$), we prepare some notation.

Put $m = \sum_{k=1}^r m_k$. Let $P = \bigoplus_{i=1}^m \mathbb{Z}\varepsilon_i$ be the weight lattice of $\mathfrak{gl}_m$, and $P^\vee = \bigoplus_{i=1}^m \mathbb{Z}h_i$ be the dual weight lattice with the natural pairing $\langle\, ,\, \rangle : P \times P^\vee \to \mathbb{Z}$ such that $\langle \varepsilon_i, h_j\rangle = \delta_{ij}$. Set $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \cdots, m-1$, then $\Pi = \{\alpha_i \,|\, 1 \leq i \leq m-1\}$ is the set of simple roots, and $Q = \bigoplus_{i=1}^{m-1}\mathbb{Z}\,\alpha_i$ is the root lattice of $\mathfrak{gl}_m$. Put $Q^+ = \bigoplus_{i=1}^{m-1}\mathbb{Z}_{\geq 0}\,\alpha_i$. We define a partial order "$\geq$" on P, so called dominance order, by $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$.

We identify the set $\Gamma(\mathbf{m})$ with the set $\{1, \cdots, m\}$ by the bijection $\gamma : \Gamma(\mathbf{m}) \to \{1, \cdots, m\}$ given by $\gamma((i,k)) = \sum_{j=1}^{k-1} m_j + i$. Put $\Gamma'(\mathbf{m}) = \Gamma(\mathbf{m}) \setminus \{(m_r, r)\}$. Under this identification, we have $P = \bigoplus_{i=1}^m \mathbb{Z}\varepsilon_i = \bigoplus_{(i,k)\in\Gamma(\mathbf{m})}\mathbb{Z}\varepsilon_{(i,k)}$ and $Q = \bigoplus_{i=1}^{m-1}\mathbb{Z}\,\alpha_i = \bigoplus_{(i,k)\in\Gamma'(\mathbf{m})}\mathbb{Z}\,\alpha_{(i,k)}$. Then we regard $\Lambda_{n,r}(\mathbf{m})$ as a subset of P by the injective map $\lambda \mapsto \sum_{(i,k)\in\Gamma(\mathbf{m})}\lambda_i^{(k)}\varepsilon_{(i,k)}$. For convenience, we consider $(m_k + 1, k) = (1, k+1)$ for $(m_k, k) \in \Gamma'(\mathbf{m})$ (resp. $(1-1, k) = (m_{k-1}, k-1)$ for $(1,k) \in \Gamma(\mathbf{m}) \setminus \{(1,1)\}$).

Now we have the following two presentations of cyclotomic q-Schur algebras.

Theorem 1.5 ([**W**, Theorem 7.16]). *Assume that $m_k \geq n$ for any $k = 1, \cdots, r$, we have the following presentations of ${}_\mathcal{K}\mathscr{S}_{n,r}$ and ${}_\mathcal{A}\mathscr{S}_{n,r}$.*

(i) *${}_\mathcal{K}\mathscr{S}_{n,r}$ is isomorphic to the algebra over $\mathcal{K}$ defined by the generators $e_{(i,k)}, f_{(i,k)}$ $\big((i,k) \in \Gamma'(\mathbf{m})\big)$ and $K^\pm_{(i,k)}$ $\big((i,k) \in \Gamma(\mathbf{m})\big)$ with the following defining relations :*

$$K_{(i,k)}K_{(j,l)} = K_{(j,l)}K_{(i,k)}, \quad K_{(i,k)}K^-_{(i,k)} = K^-_{(i,k)}K_{(i,k)} = 1 \tag{1.5.1}$$

$$K_{(i,k)}e_{(j,l)}K^-_{(i,k)} = q^{\langle \alpha_{(j,l)}, h_{(i,k)}\rangle}e_{(j,l)}, \tag{1.5.2}$$

$$K_{(i,k)}f_{(j,l)}K^-_{(i,k)} = q^{-\langle \alpha_{(j,l)}, h_{(i,k)}\rangle}f_{(j,l)}, \tag{1.5.3}$$

$$e_{(i,k)}f_{(j,l)} - f_{(j,l)}e_{(i,k)} = \delta_{(i,k),(j,l)}\eta_{(i,k)}, \tag{1.5.4}$$

$$\text{where } \eta_{(i,k)} = \begin{cases} \dfrac{K_{(i,k)}K^-_{(i+1,k)} - K^-_{(i,k)}K_{(i+1,k)}}{q-q^{-1}} & \text{if } i \neq m_k, \\ -Q_{k+1}\dfrac{K_{(m_k,k)}K^-_{(1,k+1)} - K^-_{(m_k,k)}K_{(1,k+1)}}{q-q^{-1}} & \\ \quad +K_{(m_k,k)}K^-_{(1,k+1)}(q^{-1}g_{(m_k,k)}(f,e) - qg_{(1,k+1)}(f,e)) & \text{if } i = m_k, \end{cases}$$

(1.5.5) $$e_{(i\pm1,k)}e^2_{(i,k)} - (q+q^{-1})e_{(i,k)}e_{(i\pm1,k)}e_{(i,k)} + e^2_{(i,k)}e_{(i\pm1,k)} = 0,$$
$$e_{(i,k)}e_{(j,l)} = e_{(j,l)}e_{(i,k)} \quad (|\gamma((i,k)) - \gamma((j,l))| \geq 2),$$

(1.5.6) $$f_{(i\pm1,k)}f^2_{(i,k)} - (q+q^{-1})f_{(i,k)}f_{(i\pm1,k)}f_{(i,k)} + f^2_{(i,k)}f_{(i\pm1,k)} = 0,$$
$$f_{(i,k)}f_{(j,l)} = f_{(j,l)}f_{(i,k)} \quad (|\gamma((i,k)) - \gamma((j,l))| \geq 2),$$

(1.5.7) $$\prod_{(i,k)\in\Gamma(\mathbf{m})} K_{(i,k)} = q^n,$$

(1.5.8) $$(K_{(i,k)}-1)(K_{(i,k)}-q)(K_{(i,k)}-q^2)\cdots(K_{(i,k)}-q^n) = 0,$$

The elements $g_{(m_k,k)}(f,e)$, $g_{(1,k+1)}(f,e)$ in (1.5.4) coincide with the Jucys-Murphy elements $\sigma_{(m_k,k)}$, $\sigma_{(1,k+1)}$ respectively, which are described by generators $e_{(i,k)}, f_{(i,k)}$ $\big((i,k)\in\Gamma'(\mathbf{m})\big)$ and $K^{\pm}_{(i,k)}$ $\big((i,k)\in\Gamma(\mathbf{m})\big)$ (see [**W**, 7.11]).

Moreover, ${}_{\mathcal{A}}\mathscr{S}_{n,r}$ is isomorphic to the $\mathcal{A}$-subalgebra of ${}_{\mathcal{K}}\mathscr{S}_{n,r}$ generated by $e^l_{(i,k)}/[l]!$, $f^l_{(i,k)}/[l]!$ $\big((i,k)\in\Gamma'(\mathbf{m}),\ l\geq 1\big)$, $K^{\pm}_{(i,k)}$ $\big((i,k)\in\Gamma(\mathbf{m})\big)$, $\prod_{s=1}^t \frac{K_{(i,k)}q^{-s+1}-K^{-1}_{(i,k)}q^{s-1}}{q^s-q^{-s}}$ $\big((i,k)\in\Gamma(\mathbf{m}), t\geq 1\big)$, where $[l]=\frac{q^l-q^{-l}}{q-q^{-1}}$ and $[l]! = [l][l-1]\cdots[1]$.

(ii) ${}_{\mathcal{K}}\mathscr{S}_{n,r}$ is isomorphic to the algebra over $\mathcal{K}$ defined by the generators $E_{(i,k)}$, $F_{(i,k)}$ $((i,k)\in\Gamma'(\mathbf{m}))$, 1_λ $(\lambda\in\Lambda_{n,r}(\mathbf{m}))$ with the following defining relations:

(1.5.9) $$1_\lambda 1_\mu = \delta_{\lambda,\mu}1_\lambda, \quad \sum_{\lambda\in\Lambda_{n,r}(\mathbf{m})} 1_\lambda = 1,$$

(1.5.10) $$E_{(i,k)}1_\lambda = \begin{cases} 1_{\lambda+\alpha_{(i,k)}}E_{(i,k)} & \text{if } \lambda+\alpha_{(i,k)}\in\Lambda_{n,r}(\mathbf{m}), \\ 0 & \text{otherwise,} \end{cases}$$

(1.5.11) $$F_{(i,k)}1_\lambda = \begin{cases} 1_{\lambda-\alpha_{(i,k)}}F_{(i,k)} & \text{if } \lambda-\alpha_{(i,k)}\in\Lambda_{n,r}(\mathbf{m}), \\ 0 & \text{otherwise,} \end{cases}$$

(1.5.12) $$1_\lambda E_{(i,k)} = \begin{cases} E_{(i,k)}1_{\lambda-\alpha_{(i,k)}} & \text{if } \lambda-\alpha_{(i,k)}\in\Lambda_{n,r}(\mathbf{m}), \\ 0 & \text{otherwise,} \end{cases}$$

(1.5.13) $$1_\lambda F_{(i,k)} = \begin{cases} F_{(i,k)}1_{\lambda+\alpha_{(i,k)}} & \text{if } \lambda+\alpha_{(i,k)}\in\Lambda_{n,r}(\mathbf{m}), \\ 0 & \text{otherwise,} \end{cases}$$

(1.5.14) $$E_{(i,k)}F_{(j,l)} - F_{(j,l)}E_{(i,k)} = \delta_{(i,k),(j,l)}\sum_{\lambda\in\Lambda_{n,r}} \eta^\lambda_{(i,k)},$$

$$\text{where } \eta^{\lambda}_{(i,k)} = \begin{cases} [\lambda^{(k)}_i - \lambda^{(k)}_{i+1}]1_\lambda & \text{if } i \neq m_k, \\ \Big(- Q_{k+1}[\lambda^{(k)}_{m_k} - \lambda^{(k+1)}_1] \\ \quad +q^{\lambda^{(k)}_{m_k} - \lambda^{(k+1)}_1}(q^{-1}g^{\lambda}_{(m_k,k)}(F,E) - qg^{\lambda}_{(1,k+1)}(F,E))\Big)1_\lambda & \text{if } i = m_k, \end{cases}$$

$$\text{(1.5.15)} \quad \begin{aligned} & E_{(i\pm 1,k)}(E_{(i,k)})^2 - (q+q^{-1})E_{(i,k)}E_{(i\pm1,k)}E_{(i,k)} + (E_{(i,k)})^2 E_{(i\pm1,k)} = 0, \\ & E_{(i,k)}E_{(j,l)} = E_{(j,l)}E_{(i,k)} \qquad (|\gamma((i,k)) - \gamma((j,l))| \geq 2), \end{aligned}$$

$$\text{(1.5.16)} \quad \begin{aligned} & F_{(i\pm 1,k)}(F_{(i,k)})^2 - (q+q^{-1})F_{(i,k)}F_{(i\pm1,k)}F_{(i,k)} + (F_{(i,k)})^2 F_{(i\pm1,k)} = 0, \\ & F_{(i,k)}F_{(j,l)} = F_{(j,l)}F_{(i,k)} \qquad (|\gamma((i,k)) - \gamma((j,l))| \geq 2), \end{aligned}$$

The elements $g^{\lambda}_{(m_k,k)}(F,E)$, $g^{\lambda}_{(1,k+1)}(F,E)$ in (1.5.14) coincide with $\sigma^{\lambda}_{(m_k,k)}$, $\sigma^{\lambda}_{(1,k+1)}$ respectively, which are described by generators $E_{(i,k)}, F_{(i,k)}$ $\big((i,k) \in \Gamma'(\mathbf{m})\big)$ (see [**W**, 7.1-7.4]).

Moreover, ${}_{\mathcal{A}}\mathscr{S}_{n,r}$ is isomorphic to the $\mathcal{A}$-subalgebra of ${}_{\mathcal{K}}\mathscr{S}_{n,r}$ generated by $E^{l}_{(i,k)}/[l]!$, $F^{l}_{(i,k)}/[l]!$ $\big((i,k) \in \Gamma'(\mathbf{m}),\, l \geq 1\big)$, 1_λ $(\lambda \in \Lambda_{n,r}(\mathbf{m}))$.

Remark 1.6. In [**W**], we treated only the case where $m_k = n$ for any $k = 1, \cdots, r$. We can obtain Theorem 1.5 for the general case in the same way under the condition $m_k \geq n$ for any $k = 1, \cdots, r$. However, in the case where $m_k < n$ for some k, we do not have the presentation of $\mathscr{S}_{n,r}(\Lambda_{n,r}(\mathbf{m}))$ as in the above theorem. In such a case, we have the following realization of $\mathscr{S}_{n,r}(\Lambda_{n,r}(\mathbf{m}))$. First, we take $\widetilde{\mathbf{m}} = (\widetilde{m}_1, \cdots, \widetilde{m}_r) \in \mathbb{Z}^r_{>0}$ such that $\widetilde{m}_k \geq n$ and $\widetilde{m}_k \geq m_k$ for any $k = 1, \cdots, r$. Then, we can regard $\Lambda_{n,r}(\mathbf{m})$ as a subset of $\Lambda_{n,r}(\widetilde{\mathbf{m}})$ in the natural way. We have the presentation of $\mathscr{S}_{n,r}(\Lambda_{n,r}(\widetilde{\mathbf{m}}))$ by the theorem, and we have $\mathscr{S}_{n,r}(\Lambda_{n,r}(\mathbf{m})) \cong 1_{\mathbf{m}}\mathscr{S}_{n,r}(\Lambda_{n,r}(\widetilde{\mathbf{m}}))1_{\mathbf{m}}$, where $1_{\mathbf{m}} = \sum_{\lambda \in \Lambda_{n,r}(\mathbf{m})} 1_\lambda \in \mathscr{S}_{n,r}(\Lambda_{n,r}(\widetilde{\mathbf{m}}))$.

1.7. *Weyl modules* (see [**W**] and [**DJM**] for more details). Let ${}_{\mathcal{A}}\mathscr{S}^{+}_{n,r}$ (resp. ${}_{\mathcal{A}}\mathscr{S}^{-}_{n,r}$) be the subalgebra of ${}_{\mathcal{A}}\mathscr{S}_{n,r}$ generated by $E^{l}_{(i,k)}/[l]!$ (resp. $F^{l}_{(i,k)}/[l]!$) for $(i,k) \in \Gamma'(\mathbf{m})$ and $l \geq 1$. Let ${}_{\mathcal{A}}\mathscr{S}^{0}_{n,r}$ be the subalgebra of ${}_{\mathcal{A}}\mathscr{S}_{n,r}$ generated by 1_λ for $\lambda \in \Lambda_{n,r}(\mathbf{m})$. Then ${}_{\mathcal{A}}\mathscr{S}_{n,r}$ has the triangular decomposition ${}_{\mathcal{A}}\mathscr{S}_{n,r} = {}_{\mathcal{A}}\mathscr{S}^{-}_{n,r}\, {}_{\mathcal{A}}\mathscr{S}^{0}_{n,r}\, {}_{\mathcal{A}}\mathscr{S}^{+}_{n,r}$ by [**W**, Proposition 3.2, Theorem 4.12, Theorem 5.6, Proposition 6.4, Proposition 7.7 and Theorem 7.16] . We denote by ${}_{\mathcal{A}}\mathscr{S}^{\geq 0}_{n,r}$ the subalgebra of ${}_{\mathcal{A}}\mathscr{S}_{n,r}$ generated by ${}_{\mathcal{A}}\mathscr{S}^{+}_{n,r}$ and ${}_{\mathcal{A}}\mathscr{S}^{0}_{n,r}$.

Note that ${}_{R}\mathscr{S}_{n,r}$ is obtained from ${}_{\mathcal{A}}\mathscr{S}_{n,r}$ by the specialization. Then ${}_{R}\mathscr{S}_{n,r}$ also has the triangular decomposition ${}_{R}\mathscr{S}_{n,r} = {}_{R}\mathscr{S}^{-}_{n,r}\, {}_{R}\mathscr{S}^{0}_{n,r}\, {}_{R}\mathscr{S}^{+}_{n,r}$ which comes from the triangular decomposition of ${}_{\mathcal{A}}\mathscr{S}_{n,r}$.

For $\lambda \in \Lambda^{+}_{n,r}(\mathbf{m})$, we define the one-dimensional ${}_{R}\mathscr{S}^{\geq 0}_{n,r}$-module $\theta_\lambda = Rv_\lambda$ by $E_{(i,k)} \cdot v_\lambda = 0$ $\big((i,k) \in \Gamma'(\mathbf{m})\big)$ and $1_\mu \cdot v_\lambda = \delta_{\lambda,\mu}v_\lambda$ $(\mu \in \Lambda_{n,r}(\mathbf{m}))$. Then the Weyl module ${}_{R}W(\lambda)$ of ${}_{R}\mathscr{S}_{n,r}$ is defined as the induced module ${}_{R}\mathscr{S}_{n,r} \otimes_{{}_{R}\mathscr{S}^{\geq 0}_{n,r}} \theta_\lambda$ of θ_λ for $\lambda \in \Lambda^{+}_{n,r}(\mathbf{m})$. Note that this Weyl module ${}_{R}W(\lambda)$ coincides with the ordinal Weyl module of ${}_{R}\mathscr{S}_{n,r}$ defined in [**DJM**] thanks to [**DR**, Theorem 5.16].

When R is a field, it is known that ${}_{R}W(\lambda)$ has the unique simple top ${}_{R}L(\lambda)$, and that $\{\, {}_{R}L(\lambda) \mid \lambda \in \Lambda^{+}_{n,r}(\mathbf{m})\}$ gives a complete set of non-isomorphic (left) simple ${}_{R}\mathscr{S}_{n,r}$-modules. Moreover, it is known that ${}_{\mathcal{K}}\mathscr{S}_{n,r}$ is semi-simple, and that $\{{}_{\mathcal{K}}W(\lambda) \mid \lambda \in \Lambda^{+}_{n,r}(\mathbf{m})\}$ gives a complete set of non-isomorphic (left) simple ${}_{\mathcal{K}}\mathscr{S}_{n,r}$-modules.

1.8. By (1.5.9), the identity element 1 of ${}_R\mathscr{S}_{n,r}$ decomposes to a sum of pairwise orthogonal idempotents indexed by $\varLambda_{n,r}(\mathbf{m})$, namely we have $1 = \sum_{\lambda \in \varLambda_{n,r}(\mathbf{m})} 1_\lambda$. Thanks to this decomposition, for ${}_R\mathscr{S}_{n,r}$-module M, we have the decomposition $M = \bigoplus_{\lambda \in \varLambda_{n,r}(\mathbf{m})} 1_\lambda M$ as R-modules. By the isomorphism between the first presentation and the second presentation of $\mathscr{S}_{n,r}$ in Theorem 1.5 (see [**W**, Proposition 7.12] for this isomorphism), we see that $K_{(i,k)}$ acts on $1_\lambda M$ by multiplying the scalar $q^{\lambda_i^{(k)}}$, namely we have $1_\lambda M = \{m \in M \mid K_{(i,k)} \cdot m = q^{\lambda_i^{(k)}} m \text{ for } (i,k) \in \varGamma(\mathbf{m})\}$. We call $1_\lambda M$ the weight space of weight λ (or λ-weight space simply), and denote it by M_λ.

2. $U_q(\mathfrak{g})$-crystal structure on Weyl module $W(\lambda)$ of $\mathscr{S}_{n,r}$

Throughout the rest of paper except the section 4, we assume the following condition for $\mathbf{m} = (m_1, \cdots, m_r)$:

$$m_k \geq n \text{ for any } k = 1, \cdots, r. \tag{2.0.1}$$

In the section 2 - section 4, we consider only the cyclotomic q-Schur algebra ${}_{\mathcal{K}}\mathscr{S}_{n,r}$ over $\mathcal{K}$, and we omit the subscript $\mathcal{K}$.

2.1. Let $\mathfrak{g} = \mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}$ be the Levi subalgebra of $\mathfrak{gl}_m$, and $U_q(\mathfrak{g}) \cong U_q(\mathfrak{gl}_{m_1}) \otimes \cdots \otimes U_q(\mathfrak{gl}_{m_r})$ be the quantum group over $\mathcal{K}$ corresponding to $\mathfrak{g}$. Put $\varGamma'_{\mathfrak{g}}(\mathbf{m}) = \varGamma(\mathbf{m}) \setminus \{(m_k, k) \mid 1 \leq k \leq r\}$. Let $e_{(i,k)}, f_{(i,k)}$ $((i,k) \in \varGamma'_{\mathfrak{g}}(\mathbf{m}))$, $K^{\pm}_{(i,k)}$ $((i,k) \in \varGamma(\mathbf{m}))$ be the generators of $U_q(\mathfrak{g})$, where $e_{(i,k)}, f_{(i,k)}, K^{\pm}_{(j,k)}$ $(1 \leq i \leq m_k - 1$, $1 \leq j \leq m_k)$ are the usual Chevalley generators of $U_q(\mathfrak{gl}_{m_k})$.

By the presentation of $\mathscr{S}_{n,r}$ (Theorem 1.5), we can define the algebra homomorphism $\Phi_{\mathfrak{g}} : U_q(\mathfrak{g}) \to \mathscr{S}_{n,r}$ sending generators of $U_q(\mathfrak{g})$ to the corresponding generators of $\mathscr{S}_{n,r}$ denoted by the same symbol. Note that $\Phi_{\mathfrak{g}}$ is not surjective without the case where $r = 1$. We have the following lemma which describes the image of $\Phi_{\mathfrak{g}}$.

Lemma 2.2.

(i) $$\Phi_{\mathfrak{g}}(U_q(\mathfrak{g})) \cong \bigoplus_{\substack{\eta = (n_1, \cdots, n_r) \\ n_1 + \cdots + n_r = n}} \mathscr{S}^{\eta}_{n_1,1}(\varLambda_{n_1,1}(m_1)) \otimes \cdots \otimes \mathscr{S}^{\eta}_{n_r,1}(\varLambda_{n_r,1}(m_r)),$$
where $\mathscr{S}^{\eta}_{n_k,1}(\varLambda_{n_k,1}(m_k))$ is the q-Schur algebra associated to the symmetric group $\mathfrak{S}_{n_k}$ of degree n_k.

(ii) Let ${}_{\mathcal{A}}U_q(\mathfrak{g})$ be the $\mathcal{A}$-form of $U_q(\mathfrak{g})$ by taking the divided powers. Then we have

$$\Phi_{\mathfrak{g}}({}_{\mathcal{A}}U_q(\mathfrak{g})) \cong \bigoplus_{\substack{\eta = (n_1, \cdots, n_r) \\ n_1 + \cdots + n_r = n}} {}_{\mathcal{A}}\mathscr{S}^{\eta}_{n_1,1}(\varLambda_{n_1,1}(m_1)) \otimes \cdots \otimes {}_{\mathcal{A}}\mathscr{S}^{\eta}_{n_r,1}(\varLambda_{n_r,1}(m_r)).$$

Proof. Put $\mathscr{S}_{\mathfrak{g}} = \bigoplus_{\substack{\eta = (n_1, \cdots, n_r) \\ n_1 + \cdots + n_r = n}} \mathscr{S}^{\eta}_{n_1,1}(\varLambda_{n_1,1}(m_1)) \otimes \cdots \otimes \mathscr{S}^{\eta}_{n_r,1}(\varLambda_{n_r,1}(m_r))$. Let $e_i^{\eta,k}$, $f_i^{\eta,k}$ $(1 \leq i \leq m_k - 1)$, $K_i^{\eta,k\pm}$ $(1 \leq i \leq m_k)$ be the generators of

$\mathscr{S}^{\eta}_{n_k,1}(\Lambda_{n_k,1}(m_k))$ in Theorem 1.5 (i). Then, we define the homomorphism of algebras $\varphi : U_q(\mathfrak{g}) \to \mathscr{S}_{\mathfrak{g}}$ by

$$\varphi(e_{(i,k)}) = \sum_{\substack{\eta=(n_1,\cdots,n_r)\\ n_1+\cdots+n_r=n}} \underbrace{1\otimes\cdots\otimes 1}_{k-1}\otimes e_i^{\eta,k}\otimes 1\otimes\cdots\otimes 1,$$

$$\varphi(f_{(i,k)}) = \sum_{\substack{\eta=(n_1,\cdots,n_r)\\ n_1+\cdots+n_r=n}} \underbrace{1\otimes\cdots\otimes 1}_{k-1}\otimes f_i^{\eta,k}\otimes 1\otimes\cdots\otimes 1,$$

$$\varphi(K^{\pm}_{(i,k)}) = \sum_{\substack{\eta=(n_1,\cdots,n_r)\\ n_1+\cdots+n_r=n}} \underbrace{1\otimes\cdots\otimes 1}_{k-1}\otimes K_i^{\eta,k\pm}\otimes 1\otimes\cdots\otimes 1$$

for generators $e_{(i,k)}$, $f_{(i,k)}$ $((i,k)\in\Gamma'_{\mathfrak{g}}(\mathbf{m}))$, $K^{\pm}_{(i,k)}$ $((i,k)\in\Gamma(\mathbf{m}))$ of $U_q(\mathfrak{g})$. (We can easily check that φ is well-defined by Theorem 1.5 (i).)

We also define the homomorphism of algebras $\psi : \mathscr{S}_{\mathfrak{g}} \to \mathscr{S}_{n,r}$ by

$$\psi(\underbrace{1\otimes\cdots\otimes 1}_{k-1}\otimes e_i^{\eta,k}\otimes 1\otimes\cdots\otimes 1) = (\sum_{\substack{\mu\in\Lambda_{n,r}(\mathbf{m})\\ \zeta(\mu)=\eta}} 1_\mu)\cdot e_{(i,k)}\cdot(\sum_{\substack{\mu\in\Lambda_{n,r}(\mathbf{m})\\ \zeta(\mu)=\eta}} 1_\mu),$$

$$\psi(\underbrace{1\otimes\cdots\otimes 1}_{k-1}\otimes f_i^{\eta,k}\otimes 1\otimes\cdots\otimes 1) = (\sum_{\substack{\mu\in\Lambda_{n,r}(\mathbf{m})\\ \zeta(\mu)=\eta}} 1_\mu)\cdot f_{(i,k)}\cdot(\sum_{\substack{\mu\in\Lambda_{n,r}(\mathbf{m})\\ \zeta(\mu)=\eta}} 1_\mu),$$

$$\psi(\underbrace{1\otimes\cdots\otimes 1}_{k-1}\otimes K_i^{\eta,k\pm}\otimes 1\otimes\cdots\otimes 1) = (\sum_{\substack{\mu\in\Lambda_{n,r}(\mathbf{m})\\ \zeta(\mu)=\eta}} 1_\mu)\cdot K^{\pm}_{(i,k)}\cdot(\sum_{\substack{\mu\in\Lambda_{n,r}(\mathbf{m})\\ \zeta(\mu)=\eta}} 1_\mu)$$

for each generators of $\mathscr{S}_{\mathfrak{g}}$. (We can check the well-definedness by direct calculations.) From the definitions, we see that $\psi\circ\varphi = \Phi_{\mathfrak{g}}$. Thus, ψ induces the surjective homomorphism $\psi' : \mathscr{S}_{\mathfrak{g}} \to \Phi_{\mathfrak{g}}(U_q(\mathfrak{g}))$. We prove ψ' is an isomorphism.

We easily see that simple $\mathscr{S}_{\mathfrak{g}}$-modules are indexed by $\Lambda^+_{n,r}(\mathbf{m})$, and the simple $\mathscr{S}_{\mathfrak{g}}$-module corresponding λ ($\lambda\in\Lambda^+_{n,r}(\mathbf{m})$) is regarded as the simple highest weight $U_q(\mathfrak{g})$-module $W_{\mathfrak{g}}(\lambda)$ of highest weight λ through φ.

On the other hand, by investigating the appearing weights in $\mathscr{S}_{n,r}$ as a $U_q(\mathfrak{g})$-module through $\Phi_{\mathfrak{g}}$, we see that the simple $\Phi_{\mathfrak{g}}(U_q(\mathfrak{g}))$-modules are indexed by a subset of $\Lambda^+_{n,r}(\mathbf{m})$. Moreover, for $\lambda\in\Lambda^+_{n,r}(\mathbf{m})$, we see that the Weyl module $W(\lambda)$ of $\mathscr{S}_{n,r}$ contains the simple highest weight $U_q(\mathfrak{g})$-module $W_{\mathfrak{g}}(\lambda)$ of highest weight λ through $\Phi_{\mathfrak{g}}$. As a consequence, the simple $\Phi_{\mathfrak{g}}(U_q(\mathfrak{g}))$-modules are indexed by $\Lambda^+_{n,r}(\mathbf{m})$, and the simple $\Phi_{\mathfrak{g}}(U_q(\mathfrak{g}))$-module corresponding λ ($\lambda\in\Lambda^+_{n,r}(\mathbf{m})$) is regarded as the simple highest weight $U_q(\mathfrak{g})$-module $W_{\mathfrak{g}}(\lambda)$ of highest weight λ through $\Phi_{\mathfrak{g}}$.

Note that both $\mathscr{S}_{\mathfrak{g}}$ and $\Phi_{\mathfrak{g}}(U_q(\mathfrak{g}))$ are semi-simple, by Wedderburn's theorem, we have

$$\dim\mathscr{S}_{\mathfrak{g}} = \dim\Phi_{\mathfrak{g}}(U_q(\mathfrak{g})) = \sum_{\lambda\in\Lambda^+_{n,r}(\mathbf{m})} (\dim W_{\mathfrak{g}}(\lambda))^2.$$

Thus, we have that ψ' is an isomorphism. (ii) follows from (i) by restricting $\Phi_{\mathfrak{g}}$ to ${}_{\mathcal{A}}U_q(\mathfrak{g})$. □

2.3. For an $\mathscr{S}_{n,r}$-module M, we regard M as a $U_q(\mathfrak{g})$-module through the homomorphism $\Phi_{\mathfrak{g}}$. Then, by Lemma 2.2 (or by investigating weights directly), we see that a simple $U_q(\mathfrak{g})$-module appearing in M as a composition factor is of the form $W(\lambda^{(1)})\boxtimes\cdots\boxtimes W(\lambda^{(r)})$ for some $\lambda\in\Lambda^+_{n,r}(\mathbf{m})$, where $W(\lambda^{(k)})$ is the simple highest weight $U_q(\mathfrak{gl}_{m_k})$-module of highest weight $\lambda^{(k)}$. Hence, the Weyl module $W(\lambda)$ of $\mathscr{S}_{n,r}$ decomposes as follows:

$$W(\lambda) \cong \bigoplus_{\mu \in \Lambda^+_{n,r}(\mathbf{m})} \Big(W(\mu^{(1)}) \boxtimes \cdots \boxtimes W(\mu^{(r)}) \Big)^{\oplus \beta_{\lambda\mu}} \quad \text{as } U_q(\mathfrak{g})\text{-modules.} \tag{2.3.1}$$

2.4. In order to compute the multiplicity $\beta_{\lambda\mu}$ in (2.3.1), we will describe the $U_q(\mathfrak{g})$-crystal structure of $W(\lambda)$. For such a purpose, we prepare some notation of combinatorics.

For $\mu \in \Lambda_{n,r}(\mathbf{m})$, the diagram $[\mu]$ of μ is the set

$$[\mu] = \{(i,j,k) \in \mathbb{Z}^3 \,|\, 1 \le i \le m_k,\, 1 \le j \le \mu_i^{(k)},\, 1 \le k \le r\}.$$

For $\lambda \in \Lambda^+_{n,r}(\mathbf{m})$ and $\mu \in \Lambda_{n,r}(\mathbf{m})$, a tableau of shape λ with weight μ is a map

$$T : [\lambda] \to \{(a,c) \in \mathbb{Z} \times \mathbb{Z} \,|\, a \ge 1, 1 \le c \le r\}$$

such that $\mu_i^{(k)} = \sharp\{x \in [\lambda] \,|\, T(x) = (i,k)\}$. We define the order on $\mathbb{Z} \times \mathbb{Z}$ by $(a,c) \ge (a',c')$ if either $c > c'$, or $c = c'$ and $a \ge a'$. For a tableau T of shape λ with weight μ, we say that T is semi-standard if T satisfies the following conditions:

(i) If $T((i,j,k)) = (a,c)$, then $k \le c$,
(ii) $T((i,j,k)) \le T((i,j+1,k))$ if $(i,j+1,k) \in [\lambda]$,
(iii) $T((i,j,k)) < T((i+1,j,k))$ if $(i+1,j,k) \in [\lambda]$.

For $\lambda \in \Lambda^+_{n,r}(\mathbf{m})$, $\mu \in \Lambda_{n,r}(\mathbf{m})$, we denote by $\mathcal{T}_0(\lambda,\mu)$ the set of semi-standard tableaux of shape λ with weight μ. Put $\mathcal{T}_0(\lambda) = \bigcup_{\mu \in \Lambda_{n,r}(\mathbf{m})} \mathcal{T}_0(\lambda,\mu)$. We identify a semi-standard tableau with a Young tableau as the following example.
For $\lambda = ((3,2),(3,1),(1,1))$, $\mu = ((2,1),(2,2),(3,1))$

$$T = \left(\begin{array}{|c|c|c|} \hline (1,1) & (1,1) & (1,2) \\ \hline (2,1) & (1,3) \\ \cline{1-2} \end{array}\,,\ \begin{array}{|c|c|c|} \hline (1,2) & (2,2) & (1,3) \\ \hline (2,2) \\ \cline{1-1} \end{array}\,,\ \begin{array}{|c|} \hline (1,3) \\ \hline (2,3) \\ \hline \end{array} \right) \in \mathcal{T}_0(\lambda,\mu),$$

where $T((1,1,1)) = (1,1)$, $T((1,2,1)) = (1,1)$, $\cdots$, $T((2,1,3)) = (2,3)$.

By [**DJM**], it is known that there exists a bijection between $\mathcal{T}_0(\lambda,\mu)$ and a basis of $W(\lambda)_\mu$. Hence, we will describe a $U_q(\mathfrak{g})$-crystal structure on $\mathcal{T}_0(\lambda)$ which is isomorphic to the $U_q(\mathfrak{g})$-crystal basis of $W(\lambda)$.

2.5. By (2.3.1), for $\lambda \in \Lambda^+_{n,r}(\mathbf{m})$, $\mu \in \Lambda_{n,r}(\mathbf{m})$, we have

$$\begin{aligned} \sharp\mathcal{T}_0(\lambda,\mu) &= \dim W(\lambda)_\mu \\ &= \sum_{\nu \in \Lambda^+_{n,r}(\mathbf{m})} \beta_{\lambda\nu} \cdot \dim \Big(W(\nu^{(1)}) \boxtimes \cdots \boxtimes W(\nu^{(r)}) \Big)_\mu \\ &= \sum_{\nu \in \Lambda^+_{n,r}(\mathbf{m})} \beta_{\lambda\nu} \prod_{k=1}^r \dim W(\nu^{(k)})_{\mu^{(k)}} \\ &= \sum_{\nu \in \Lambda^+_{n,r}(\mathbf{m})} \beta_{\lambda\nu} \prod_{k=1}^r \sharp\mathcal{T}_0(\nu^{(k)},\mu^{(k)}) \\ &= \sum_{\nu \in \Lambda^+_{n,r}(\mathbf{m})} \beta_{\lambda\nu} \prod_{k=1}^r K_{\nu^{(k)}\mu^{(k)}}, \end{aligned} \tag{2.5.1}$$

where $K_{\nu^{(k)}\mu^{(k)}}$ is the Kostka number. We have the following properties of $\beta_{\lambda\mu}$.

LEMMA 2.6.
(i) For $\lambda \in \Lambda^+_{n,r}(\mathbf{m})$, $\beta_{\lambda\lambda} = 1$.
(ii) For $\lambda, \mu \in \Lambda^+_{n,r}(\mathbf{m})$, if $\beta_{\lambda\mu} \neq 0$, we have $\lambda \geq \mu$.
(iii) For $\lambda, \mu \in \Lambda^+_{n,r}(\mathbf{m})$, if $\lambda \neq \mu$ and $\zeta(\lambda) = \zeta(\mu)$, we have $\beta_{\lambda\mu} = 0$.
(iv) For $\lambda, \mu \in \Lambda^+_{n,r}(\mathbf{m})$ such that $\zeta(\lambda) \neq \zeta(\mu)$, if $\mathcal{T}_0(\lambda, \nu) = \emptyset$ for any $\nu \in \Lambda^+_{n,r}(\mathbf{m})$ such that $\zeta(\nu) = \zeta(\mu)$ and $\nu > \mu$, then we have $\beta_{\lambda\mu} = \sharp\mathcal{T}_0(\lambda, \mu)$.

PROOF. (i) From the definition of $W(\lambda)$, we have $W(\lambda) = \mathscr{S}^-_{n,r} \cdot v_\lambda$, where we denote $1 \otimes v_\lambda \in \mathscr{S}_{n,r} \otimes_{\mathscr{S}^{\geq 0}_{n,r}} \theta_\lambda$ by v_λ simply. Thus, we have that $W(\lambda)_\lambda = \mathcal{K} v_\lambda$, and that v_λ is a highest weight vector of highest weight λ in $U_q(\mathfrak{g})$-module $W(\lambda)$. This implies that $\beta_{\lambda\lambda} = 1$.

(ii) $\beta_{\lambda\mu} \neq 0 \Rightarrow W(\lambda)_\mu \neq 0 \Rightarrow \lambda \geq \mu$.

(iii) Assume that $\lambda \neq \mu$ and $\zeta(\lambda) = \zeta(\mu)$. By (2.5.1), we have

$$
\begin{aligned}
\sharp\mathcal{T}_0(\lambda, \mu) =& \beta_{\lambda\lambda} \prod_{k=1}^{r} \sharp\mathcal{T}_0(\lambda^{(k)}, \mu^{(k)}) + \beta_{\lambda\mu} \prod_{k=1}^{r} \sharp\mathcal{T}_0(\mu^{(k)}, \mu^{(k)}) \\
&+ \sum_{\substack{\nu \in \Lambda^+_{n,r}(\mathbf{m}) \\ \nu \neq \lambda, \mu}} \beta_{\lambda\nu} \prod_{k=1}^{r} \sharp\mathcal{T}_0(\nu^{(k)}, \mu^{(k)}).
\end{aligned}
\tag{2.6.1}
$$

This implies that $\beta_{\lambda\mu} = 0$ since $\sharp\mathcal{T}_0(\mu^{(k)}, \mu^{(k)}) = 1$, and $\sharp\mathcal{T}_0(\lambda, \mu) = \prod_{k=1}^r \sharp\mathcal{T}_0(\lambda^{(k)}, \mu^{(k)})$ if $\zeta(\lambda) = \zeta(\mu)$.

(iv) Note that $\prod_{k=1}^r \sharp\mathcal{T}_0(\nu^{(k)}, \mu^{(k)}) = 0$ if $\zeta(\nu) \neq \zeta(\mu)$ or $\nu \not\geq \mu$, and that $\prod_{k=1}^r \sharp\mathcal{T}_0(\nu^{(k)}, \mu^{(k)}) = \mathcal{T}_0(\nu, \mu)$ if $\zeta(\nu) = \zeta(\mu)$. Then (2.5.1) combining with the assumption of (iv) implies $\sharp\mathcal{T}_0(\lambda, \mu) = \beta_{\lambda\mu}\sharp\mathcal{T}_0(\mu, \mu) = \beta_{\lambda\mu}$ since $\beta_{\lambda\nu} = 0$ if $\mathcal{T}_0(\lambda, \nu) = \emptyset$. □

2.7. For $\lambda \in \Lambda_{n,r}(\mathbf{m})$, we define the total order "$\succeq$" on the diagram $[\lambda]$ by $(i, j, k) \succ (i', j', k')$ if $k > k'$, $k = k'$ and $j > j'$ or if $k = k', j = j'$ and $i < i'$. For an example, we have

$$(5, 4, 2) \succ (2, 3, 2) \succ (5, 3, 2) \succ (6, 4, 1).$$

2.8. We define the equivalence relation "$\sim$" on $\mathcal{T}_0(\lambda)$ by $T \sim T'$ if

$$
\begin{aligned}
&\{x \in [\lambda] \mid T(x) = (i, k) \text{ for some } i = 1, \cdots, m_k\} \\
&= \{y \in [\lambda] \mid T'(y) = (j, k) \text{ for some } j = 1, \cdots, m_k\}
\end{aligned}
$$

for any $k = 1, \cdots, r$. By the definition, for $T \in \mathcal{T}_0(\lambda, \mu)$ and $T' \in \mathcal{T}_0(\lambda, \nu)$, we have

$$\zeta(\mu) = \zeta(\nu) \text{ if } T \sim T'. \tag{2.8.1}$$

EXAMPLE 2.1. Put

$$
T_1 = \left(\begin{array}{|c|c|} \hline (1,1) & (1,1) \\ \hline (1,2) & (2,2) \\ \hline \end{array}, \begin{array}{|c|c|} \hline (1,2) & (2,2) \\ \hline (3,2) & \\ \cline{1-1} \end{array} \right), \quad T_2 = \left(\begin{array}{|c|c|} \hline (1,1) & (2,1) \\ \hline (1,2) & (3,2) \\ \hline \end{array}, \begin{array}{|c|c|} \hline (2,2) & (2,2) \\ \hline (4,2) & \\ \cline{1-1} \end{array} \right),
$$

$$
T_3 = \left(\begin{array}{|c|c|} \hline (1,1) & (1,2) \\ \hline (2,1) & (3,2) \\ \hline \end{array}, \begin{array}{|c|c|} \hline (2,2) & (2,2) \\ \hline (4,2) & \\ \cline{1-1} \end{array} \right), \quad T_4 = \left(\begin{array}{|c|c|} \hline (1,1) & (2,2) \\ \hline (3,1) & (3,2) \\ \hline \end{array}, \begin{array}{|c|c|} \hline (1,2) & (1,2) \\ \hline (2,2) & \\ \cline{1-1} \end{array} \right).
$$

Then, we have $T_1 \sim T_2$, $T_2 \not\sim T_3$ and $T_3 \sim T_4$.

2.9. Let V_{m_k} be the vector representation of $U_q(\mathfrak{gl}_{m_k})$ with a natural basis $\{v_1, v_2, \cdots, v_{m_k}\}$. Let $\mathcal{A}_0$ be the localization of $\mathbb{Q}(Q_1, \cdots, Q_r)[q]$ at $q = 0$. Put

$\mathcal{L}_{m_k}=\bigoplus_{j=1}^{m_k}\mathcal{A}_0\cdot v_j$, $\boxed{j}=v_j+q\mathcal{L}_{m_k}\in\mathcal{L}_{m_k}/q\mathcal{L}_{m_k}$ and $\mathcal{B}_{m_k}=\{\boxed{j}\,|\,1\leq j\leq m_k\}$. Then $(\mathcal{L}_{m_k},\mathcal{B}_{m_k})$ gives the crystal basis of V_{m_k}. Then the $U_q(\mathfrak{g})$-crystal $\mathcal{B}_{m_1}^{\otimes n_1}\boxtimes\cdots\boxtimes\mathcal{B}_{m_r}^{\otimes n_r}$ is the crystal basis of $V_{m_1}^{\otimes n_1}\boxtimes\cdots\boxtimes V_{m_r}^{\otimes n_r}$.

Let $\mathcal{T}_0(\lambda)/_\sim$ be the set of equivalence classes with respect to the relation $\sim$. To avoid confusion, we also use a notation $\mathcal{T}_0(\lambda)[t]$ for the equivalence class $t\in\mathcal{T}_0(\lambda)/_\sim$. Hence we have the disjoint union $\mathcal{T}_0(\lambda)=\bigcup_{t\in\mathcal{T}_0(\lambda)/_\sim}\mathcal{T}_0(\lambda)[t]$.

For each equivalence class $\mathcal{T}_0(\lambda)[t]$, put $(n_1,\cdots,n_r)=\zeta(\mu)$ for some μ such that $\mathcal{T}_0(\lambda,\mu)\cap\mathcal{T}_0(\lambda)[t]\neq\emptyset$ (note (2.8.1)), and we define the map

$$\Psi_t^\lambda:\mathcal{T}_0(\lambda)[t]\to\mathcal{B}_{m_1}^{\otimes n_1}\boxtimes\cdots\boxtimes\mathcal{B}_{m_r}^{\otimes n_r}$$

as

$$\Psi_t^\lambda(T)=\Big(\boxed{i_1^{(1)}}\otimes\cdots\otimes\boxed{i_{n_1}^{(1)}}\Big)\boxtimes\cdots\boxtimes\Big(\boxed{i_1^{(r)}}\otimes\cdots\otimes\boxed{i_{n_r}^{(r)}}\Big)$$

satisfying the following three conditions:

(i) $\{x\in[\lambda]\,|\,T(x)=(i,k)\text{ for some }i=1,\cdots,m_k\}=\{x_1^{(k)},x_2^{(k)},\cdots,x_{n_k}^{(k)}\}$ for $k=1,\cdots,r$.
(ii) $x_1^{(k)}\succ x_2^{(k)}\succ\cdots\succ x_{n_k}^{(k)}$ for $k=1,\cdots,r$.
(iii) $T(x_j^{(k)})=(i_j^{(k)},k)$ for $1\leq j\leq n_k$, $1\leq k\leq r$.

Namely, $\Big(\boxed{i_1^{(k)}}\otimes\cdots\otimes\boxed{i_{n_k}^{(k)}}\Big)$ in $\Psi_t^\lambda(T)$ is obtained by reading the first coordinate of $T(x)$ for $x\in[\lambda]$ such that $T(x)=(i,k)$ for some $i=1,\cdots,m_k$ in the order $\succeq$ on $[\lambda]$.

Example 2.2. For

$$T=\left(\begin{array}{|c|c|c|}\hline (1,1)&(1,1)&(1,2)\\\hline (2,1)&(1,3)\\\cline{1-2}\end{array}\,,\ \begin{array}{|c|c|c|}\hline (1,2)&(2,2)&(1,3)\\\hline (2,2)\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|}\hline (1,3)\\\hline (2,3)\\\hline\end{array}\right)\in\mathcal{T}_0(\lambda)[t],$$

we have $\Psi_t^\lambda(T)=\big(\boxed{1}\otimes\boxed{1}\otimes\boxed{2}\big)\boxtimes\big(\boxed{2}\otimes\boxed{1}\otimes\boxed{2}\otimes\boxed{1}\big)\boxtimes\big(\boxed{1}\otimes\boxed{2}\otimes\boxed{1}\otimes\boxed{1}\big)$.

Remark 2.10. In the case where $r=1$, $\mathcal{T}_0(\lambda)$ has only one equivalence class (itself) with respect to $\sim$, and Ψ^λ coincides with the Far-Eastern reading given in [**KN**, §3] (see also [**HK**, Ch. 7]).

2.11. Let $\widetilde{e}_{(i,k)}$, $\widetilde{f}_{(i,k)}$ ($(i,k)\in\varGamma'_{\mathfrak{g}}(\mathbf{m})$) be the Kashiwara operators on $U_q(\mathfrak{g})$-crystal $\mathcal{B}_{m_1}^{\otimes n_1}\boxtimes\cdots\boxtimes\mathcal{B}_{m_r}^{\otimes n_r}$. Then we have the following proposition.

Proposition 2.12. For each equivalence class $\mathcal{T}_0(\lambda)[t]$ of $\mathcal{T}_0(\lambda)$, we have the followings.

(i) The map $\Psi_t^\lambda:\mathcal{T}_0(\lambda)[t]\to\mathcal{B}_{m_1}^{\otimes n_1}\boxtimes\cdots\boxtimes\mathcal{B}_{m_r}^{\otimes n_r}$ is injective.
(ii) $\Psi_t^\lambda\big(\mathcal{T}_0(\lambda)[t]\big)\cup\{0\}$ is stable under the Kashiwara operators $\widetilde{e}_{(i,k)}$, $\widetilde{f}_{(i,k)}$ $\big((i,k)\in\varGamma'_{\mathfrak{g}}(\mathbf{m})\big)$.

Proof. (i) is clear from the definitions. We prove (ii). For $b\in\mathcal{B}_{m_1}^{\otimes n_1}\boxtimes\cdots\boxtimes\mathcal{B}_{m_r}^{\otimes n_r}$, we can obtain the tableau $T_t^\lambda(b)$ of shape λ by taking the reverse process of the definition of Ψ_t^λ. Note that $T_t^\lambda(b)$ may not be semi-standard. If $T_t^\lambda(b)$ is semi-standard, we have that $T_t^\lambda(b)\in\mathcal{T}_0(\lambda)[t]$, and that $\Psi_t^\lambda(T_t^\lambda(b))=b$ from the definitions. Hence, in order to prove (ii), it is enough to show that, $T_t^\lambda(\widetilde{e}_{(i,k)}\cdot\Psi_t^\lambda(T))$ (resp. $T_t^\lambda(\widetilde{f}_{(i,k)}\cdot\Psi_t^\lambda(T))$) is semi-standard for $T\in\mathcal{T}_0(\lambda)[t]$ and $(i,k)\in\varGamma'_{\mathfrak{g}}(\mathbf{m})$ such that $\widetilde{e}_{(i,k)}\cdot\Psi_t^\lambda(T)\neq 0$ (resp. $\widetilde{f}_{(i,k)}\cdot\Psi_t^\lambda(T)\neq 0$). This can be proven in a similar way as in the case of type A ($r=1$) (see [**KN**] or [**HK**, Theorem 7.3.6]), and we obtain (ii). $\square$

2.13. By Proposition 2.12, we define the $U_q(\mathfrak{g})$-crystal structure on $\mathcal{T}_0(\lambda)[t]$ through Ψ_t^λ, and also define the $U_q(\mathfrak{g})$-crystal structure on $\mathcal{T}_0(\lambda)$. Note that the $U_q(\mathfrak{g})$-crystal graphs of $\mathcal{T}_0(\lambda)[t]$ and of $\mathcal{T}_0(\lambda)[t']$ are disconnected in the $U_q(\mathfrak{g})$-crystal graph of $\mathcal{T}_0(\lambda)$ if $\mathcal{T}_0(\lambda)[t]$ is a different equivalence class from $\mathcal{T}_0(\lambda)[t']$. For $T \in \mathcal{T}_0(\lambda)$, we say that T is $U_q(\mathfrak{g})$-singular if $\widetilde{e}_{(i,k)} \cdot T = 0$ for any $(i,k) \in \Gamma'_{\mathfrak{g}}(\mathbf{m})$. Put

$$\mathcal{T}_{sing}(\lambda,\mu) = \{T \in \mathcal{T}_0(\lambda,\mu) \,|\, T : U_q(\mathfrak{g})\text{-singular}\}.$$

Remark 2.14. We should define the map Ψ_t^λ for each equivalence class $\mathcal{T}_0(\lambda)[t]$ of $\mathcal{T}_0(\lambda)$ since it may happen that $\Psi_t^\lambda(T) = \Psi_{t'}^\lambda(T')$ for different equivalence classes $\mathcal{T}_0(\lambda)[t]$ and $\mathcal{T}_0(\lambda)[t']$. For an example, put

$$T = \left(\begin{array}{|c|c|c|}\hline (1,1) & (1,1) & (\mathbf{1},\mathbf{2}) \\ \hline (2,1) & (\mathbf{1},\mathbf{3}) \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|}\hline (1,2) & (2,2) & (1,3) \\ \hline (2,2) \\ \cline{1-1} \end{array}, \begin{array}{|c|}\hline (1,3) \\ \hline (2,3) \\ \hline \end{array} \right) \in \mathcal{T}_0(\lambda)[t],$$

$$T' = \left(\begin{array}{|c|c|c|}\hline (1,1) & (1,1) & (\mathbf{1},\mathbf{3}) \\ \hline (2,1) & (\mathbf{1},\mathbf{2}) \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|}\hline (1,2) & (2,2) & (1,3) \\ \hline (2,2) \\ \cline{1-1} \end{array}, \begin{array}{|c|}\hline (1,3) \\ \hline (2,3) \\ \hline \end{array} \right) \in \mathcal{T}_0(\lambda)[t'].$$

Then we have

$$\Psi_t^\lambda(T) = \Psi_{t'}^\lambda(T') = \left(\boxed{1} \otimes \boxed{1} \otimes \boxed{2}\right) \boxtimes \left(\boxed{2} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{1}\right) \boxtimes \left(\boxed{1} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{1}\right).$$

Now, we have the following theorem.

Theorem 2.15.

(i) The $U_q(\mathfrak{g})$-crystal structure on $T_0(\lambda)$ is isomorphic to the $U_q(\mathfrak{g})$-crystal basis of $W(\lambda)$ as crystals.

(ii) For $\lambda, \mu \in \Lambda^+_{n,r}(\mathbf{m})$, we have $\beta_{\lambda\mu} = \sharp \mathcal{T}_{sing}(\lambda,\mu)$.

Proof. From the definition, the $U_q(\mathfrak{g})$-crystal structure on $\mathcal{T}_0(\lambda)[t]$ is isomorphic to the crystal basis of a $U_q(\mathfrak{g})$-submodule of $V_{m_1}^{\otimes n_1} \boxtimes \cdots \boxtimes V_{m_r}^{n_r}$. Thus the $U_q(\mathfrak{g})$-crystal $\mathcal{T}_0(\lambda)$ is isomorphic to the crystal basis of a certain finite dimensional $U_q(\mathfrak{g})$-module and the crystal basis of a finite dimensional $U_q(\mathfrak{g})$-module is uniquely determined by its character up to isomorphism. We see that the weight of $\Psi_t^\lambda(T)$ is μ if $T \in \mathcal{T}_0(\lambda,\mu)$. We also see that the dimension of the μ-weight space of $W(\lambda)$ (as $U_q(\mathfrak{g})$-module) is the cardinality of $\mathcal{T}_0(\lambda,\mu)$. Thus, the character of $\mathcal{T}_0(\lambda)$ coincides with the character of $W(\lambda)$. This implies (i). (ii) follows from (i) immediately. $\square$

3. Some properties of the number $\beta_{\lambda\mu}$

In this section, we collect some properties of the number $\beta_{\lambda\mu}$.

3.1. For r-partitions λ and μ, we denote by $\lambda \supset \mu$ if $[\lambda] \supset [\mu]$. For r-partitions λ and μ such that $\lambda \supset \mu$, we define the skew Young diagram by $\lambda/\mu = [\lambda] \setminus [\mu]$. One can naturally identify λ/μ with $(\lambda^{(1)}/\mu^{(1)}, \cdots, \lambda^{(r)}/\mu^{(r)})$, where $\lambda^{(k)}/\mu^{(k)}$ $(1 \leq k \leq r)$ is the usual skew Young diagram for $\lambda^{(k)} \supset \mu^{(k)}$. For a skew Young diagram λ/μ, we define a semi-standard tableau of shape λ/μ in a similar manner as in the case where the shape is an r-partition. We denote by $\mathcal{T}_0(\lambda/\mu,\nu)$ the set of semi-standard tableaux of shape λ/μ with weight ν. Put $\mathcal{T}_0(\lambda/\mu) = \bigcup_{\nu \in \Lambda_{n',r}(\mathbf{m})} \mathcal{T}_0(\lambda/\mu,\nu)$, where $n' = |\lambda/\mu|$. Then, we can describe the $U_q(\mathfrak{g})$-crystal structure on $\mathcal{T}_0(\lambda/\mu)$ in a similar way as in the paragraphs 2.7 - 2.13. Namely, we define the equivalence relation "$\sim$" on $\mathcal{T}_0(\lambda/\mu)$ in a similar way as in 2.8, and define the map $\Psi_t^{\lambda/\mu} : \mathcal{T}_0(\lambda/\mu)[t] \to \mathcal{B}_{m_1}^{\otimes n_1} \boxtimes \cdots \boxtimes \mathcal{B}_{m_r}^{\otimes n_r}$ for each equivalence class $\mathcal{T}_0(\lambda/\mu)[t]$ of $\mathcal{T}_0(\lambda/\mu)$ as in 2.9. Then we can show that $\Psi_t^{\lambda/\mu}$ is injective, and that $\Psi_t^{\lambda/\mu}(\mathcal{T}_0(\lambda/\mu)[t])\cup\{0\}$ is stable under the Kashiwara operators $\widetilde{e}_{(i,k)}$, $\widetilde{f}_{(i,k)}$ for $(i,k)\in\Gamma'_{\mathfrak{g}}(\mathbf{m})$ (cf. Proposition 2.12). Put

$$\mathcal{T}_{sing}(\lambda/\mu,\nu)=\{T\in\mathcal{T}_0(\lambda/\mu,\nu)\,|\,T:\,U_q(\mathfrak{g})\text{-singular}\}.$$

From the tensor product rule for $U_q(\mathfrak{g})$-crystals, we have the following criterion on whether $T\in\mathcal{T}_0(\lambda/\mu)$ is $U_q(\mathfrak{g})$-singular or not (note that $\mathcal{T}_0(\lambda/\mu)=\mathcal{T}_0(\lambda)$ if $\mu=\emptyset$).

Lemma 3.2. For $T\in\mathcal{T}_0(\lambda/\mu)[t]$, let

$$\Psi_t^{\lambda/\mu}(T)=\Big(\boxed{i_1^{(1)}}\otimes\cdots\otimes\boxed{i_{n_1}^{(1)}}\Big)\boxtimes\cdots\boxtimes\Big(\boxed{i_1^{(r)}}\otimes\cdots\otimes\boxed{i_{n_r}^{(r)}}\Big). \tag{3.2.1}$$

Then, T is $U_q(\mathfrak{g})$-singular if and only if the weight of $\Big(\boxed{i_1^{(k)}}\otimes\cdots\otimes\boxed{i_j^{(k)}}\Big)\in\mathcal{B}_{m_k}^{\otimes j}$ is a partition (i.e. dominant integral weight of $\mathfrak{gl}_{m_k}$) for any $1\le j\le n_k$ and any $1\le k\le r$.

Proof. It is clear that, for $T\in\mathcal{T}_0(\lambda/\mu)[t]$ satisfying (3.2.1), T is $U_q(\mathfrak{g})$-singular if and only if $\Big(\boxed{i_1^{(k)}}\otimes\cdots\otimes\boxed{i_{n_k}^{(k)}}\Big)\in\mathcal{B}_{m_k}^{\otimes n_k}$ is $U_q(\mathfrak{gl}_{m_k})$-singular for any $k=1,\cdots,r$. Hence, the lemma follows from [**N**, Lemma 6.1.1] (see also [**HK**, Corollary 4.4.4]). $\square$

Remark 3.3. By Lemma 3.2, if $T\in\mathcal{T}_0(\lambda)$ is $U_q(\mathfrak{g})$-singular, the weight of T must be an r-partition. Moreover, we see that the number $\beta_{\lambda\mu}$ is independent of a choice of $\mathbf{m}$ satisfying the condition (2.0.1) from Lemma 3.2.

For some special partitions, we have the following lemma.

Lemma 3.4.

(i) If $\lambda=((n),\emptyset,\cdots,\emptyset)$,
$$\beta_{\lambda\mu}=\begin{cases}1 & \text{if } \mu=((n_1),(n_2),\cdots,(n_r)) \text{ for some } (n_1,\cdots,n_r)\in\mathbb{Z}_{\ge0}^r\\ 0 & \text{otherwise}\end{cases}.$$
(ii) If $\lambda=((1^n),\emptyset,\cdots,\emptyset)$,
$$\beta_{\lambda\mu}=\begin{cases}1 & \text{if } \mu=((1^{n_1}),(1^{n_2}),\cdots,(1^{n_r})) \text{ for some } (n_1,\cdots,n_r)\in\mathbb{Z}_{\ge0}^r\\ 0 & \text{otherwise}\end{cases}.$$
(iii) If $\mu=(\emptyset,\cdots,\emptyset,(n))$,
$$\beta_{\lambda\mu}=\begin{cases}1 & \text{if } \lambda=((n_1),(n_2),\cdots,(n_r)) \text{ for some } (n_1,\cdots,n_r)\in\mathbb{Z}_{\ge0}^r\\ 0 & \text{otherwise}\end{cases}.$$
(iv) If $\mu=(\emptyset,\cdots,\emptyset,(1^n))$,
$$\beta_{\lambda\mu}=\begin{cases}1 & \text{if } \lambda=((1^{n_1}),(1^{n_2}),\cdots,(1^{n_r})) \text{ for some } (n_1,\cdots,n_r)\in\mathbb{Z}_{\ge0}^r\\ 0 & \text{otherwise}\end{cases}.$$

Proof. One can easily check them by using Theorem 2.15 and Lemma 3.2. $\square$

3.5. For an integer g $(1<g<r)$, fix $\mathbf{p}=(r_1,\dots,r_g)\in\mathbb{Z}_{>0}^g$ such that $\sum_{k=1}^g r_k=r$. For $\lambda=(\lambda^{(1)},\cdots,\lambda^{(r)})\in\Lambda_{n,r}^+(\mathbf{m})$, put $\lambda^{[k]_\mathbf{p}}=(\lambda^{(p_k+1)},\cdots,\lambda^{(p_k+r_k)})$, where $p_k=\sum_{j=1}^{k-1}r_j$ with $p_1=0$. We define the map $\zeta^\mathbf{p}:\Lambda_{n,r}^+(\mathbf{m})\to\mathbb{Z}_{\ge0}^g$ by $\zeta^\mathbf{p}(\lambda)=(|\lambda^{[1]_\mathbf{p}}|,\cdots,|\lambda^{[g]_\mathbf{p}}|)$. Then, we have the following lemma.

Lemma 3.6. For $\lambda, \mu \in \Lambda^+_{n,r}(\mathbf{m})$ such that $\zeta^{\mathbf{p}}(\lambda) = \zeta^{\mathbf{p}}(\mu)$, we have

$$\beta_{\lambda\mu} = \prod_{k=1}^{g} \beta_{\lambda^{[k]_{\mathbf{p}}}\mu^{[k]_{\mathbf{p}}}}.$$

Proof. It is enough to show the case where $\mathbf{p} = (r_1, r_2)$ since we can obtain the claim for general cases by the induction on g. If $\zeta^{\mathbf{P}}(\lambda) = \zeta^{\mathbf{p}}(\mu)$ for $\mathbf{p} = (r_1, r_2)$, then we have the bijection

$$\mathcal{T}_0(\lambda,\mu) \to \mathcal{T}_0(\lambda^{[1]_{\mathbf{p}}}, \mu^{[1]_{\mathbf{p}}}) \times \mathcal{T}_0(\lambda^{[2]_{\mathbf{p}}}, \mu^{[2]_{\mathbf{p}}}) \text{ such that } T \mapsto (T^{[1]_{\mathbf{p}}}, T^{[2]_{\mathbf{p}}}), \tag{3.6.1}$$

where $T^{[1]_{\mathbf{p}}}((i,j,k)) = T((i,j,k))$ for $(i,j,k) \in [\lambda^{[1]_{\mathbf{p}}}]$, and $T^{[2]_{\mathbf{p}}}((i,j,k)) = (a, c-r_1)$ if $T((i,j,r_1+k)) = (a,c)$ for $(i,j,k) \in [\lambda^{[2]_{\mathbf{p}}}]$. In this case, by the definition of Ψ^λ_t and Lemma 3.2, it is clear that $T \in \mathcal{T}_0(\lambda,\mu)$ is $U_q(\mathfrak{g})$-singular if and only if $T^{[1]_{\mathbf{p}}}$ (resp. $T^{[2]_{\mathbf{p}}}$) is $U_q(\mathfrak{g}^{[1]})$-singular (resp. $U_q(\mathfrak{g}^{[2]})$-singular), where $\mathfrak{g}^{[1]} = \mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_{r_1}}$ (resp. $\mathfrak{g}^{[2]} = \mathfrak{gl}_{m_{r_1+1}} \oplus \cdots \oplus \mathfrak{gl}_{m_r}$). Then, by Theorem 2.15 (i) together with (3.6.1), we have $\beta_{\lambda\mu} = \beta_{\lambda^{[1]_{\mathbf{p}}}\mu^{[1]_{\mathbf{p}}}} \beta_{\lambda^{[2]_{\mathbf{p}}}\mu^{[2]_{\mathbf{p}}}}$. □

3.7. For $\lambda, \mu \in \Lambda^+_{n,r}(\mathbf{m})$, we define the following set of sequences of r-partitions:

$$\Theta(\lambda,\mu) := \Big\{ \lambda = \lambda_{\langle r\rangle} \supset \lambda_{\langle r-1\rangle} \supset \cdots \supset \lambda_{\langle 1\rangle} \supset \lambda_{\langle 0\rangle} = (\emptyset, \cdots, \emptyset) \;\Big|\; (\lambda_{\langle k\rangle})^{(k+1)} = \emptyset,\ \ |\lambda_{\langle k\rangle}/\lambda_{\langle k-1\rangle}| = |\mu^{(k)}| \text{ for } k = 1, \cdots, r \Big\}.$$

It is clear that, for $\lambda_{\langle r\rangle} \supset \cdots \supset \lambda_{\langle 0\rangle} \in \Theta(\lambda,\mu)$, $(\lambda_{\langle k\rangle})^{(l)} = \emptyset$ if $l > k$, and that $|\lambda_{\langle k\rangle}| = \sum_{j=1}^{k} |\mu^{(k)}|$. Then, we can rewrite Theorem 2.15 (i) as the following corollary.

Corollary 3.8. For $\lambda, \mu \in \Lambda^+_{n,r}(\mathbf{m})$, we have

$$\beta_{\lambda\mu} = \sum_{\lambda_{\langle r\rangle} \supset \cdots \supset \lambda_{\langle 0\rangle} \in \Theta(\lambda,\mu)} \prod_{k=1}^{r} \sharp \mathcal{T}_{sing}\big(\lambda_{\langle k\rangle}/\lambda_{\langle k-1\rangle}, (\emptyset, \cdots, \emptyset, \mu^{(k)}, \emptyset, \cdots, \emptyset)\big). \tag{3.8.1}$$

In particular, if $\lambda = (\emptyset, \cdots, \emptyset, \lambda^{(t)}, \emptyset, \cdots, \emptyset)$ for some t, then we have

$$\beta_{\lambda\mu} = \sum_{\lambda_{\langle r\rangle} \supset \cdots \supset \lambda_{\langle 0\rangle} \in \Theta(\lambda,\mu)} \prod_{k=1}^{r} \mathrm{LR}^{\lambda^{(t)}_{\langle k\rangle}}_{\lambda^{(t)}_{\langle k-1\rangle}, \mu^{(k)}}, \tag{3.8.2}$$

where $\mathrm{LR}^{\lambda^{(t)}_{\langle k\rangle}}_{\lambda^{(t)}_{\langle k-1\rangle}, \mu^{(k)}}$ is the Littlewood-Richardson coefficient for $\lambda^{(t)}_{\langle k-1\rangle}$, $\mu^{(k)}$ and $\lambda^{(t)}_{\langle k\rangle}$ with $\mathrm{LR}^{\emptyset}_{\emptyset,\emptyset} = 1$.

Proof. Note that we can identify the set $\Theta(\lambda,\mu)$ with the set of equivalence classes of $\mathcal{T}_0(\lambda,\mu)$ with respect to the relation $\sim$ by corresponding $\lambda_{\langle r\rangle} \supset \cdots \supset \lambda_{\langle 0\rangle} \in \Theta(\lambda,\mu)$ to the equivalence class of $\mathcal{T}_0(\lambda,\mu)$ containing $T \in \mathcal{T}_0(\lambda,\mu)$ such that

$$[\lambda_{\langle k\rangle}] = \{(i,j,l) \in [\lambda] \,|\, T((i,j,l)) = (a,c) \text{ for some } 1 \le a \le m_c, 1 \le c \le k\}$$

for any $k = 1, \cdots, r$. Then Lemma 3.2 and Theorem 2.15 (i) imply the equation (3.8.1).

Assume that $\lambda^{(k)} = \emptyset$ if $k \neq t$ for some t. Then, for $\lambda_{\langle r\rangle} \supset \cdots \supset \lambda_{\langle 0\rangle} \in \Theta(\lambda,\mu)$, we have

$$\begin{aligned}\sharp\mathcal{T}_{sing}\big(\lambda_{\langle k\rangle}/\lambda_{\langle k-1\rangle}, (\emptyset,\cdots,\emptyset,\mu^{(k)},\emptyset,\cdots,\emptyset)\big) &= \sharp\mathcal{T}_{sing}\big(\lambda^{(t)}_{\langle k\rangle}/\lambda^{(t)}_{\langle k-1\rangle}, \mu^{(k)}\big)\\ &= \mathrm{LR}^{\lambda^{(t)}_{\langle k\rangle}}_{\lambda^{(t)}_{\langle k-1\rangle},\mu^{(k)}},\end{aligned}$$

where the last equation follows from the original Littlewood-Richardson rule ([**Mac**, Ch. I (9.2)]). (Note that, for partitions λ,μ (not multi-partitions) such that $\lambda\supset\mu$, the $U_q(\mathfrak{gl}_m)$-crystal structure on $\mathcal{T}_0(\lambda/\mu)$ does not depend on the choice of admissible reading (see [**HK**, Theorem 7.3.6]). Then a similar statement as in Lemma 3.2 for $\mathcal{T}_0(\lambda/\mu)$ under the Middle-Eastern reading coincides with the Littlewood-Richardson rule.) Then (3.8.1) implies (3.8.2). □

Remark 3.9. In the case where $r=2$ and $\lambda = (\lambda^{(1)},\emptyset)$, by (3.8.2), we have

$$\begin{aligned}\beta_{\lambda\mu} &= \sum_{\lambda^{(1)}_{\langle 1\rangle}} \mathrm{LR}^{\lambda^{(1)}}_{\lambda^{(1)}_{\langle 1\rangle},\mu^{(1)}}\,\mathrm{LR}^{\lambda^{(1)}_{\langle 1\rangle}}_{\emptyset,\mu^{(2)}}\\ &= \mathrm{LR}^{\lambda^{(1)}}_{\mu^{(2)},\mu^{(1)}},\end{aligned}$$

where the last equation follows from $\mathrm{LR}^{\lambda^{(1)}_{\langle 1\rangle}}_{\emptyset,\mu^{(2)}} = \delta_{\lambda^{(1)}_{\langle 1\rangle},\mu^{(2)}}$. Thus, the Littlewood-Richardson coefficient $\mathrm{LR}^{\lambda}_{\mu,\nu}$ for partitions λ,μ,ν is obtained as the number $\beta_{(\lambda,\emptyset)(\mu,\nu)}$. Moreover, thanks to Lemma 3.2 together with the reading $\Psi^{\lambda/\mu}_t$, we can regard (3.8.1) as a generalization of the Littlewood-Richardson rule.

We also remark the following classical fact. Let $\mathbf{GL}_n$ be the general linear group of rank n, and V_λ be the simple $\mathbf{GL}_n$-module corresponding to a partition λ. For $m<n$, we can regard $\mathbf{GL}_m\times\mathbf{GL}_{n-m}$ as a subgroup of $\mathbf{GL}_n$ in the natural way. Let $[\mathrm{Res}^{\mathbf{GL}_n}_{\mathbf{GL}_m\times\mathbf{GL}_{n-m}} V_\lambda : V_\mu\boxtimes V_\nu]_{\mathbf{GL}_m\times\mathbf{GL}_n}$ be the multiplicity of the simple $\mathbf{GL}_m\boxtimes\mathbf{GL}_{n-m}$-module $V_\mu\boxtimes V_\nu$ in the simple $\mathbf{GL}_n$-module V_λ through the restriction. Then we have

$$[\mathrm{Res}^{\mathbf{GL}_n}_{\mathbf{GL}_m\times\mathbf{GL}_{n-m}} V_\lambda : V_\mu\boxtimes V_\nu]_{\mathbf{GL}_m\times\mathbf{GL}_n} = \mathrm{LR}^{\lambda}_{\mu,\nu}\,. \tag{3.9.1}$$

Comparing (2.3.1) with (3.9.1), we may regard the number $\beta_{\lambda\mu}$ as a generalization of Littlewood-Richardson coefficients.

4. Characters of the Weyl modules and symmetric functions

In this section, for the completeness about symmetric polynomials, we do not assume the condition (2.0.1) for $\mathbf{m}$. We remark that, in the case where $\mathbf{m}$ does not satisfy (2.0.1), we can not define the number $\beta_{\lambda\mu}$ by (2.3.1) since we can not define the map $\Phi_{\mathfrak{g}}$ in this case. Hence, for $\lambda,\mu\in\Lambda^+_{n,r}(\mathbf{m})$, we redefine the number $\beta_{\lambda\mu}$ without any conditions for $\mathbf{m}$ as follows. When $\mathbf{m}$ satisfies the condition (2.0.1), we denote by $\beta_{\lambda\mu}(\mathbf{m})$ the multiplicity $\beta_{\lambda\mu}$ in (2.3.1). Then, for $\lambda,\mu\in\Lambda^+_{n,r}(\mathbf{m})$ (without any conditions for $\mathbf{m}$), we redefine the number $\beta_{\lambda\mu}$ as the number $\beta_{\lambda\mu}(\mathbf{m}')$ for some $\mathbf{m}'$ satisfying the condition (2.0.1). Note that this definition does not depend on a choice of $\mathbf{m}'$ satisfying the condition (2.0.1) (see Remark 3.3).

4.1. For $\mathbf{m}=(m_1,\cdots,m_r)\in\mathbb{Z}^r_{>0}$, we denote by $\Xi_{\mathbf{m}} = \bigotimes_{k=1}^r \mathbb{Z}[x^{(k)}_1,\cdots,x^{(k)}_{m_k}]^{\mathfrak{S}_{m_k}}$ the ring of symmetric polynomials (with respect to $\mathfrak{S}_{m_1}\times\cdots\times\mathfrak{S}_{m_r}$) with variables

$x_i^{(k)}$ $(1 \le i \le m_k,\ 1 \le k \le r)$. We denote by $x_{\mathbf{m}}^{(k)} = (x_1^{(k)}, x_2^{(k)}, \cdots, x_{m_k}^{(k)})$ the set of m_k independent variables for $k = 1, \cdots, r$, and denote by $\mathbf{x_m} = (x_{\mathbf{m}}^{(1)}, \cdots, x_{\mathbf{m}}^{(r)})$ the whole variables. Let $\Xi_{\mathbf{m}}^n$ be the subset of $\Xi_{\mathbf{m}}$ which consists of homogeneous symmetric polynomials of degree n. We also consider the inverse limit $\Xi^n = \varprojlim_{\mathbf{m}} \Xi_{\mathbf{m}}^n$ with respect to $\mathbf{m}$. Put $\Xi = \bigoplus_{n \ge 0} \Xi^n$. Then Ξ becomes the ring of symmetric functions $\Xi = \bigotimes_{k=1}^r \mathbb{Z}[X^{(k)}]^{\mathfrak{S}(X^{(k)})}$, where $X^{(k)} = (X_1^{(k)}, X_2^{(k)}, \cdots)$ is the set of (infinite) variables, and $\mathfrak{S}(X^{(k)})$ is the permutation group of the set $X^{(k)}$. We denote by $\mathbf{X} = (X^{(1)}, \cdots, X^{(r)})$ the whole variables of Ξ.

For $\lambda = (\lambda^{(1)}, \cdots, \lambda^{(r)}) \in \varLambda_{n,r}^+(\mathbf{m})$, put $S_\lambda(\mathbf{x_m}) = \prod_{k=1}^r S_{\lambda^{(k)}}(x_{\mathbf{m}}^{(k)})$ (resp. $S_\lambda(\mathbf{X}) = \prod_{k=1}^r S_{\lambda^{(k)}}(X^{(k)})$), where $S_{\lambda^{(k)}}(x_{\mathbf{m}}^{(k)})$ (resp. $S_{\lambda^{(k)}}(X^{(k)})$) is the Schur polynomial (resp. Schur function) associated to $\lambda^{(k)}$ $(1 \le k \le r)$ in the variables $x_{\mathbf{m}}^{(k)}$ (resp. $X^{(k)}$). Then $\{S_\lambda(\mathbf{x_m}) \mid \lambda \in \varLambda_{n,r}^+(\mathbf{m})\}$ (resp. $\{S_\lambda(\mathbf{X}) \mid \lambda \in \varLambda_{n,r}^+\}$) gives a $\mathbb{Z}$-basis of $\Xi_{\mathbf{m}}^n$ (resp. $\mathbb{Z}$-basis of Ξ^n).

4.2. For an $\mathscr{S}_{n,r}(\varLambda_{n,r}(\mathbf{m}))$-module M, we define the character of M by

$$\operatorname{ch} M = \sum_{\mu \in \varLambda_{n,r}(\mathbf{m})} \dim M_\mu \cdot x^\mu \ \in \mathbb{Z}[\mathbf{x_m}],$$

where $x^\mu = \prod_{k=1}^r (x_1^{(k)})^{\mu_1^{(k)}} (x_2^{(k)})^{\mu_2^{(k)}} \cdots (x_{m_k}^{(k)})^{\mu_{m_k}^{(k)}}$. Put $\widetilde{S}_\lambda(\mathbf{x_m}) = \operatorname{ch} W(\lambda)$ for $\lambda \in \varLambda_{n,r}^+(\mathbf{m})$. Then the character $\widetilde{S}_\lambda(\mathbf{x_m})$ of the Weyl module $W(\lambda)$ for $\mathscr{S}_{n,r}(\varLambda_{n,r}(\mathbf{m}))$ has the following properties.

Theorem 4.3.

(i) For $\lambda \in \varLambda_{n,r}^+(\mathbf{m})$, we have

$$\widetilde{S}_\lambda(\mathbf{x_m}) = \operatorname{ch} W(\lambda) = \sum_{\mu \in \varLambda_{n,r}(\mathbf{m})} \left(\sum_{\nu \in \varLambda_{n,r}^+(\mathbf{m})} \beta_{\lambda\nu} \prod_{k=1}^r K_{\nu^{(k)}\mu^{(k)}} \right) \cdot x^\mu,$$

where $K_{\nu^{(k)}\mu^{(k)}}$ is the Kostka number corresponding to partitions $\nu^{(k)}$ and $\mu^{(k)}$.

(ii) For $\lambda \in \varLambda_{n,r}^+(\mathbf{m})$, we have

$$\widetilde{S}_\lambda(\mathbf{x_m}) = \sum_{\mu \in \varLambda_{n,r}^+(\mathbf{m})} \beta_{\lambda\mu} S_\mu(\mathbf{x_m}).$$

(iii) $\{\widetilde{S}_\lambda(\mathbf{x_m}) \mid \lambda \in \varLambda_{n,r}^+(\mathbf{m})\}$ gives a $\mathbb{Z}$-basis of $\Xi_{\mathbf{m}}^n$.

Proof. Assume that $\mathbf{m} = (m_1, \cdots, m_r) \in \mathbb{Z}_{>0}^r$ does not satisfy the condition (2.0.1). In this case, we can take $\widetilde{\mathbf{m}} = (\widetilde{m}_1, \cdots, \widetilde{m}_r) \in \mathbb{Z}_{>0}^r$ satisfying (2.0.1) and $\widetilde{m}_k \ge m_k$ for any k. Then we have $\mathscr{S}_{n,r}(\varLambda_{n,r}(\mathbf{m})) \cong 1_{\mathbf{m}} \mathscr{S}_{n,r}(\varLambda_{n,r}(\widetilde{\mathbf{m}})) 1_{\mathbf{m}}$, where $1_{\mathbf{m}} = \sum_{\lambda \in \varLambda_{n,r}(\mathbf{m})} 1_\lambda \in \mathscr{S}_{n,r}(\varLambda_{n,r}(\widetilde{\mathbf{m}}))$ (see Remark 1.6). Thus, for an $\mathscr{S}_{n,r}(\varLambda_{n,r}(\widetilde{\mathbf{m}}))$-module M, $1_{\mathbf{m}} M$ turns out to be an $\mathscr{S}_{n,r}(\varLambda_{n,r}(\mathbf{m}))$-module. In particular, for the Weyl module $W(\lambda)$ of $\mathscr{S}_{n,r}(\varLambda_{n,r}(\widetilde{\mathbf{m}}))$, $1_{\mathbf{m}} W(\lambda)$ is isomorphic to the Weyl module corresponding to λ of $\mathscr{S}_{n,r}(\varLambda_{n,r}(\mathbf{m}))$ if $\lambda \in \varLambda_{n,r}^+(\mathbf{m})$.

Let $\phi : \mathbb{Z}[\mathbf{x}_{\widetilde{\mathbf{m}}}] \to \mathbb{Z}[\mathbf{x_m}]$ be the ring homomorphism such that $\phi(f(\mathbf{x}_{\widetilde{\mathbf{m}}}))$ is the polynomial obtained by setting $x_i^{(k)} = 0$ for $x_i^{(k)} \notin \mathbf{x_m}$. Then, by the definition of characters, we have that $\phi(\operatorname{ch} M) = \operatorname{ch} 1_{\mathbf{m}} M$ for $\mathscr{S}_{n,r}(\varLambda_{n,r}(\widetilde{\mathbf{m}}))$-module M. Note that $\phi(S_\lambda(\mathbf{x}_{\widetilde{\mathbf{m}}})) = 0$ if $\lambda \notin \varLambda_{n,r}^+(\mathbf{m})$, the statements in the theorem for $\mathbf{m}$ are deduced from the statements for $\widetilde{\mathbf{m}}$ through ϕ. Thus, it is enough to show the case where $\mathbf{m}$ satisfies the condition (2.0.1), and we assume the condition (2.0.1) for $\mathbf{m}$.

Since there exists a bijection between a basis of $W(\lambda)_\mu$ and $\mathcal{T}_0(\lambda,\mu)$, (i) follows from (2.5.1).

It is known that

$$S_\lambda(\mathbf{x_m}) = \sum_{\mu\in\Lambda_{n,r}(\mathbf{m})} \dim\Big(W(\lambda^{(1)})\boxtimes\cdots\boxtimes W(\lambda^{(r)})\Big)_\mu \cdot x^\mu. \tag{4.3.1}$$

Note that the μ-weight space of an $\mathscr{S}_{n,r}$-module coincides with the μ-weight space as the $U_q(\mathfrak{g})$-module via the homomorphism $\Phi_{\mathfrak{g}} : U_q(\mathfrak{g}) \to \mathscr{S}_{n,r}$. Thus, the decomposition (2.3.1) together with (4.3.1) implies (ii).

(iii) follows from (ii) since the number $\beta_{\lambda\mu}$ $(\lambda,\mu\in\Lambda^+_{n,r}(\mathbf{m}))$ has the unitriangular property by Lemma 2.6. $\square$

4.4. For $\lambda\in\Lambda^+_{n,r}(\mathbf{m})$, let $\widetilde{S}_\lambda(\mathbf{X})\in\Xi^n$ be the image of $\widetilde{S}_\lambda(\mathbf{x})$ in the inverse limit. We denote by $\Lambda^+_{\geq 0,r}=\bigcup_{n\geq 0}\Lambda^+_{n,r}$ the set of r-partitions. Then, Theorem 4.3 (iii) implies that $\{\widetilde{S}_\lambda(\mathbf{X})\mid\lambda\in\Lambda^+_{\geq 0,r}\}$ gives a $\mathbb{Z}$-basis of Ξ. For a certain special r-partition λ, $\widetilde{S}_\lambda(\mathbf{X})$ coincides with a Schur function as follows.

Proposition 4.5. Let $\lambda=(\lambda^{(1)},\cdots,\lambda^{(r)})\in\Lambda^+_{n,r}$. Assume that there exists an index $t=1,\cdots,r$ such that $\lambda^{(l)}=\emptyset$ unless $l=t$. Then we have

$$\widetilde{S}_\lambda(\mathbf{X}) = S_{\lambda^{(t)}}(X^{(t)}\cup X^{(t+1)}\cup\cdots\cup X^{(r)}),$$

where $S_{\lambda^{(t)}}(X^{(t)}\cup\cdots\cup X^{(r)})\in\mathbb{Z}[X^{(t)}\cup\cdots\cup X^{(r)}]^{\mathfrak{S}(X^{(t)}\cup\cdots\cup X^{(r)})}$ is the Schur function corresponding to the partition $\lambda^{(t)}$.

Proof. Assume that $\lambda^{(l)}=\emptyset$ unless $l=t$, then we see that the variable $X_i^{(l)}$ $(i\geq 1,\ 1\leq l\leq t-1)$ does not appear in $\widetilde{S}_\lambda(\mathbf{X})$ since $\lambda\geq\mu$ if $\dim W(\lambda)_\mu\neq 0$. Note that we can regard $\mathbb{Z}[X^{(t)}\cup\cdots\cup X^{(r)}]^{\mathfrak{S}(X^{(t)}\cup\cdots\cup X^{(r)})}$ as a subring of $\Xi=\bigotimes_{k=1}^r\mathbb{Z}[X^{(k)}]^{\mathfrak{S}(X^{(k)})}$ in the natural way. By Theorem 4.3 (ii) with (3.8.2), we have

$$\begin{aligned}
\widetilde{S}_\lambda(\mathbf{X}) &= \sum_{\mu\in\Lambda^+_{n,r}}\Bigg(\sum_{\lambda_{\langle r\rangle}\supset\cdots\supset\lambda_{\langle 0\rangle}\in\Theta(\lambda,\mu)}\prod_{k=1}^r \mathrm{LR}^{\lambda^{(t)}_{\langle k\rangle}}_{\lambda^{(t)}_{\langle k-1\rangle},\mu^{(k)}}\Bigg) S_\mu(\mathbf{X})\\
&= \sum_{\mu\in\Lambda^+_{n,r}}\sum_{(*1)}\Bigg(\prod_{k=t}^r \mathrm{LR}^{\lambda^{(t)}_{\langle k\rangle}}_{\lambda^{(t)}_{\langle k-1\rangle},\mu^{(k)}} S_{\mu^{(k)}}(X^{(k)})\Bigg)\\
&= \sum_{(*2)}\sum_{\mu\in\Lambda^+_{n,r}}\Bigg(\prod_{k=t}^r \mathrm{LR}^{\lambda^{(t)}_{\langle k\rangle}}_{\lambda^{(t)}_{\langle k-1\rangle},\mu^{(k)}} S_{\mu^{(k)}}(X^{(k)})\Bigg)\\
&= \sum_{(*2)}\prod_{k=t}^r\Bigg(\sum_{(*3)} \mathrm{LR}^{\lambda^{(t)}_{\langle k\rangle}}_{\lambda^{(t)}_{\langle k-1\rangle},\mu^{(k)}} S_{\mu^{(k)}}(X^{(k)})\Bigg)\\
&= \sum_{(*2)}\prod_{k=t}^r S_{\lambda^{(t)}_{\langle k\rangle}/\lambda^{(t)}_{\langle k-1\rangle}}(X^{(k)}) \qquad \big(\text{because of [}\mathbf{Mac}\text{, Ch. 1. (5.3)]}\big)\\
&= S_{\lambda^{(t)}}(X^{(t)}\cup X^{(t+1)}\cup\cdots\cup X^{(r)}) \qquad \big(\text{because of [}\mathbf{Mac}\text{, Ch. 1. (5.11)]}\big),
\end{aligned}$$

where the summations (∗1)-(∗3) run the following sets respectively:

$$(*1) : \big\{\lambda^{(t)} = \lambda^{(t)}_{\langle r\rangle} \supset \cdots \supset \lambda^{(t)}_{\langle t\rangle} \supset \lambda^{(t)}_{\langle t-1\rangle} = \emptyset \bigm| |\lambda^{(t)}_{\langle k\rangle}/\lambda^{(t)}_{\langle k-1\rangle}| = |\mu^{(k)}| \text{ for } k = t, \cdots, r\big\},$$
$$(*2) : \big\{\lambda^{(t)} = \lambda^{(t)}_{\langle r\rangle} \supset \cdots \supset \lambda^{(t)}_{\langle t\rangle} \supset \lambda^{(t)}_{\langle t-1\rangle} = \emptyset\big\},$$
$$(*3) : \big\{\mu^{(k)} \,:\, \text{partition}\big\}.$$

In the above equations, note that $\mathrm{LR}^{\lambda^{(t)}_{\langle k\rangle}}_{\lambda^{(t)}_{\langle k-1\rangle},\mu^{(k)}} = 0$ unless $|\lambda^{(t)}_{\langle k\rangle}| = |\lambda^{(t)}_{\langle k-1\rangle}| + |\mu^{(k)}|$.

□

4.6. Thanks to the above lemma, the symmetric function $\widetilde{S}_\lambda(\mathbf{X})$ seems to be a generalization of the Schur functions.

For $\lambda, \mu, \nu \in \Lambda^+_{\geq 0,r}$, we define the integer $c^\nu_{\lambda\mu} \in \mathbb{Z}$ by

$$\widetilde{S}_\lambda(\mathbf{X})\widetilde{S}_\mu(\mathbf{X}) = \sum_{\nu\in\Lambda^+_{\geq 0,r}} c^\nu_{\lambda\mu}\widetilde{S}_\nu(\mathbf{X}).$$

Then we determine the number $c^\nu_{\lambda\mu}$ as follows.

Proposition 4.7. For $\lambda, \mu, \nu \in \Lambda^+_{\geq 0,r}$, we have the following.

(i) $c^\nu_{\lambda\mu} = 0$ unless $|\nu| = |\lambda| + |\mu|$.

(ii) Put $\big(\beta'_{\tau\nu}\big)_{\tau,\nu\in\Lambda^+_{n,r}} = \big(\beta_{\tau\nu}\big)^{-1}_{\tau,\nu\in\Lambda^+_{n,r}}$ $(n = |\nu|)$. Then we have

$$c^\nu_{\lambda\mu} = \sum_{\xi,\eta,\tau\in\Lambda_{\geq 0,r}} \beta_{\lambda\xi}\beta_{\mu\eta}\beta'_{\tau\nu}\prod_{k=1}^{r}\mathrm{LR}^{\tau^{(k)}}_{\xi^{(k)}\eta^{(k)}}.$$

(iii) If $\zeta(\nu) = \zeta(\lambda+\mu)$, we have

$$c^\nu_{\lambda\mu} = \prod_{k=1}^{r}\mathrm{LR}^{\nu^{(k)}}_{\lambda^{(k)}\mu^{(k)}}.$$

(iv) If $\lambda^{(l)} = \emptyset$ and $\mu^{(l)} = \emptyset$ unless $l = t$ for some t, we have

$$c^\nu_{\lambda\mu} = \begin{cases}\mathrm{LR}^{\nu^{(t)}}_{\lambda^{(t)}\mu^{(t)}} & \text{if } \nu^{(l)} = \emptyset \text{ unless } l = t,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. (i) is clear from the definitions. We prove (ii). By Theorem 4.3 (ii), we have

$$\begin{aligned}
\widetilde{S}_\lambda(\mathbf{X})\widetilde{S}_\mu(\mathbf{X}) &= \Big(\sum_{\xi}\beta_{\lambda\xi}S_\xi(\mathbf{X})\Big)\Big(\sum_{\eta}\beta_{\mu\eta}S_\eta(\mathbf{X})\Big) \\
&= \sum_{\xi,\eta}\beta_{\lambda\xi}\beta_{\mu\eta}S_\xi(\mathbf{X})S_\eta(\mathbf{X}) \\
&= \sum_{\xi,\eta}\beta_{\lambda\xi}\beta_{\mu\eta}\left(\sum_{\tau}\Big(\prod_{k=1}^{r}\mathrm{LR}^{\tau^{(k)}}_{\xi^{(k)}\eta^{(k)}}\Big)S_\tau(\mathbf{X})\right) \\
&= \sum_{\xi,\eta}\beta_{\lambda\xi}\beta_{\mu\eta}\left(\sum_{\tau}\Big(\prod_{k=1}^{r}\mathrm{LR}^{\tau^{(k)}}_{\xi^{(k)}\eta^{(k)}}\Big)\Big(\sum_{\nu}\beta'_{\tau\nu}\widetilde{S}_\nu(\mathbf{X})\Big)\right) \\
&= \sum_{\nu}\left(\sum_{\xi,\eta,\tau}\beta_{\lambda\xi}\beta_{\mu\eta}\beta'_{\tau\nu}\prod_{k=1}^{r}\mathrm{LR}^{\tau^{(k)}}_{\xi^{(k)}\eta^{(k)}}\right)\widetilde{S}_\nu(\mathbf{X}).
\end{aligned}\tag{4.7.1}$$

This implies (ii).

By Lemma 2.6 and the fact that $\mathrm{LR}^{\nu^{(k)}}_{\xi^{(k)}\eta^{(k)}} = 0$ unless $|\nu^{(k)}| = |\xi^{(k)}| + |\eta^{(k)}|$, the equations (4.7.1) imply that

$$\begin{aligned}
&\widetilde{S}_\lambda(\mathbf{X})\widetilde{S}_\mu(\mathbf{X}) \\
&= \sum_{\substack{\nu\\ \zeta(\nu)=\zeta(\lambda+\mu)}}\Big(\prod_{k=1}^{r}\mathrm{LR}^{\nu^{(k)}}_{\lambda^{(k)}\mu^{(k)}}\Big)S_\nu(\mathbf{X}) + \sum_{\substack{\nu\\ \zeta(\nu)\prec\zeta(\lambda+\mu)}}\Big(\sum_{\xi,\eta}\beta_{\lambda\xi}\beta_{\mu\eta}\prod_{k=1}^{r}\mathrm{LR}^{\nu^{(k)}}_{\xi^{(k)}\eta^{(k)}}\Big)S_\nu(\mathbf{X}) \\
&= \sum_{\substack{\nu\\ \zeta(\nu)=\zeta(\lambda+\mu)}}\Big(\prod_{k=1}^{r}\mathrm{LR}^{\nu^{(k)}}_{\lambda^{(k)}\mu^{(k)}}\Big)\widetilde{S}_\nu(\mathbf{X}) + \sum_{\substack{\nu\\ \zeta(\nu)\prec\zeta(\lambda+\mu)}}a^{\nu}_{\lambda\mu}\widetilde{S}_\nu(\mathbf{X}) \qquad (a^{\nu}_{\lambda\mu}\in\mathbb{Z}).
\end{aligned}$$

This implies (iii).

Finally, we prove (iv). By Proposition 4.5, we have

$$\begin{aligned}
\widetilde{S}_\lambda(\mathbf{X})\widetilde{S}_\mu(\mathbf{X}) &= S_{\lambda^{(t)}}(X^{(t)}\cup\cdots\cup X^{(r)})S_{\mu^{(t)}}(X^{(t)}\cup\cdots\cup X^{(r)}) \\
&= \sum_{\nu^{(t)}}\mathrm{LR}^{\nu^{(t)}}_{\lambda^{(t)}\mu^{(t)}}\ S_{\nu^{(t)}}(X^{(t)}\cup\cdot\cup X^{(r)}) \\
&= \sum_{\nu^{(t)}}\mathrm{LR}^{\nu^{(t)}}_{\lambda^{(t)}\mu^{(t)}}\ \widetilde{S}_{(\emptyset,\cdots,\emptyset,\nu^{(t)},\emptyset,\cdots,\emptyset)}(\mathbf{X}).
\end{aligned}$$

This implies (iv). □

4.8. We have some conjectures for the number $c^{\nu}_{\lambda\mu}$ as follows.

Conjecture 1: For $\lambda,\mu,\nu\in\Lambda^{+}_{\geq 0,r}$, the number $c^{\nu}_{\lambda\mu}$ is a non-negative integer. More strongly, we conjecture the following.

Conjecture 2: $c^{\nu}_{\lambda\mu} = \prod_{k=1}^{r}\mathrm{LR}^{\nu^{(k)}}_{\lambda^{(k)}\mu^{(k)}}$.

Note that $\mathrm{LR}^{\nu^{(k)}}_{\lambda^{(k)}\mu^{(k)}} = 0$ if $|\nu^{(k)}| \neq |\lambda^{(k)}|+|\mu^{(k)}|$, then Conjecture 2 is equivalent to $c^{\nu}_{\lambda\mu} = 0$ unless $\zeta(\nu) = \zeta(\lambda+\mu)$ by Proposition 4.7 (iii).

We remark that Conjecture 2 is true for $\lambda,\mu\in\Lambda^{+}_{\geq 0,r}$ such that $\lambda^{(l)} = \emptyset$ and $\mu^{(l)} = \emptyset$ unless $l = t$ for some t by Proposition 4.7 (iv).

5. Decomposition matrices of cyclotomic q-Schur algebras

In this section, we consider the specialized cyclotomic q-Schur algebra ${}_F\mathscr{S}_{n,r}$ over a field F with parameters $q, Q_1, \cdots, Q_r \in F$ such that $q \neq 0$. We also denote $F \otimes_{\mathcal{A}} {}_{\mathcal{A}}U_q(\mathfrak{g})$ by ${}_F U_q(\mathfrak{g})$ simply. As declared in the beginning of section 2, throughout this and next section, we assume the condition (2.0.1) for $\mathbf{m}$.

5.1. For an ${}_F\mathscr{S}_{n,r}$-module M, we regard M as a ${}_F U_q(\mathfrak{g})$-module through the homomorphism $\Phi_{\mathfrak{g}}$. Then, by Lemma 2.2 (ii), we see that a simple ${}_F U_q(\mathfrak{g})$-module appearing in the composition series of M is of the form $L(\lambda^{(1)}) \boxtimes \cdots \boxtimes L(\lambda^{(r)})$ ($\lambda \in \Lambda^+_{n,r}(\mathbf{m})$), where $L(\lambda^{(k)})$ is the simple ${}_F U_q(\mathfrak{gl}_{m_k})$-module with highest weight $\lambda^{(k)}$.

For a simple ${}_F\mathscr{S}_{n,r}$-module $L(\lambda)$ ($\lambda \in \Lambda^+_{n,r}(\mathbf{m})$), let

$$x_{\lambda\mu} = [L(\lambda) : L(\mu^{(1)}) \boxtimes \cdots \boxtimes L(\mu^{(r)})]_{{}_F U_q(\mathfrak{g})}$$

be the multiplicity of $L(\mu^{(1)}) \boxtimes \cdots \boxtimes L(\mu^{(r)})$ ($\mu \in \Lambda^+_{n,r}(\mathbf{m})$) in the composition series of $L(\lambda)$ as ${}_F U_q(\mathfrak{g})$-modules through $\Phi_{\mathfrak{g}}$. Then we have the following lemma.

Lemma 5.2.

(i) For $\lambda \in \Lambda^+_{n,r}(\mathbf{m})$, $x_{\lambda\lambda} = 1$.
(ii) For $\lambda, \mu \in \Lambda^+_{n,r}(\mathbf{m})$, if $x_{\lambda\mu} \neq 0$, we have $\lambda \geq \mu$.
(iii) For $\lambda, \mu \in \Lambda^+_{n,r}(\mathbf{m})$, if $\lambda \neq \mu$ and $\zeta(\lambda) = \zeta(\mu)$, we have $x_{\lambda\mu} = 0$.

Proof. By the definition of Weyl modules (see 1.7), we have $W(\lambda) = {}_F\mathscr{S}^-_{n,r} \cdot v_\lambda$, and $L(\lambda)$ is the unique simple top $W(\lambda)/\operatorname{rad} W(\lambda)$ of $W(\lambda)$. Thus, by investigating the weights in $L(\lambda)$, we have (i) and (ii).

We prove (iii). We denote by $\overline{v}_\lambda$ the image of v_λ under the natural surjection $W(\lambda) \to L(\lambda)$. Then, we have $L(\lambda) = {}_F\mathscr{S}^-_{n,r} \cdot \overline{v}_\lambda$. One sees that

$$M(\lambda) = \bigoplus_{\substack{\mu \in \Lambda_{n,r}(\mathbf{m}) \\ \zeta(\lambda) \succneqq \zeta(\mu)}} L(\lambda)_\mu$$

is a ${}_F U_q(\mathfrak{g})$-submodule of $L(\lambda)$ since $\zeta(\mu \pm \alpha_{(i,k)}) = \zeta(\mu)$ for any $(i,k) \in \Gamma'_{\mathfrak{g}}(\mathbf{m})$. It is clear that $M(\lambda)$ is also an ${}_F\mathscr{S}^-_{n,r}$-submodule of $L(\lambda)$, and $L(\lambda)/M(\lambda) = {}_F\mathscr{S}^-_{n,r} \cdot (\overline{v}_\lambda + M(\lambda))$. For $F_{(i_1,k_1)} F_{(i_2,k_2)} \cdots F_{(i_l,k_l)} \in {}_F\mathscr{S}^-_{n,r}$, if $i_j = m_{k_j}$ for some j, one sees that $F_{(i_1,k_1)} \cdots F_{(i_l,k_l)} \cdot \overline{v}_\lambda \in M(\lambda)$. This implies that $L(\lambda)/M(\lambda)$ is generated by $\overline{v}_\lambda + M(\lambda)$ as a ${}_F U_q(\mathfrak{g})$-module, namely we have $L(\lambda)/M(\lambda) = {}_F U_q(\mathfrak{g}) \cdot (\overline{v}_\lambda + M(\lambda))$. Hence, we have the surjective homomorphism of ${}_F U_q(\mathfrak{g})$-modules

$$\psi : L(\lambda)/M(\lambda) \to L(\lambda^{(1)}) \boxtimes \cdots \boxtimes L(\lambda^{(r)})$$

such that $\overline{v}_\lambda + M(\lambda) \mapsto \overline{v}_{\lambda^{(1)}} \boxtimes \cdots \boxtimes \overline{v}_{\lambda^{(r)}}$, where $\overline{v}_{\lambda^{(k)}}$ is a highest weight vector of $L(\lambda^{(k)})$ with the highest weight $\lambda^{(k)}$. We claim that ψ is an isomorphism.

If ψ is not an isomorphism, there exists an element $x \in L(\lambda)_\mu$ such that $\lambda \neq \mu \in \Lambda^+_{n,r}(\mathbf{m})$, $\zeta(\mu) = \zeta(\lambda)$ and $E_{(i,k)} \cdot x \in M(\lambda)$ for any $(i,k) \in \Gamma'_{\mathfrak{g}}(\mathbf{m})$, namely $x + M(\lambda) \in L(\lambda)/M(\lambda)$ is a highest weight vector of highest weight μ as a ${}_F U_q(\mathfrak{g})$-module. On the other hand, we have $E_{(m_k,k)} \cdot x = 0$ for $k = 1, \cdots, r-1$ since $\zeta(\mu + \alpha_{(m_k,k)}) \succ \zeta(\mu) = \zeta(\lambda)$. Thus, we have that $E_{(i,k)} \cdot x \in M(\lambda)$ for any $(i,k) \in \Gamma'(\mathbf{m})$. This implies that ${}_F\mathscr{S}_{n,r} \cdot x$ is a proper ${}_F\mathscr{S}_{n,r}$-submodule of $L(\lambda)$ which contradict to the irreducibility of $L(\lambda)$ as an ${}_F\mathscr{S}_{n,r}$-module. Hence, ψ is an isomorphism. Then, the isomorphism $L(\lambda)/M(\lambda) \cong L(\lambda^{(1)}) \boxtimes \cdots \boxtimes L(\lambda^{(r)})$ together with the definition of $M(\lambda)$ implies (iii). $\square$

5.3. For an algebra $\mathscr{A}$, let $\mathscr{A}$-mod be the category of finitely generated $\mathscr{A}$-modules, and $K_0(\mathscr{A}\text{-mod})$ be the Grothendieck group of $\mathscr{A}$-mod. For $M \in \mathscr{A}$-mod, we denote by $[M]$ the image of M in $K_0(\mathscr{A}\text{-mod})$.

5.4. For $\lambda, \mu \in \varLambda^+_{n,r}(\mathbf{m})$, let

$$d_{\lambda\mu} = [W(\lambda) : L(\mu)]_{{}_F\mathscr{S}_{n,r}}$$

be the multiplicity of $L(\mu)$ in the composition series of $W(\lambda)$ as ${}_F\mathscr{S}_{n,r}$-modules, and

$$\overline{d}_{\lambda\mu} = [W(\lambda^{(1)}) \boxtimes \cdots \boxtimes W(\lambda^{(r)}) : L(\mu^{(1)}) \boxtimes \cdots \boxtimes L(\mu^{(r)})]_{{}_F U_q(\mathfrak{g})}$$

be the multiplicity of $L(\mu^{(1)}) \boxtimes \cdots \boxtimes L(\mu^{(r)})$ in the composition series of $W(\lambda^{(1)}) \boxtimes \cdots \boxtimes W(\lambda^{(r)})$ as ${}_F U_q(\mathfrak{g})$-modules. Put

$$D = \big(d_{\lambda\mu}\big)_{\lambda,\mu \in \varLambda^+_{n,r}(\mathbf{m})}, \qquad \overline{D} = \big(\overline{d}_{\lambda\mu}\big)_{\lambda,\mu \in \varLambda^+_{n,r}(\mathbf{m})},$$
$$X = \big(x_{\lambda\mu}\big)_{\lambda,\mu \in \varLambda^+_{n,r}(\mathbf{m})}, \qquad B = \big(\beta_{\lambda\mu}\big)_{\lambda,\mu \in \varLambda^+_{n,r}(\mathbf{m})}.$$

Then the decomposition matrix D of ${}_F\mathscr{S}_{n,r}$ is factorized as follows.

Theorem 5.5. We have that $B \cdot \overline{D} = D \cdot X$.

Proof. By the definitions, for $\lambda \in \varLambda^+_{n,r}(\mathbf{m})$, we have

$$\begin{aligned}
[W(\lambda)] &= \sum_{\mu \in \varLambda^+_{n,r}(\mathbf{m})} d_{\lambda\mu}[L(\mu)] \\
&= \sum_{\mu \in \varLambda^+_{n,r}(\mathbf{m})} d_{\lambda\mu}\Big(\sum_{\nu \in \varLambda^+_{n,.r}(\mathbf{m})} x_{\mu\nu}[L(\nu^{(1)}) \boxtimes \cdots \boxtimes L(\nu^{(r)})]\Big) \\
&= \sum_{\nu \in \varLambda^+_{n,r}} \Big(\sum_{\mu \in \varLambda^+_{n,r}(\mathbf{m})} d_{\lambda\mu}x_{\mu\nu}\Big)[L(\nu^{(1)}) \boxtimes \cdots \boxtimes L(\nu^{(r)})]
\end{aligned}$$

in $\mathcal{K}_0({}_F U_q(\mathfrak{g})\text{-mod})$. On the other hand, by taking a suitable modular system for ${}_F\mathscr{S}_{n,r}$, we have

$$\begin{aligned}
[W(\lambda)] &= \sum_{\mu \in \varLambda^+_{n,r}(\mathbf{m})} \beta_{\lambda\mu}[W(\mu^{(1)}) \boxtimes \cdots \boxtimes W(\mu^{(r)})] \\
&= \sum_{\mu \in \varLambda^+_{n,r}(\mathbf{m})} \beta_{\lambda\mu}\Big(\sum_{\nu \in \varLambda^+_{n,r}(\mathbf{m})} \overline{d}_{\mu\nu}[L(\nu^{(1)}) \boxtimes \cdots \boxtimes L(\nu^{(r)})]\Big) \\
&= \sum_{\nu \in \varLambda^+_{n,r}(\mathbf{m})} \Big(\sum_{\mu \in \varLambda^+_{n,r}(\mathbf{m})} \beta_{\lambda\mu}\overline{d}_{\mu\nu}\Big)[L(\nu^{(1)}) \boxtimes \cdots \boxtimes L(\nu^{(r)})]
\end{aligned}$$

in $K_0({}_F U_q(\mathfrak{g})\text{-mod})$. By comparing the coefficients of $[L(\nu^{(1)}) \boxtimes \cdots \boxtimes L(\nu^{(r)})]$, we obtain the claim of the theorem. $\square$

As a corollary of Theorem 5.5, we have the following formula. This formula has already known as the product formula for decomposition numbers of ${}_F\mathscr{S}_{n,r}$ studied in [**Saw**] by another method.

Corollary 5.6. For $\lambda, \mu \in \Lambda^+_{n,r}(\mathbf{m})$ such that $\zeta(\lambda) = \zeta(\mu)$, we have

$$d_{\lambda\mu} = \overline{d}_{\lambda\mu} = \prod_{k=1}^{r} d_{\lambda^{(k)}\mu^{(k)}},$$

where $d_{\lambda^{(k)}\mu^{(k)}} = [W(\lambda^{(k)}) : L(\mu^{(k)})]$ is the decomposition number of ${}_F U_q(\mathfrak{gl}_{m_k})$-modules.

Proof. By Lemma 2.6 (ii), for $\lambda, \mu, \nu \in \Lambda^+_{n,r}(\mathbf{m})$, if $\beta_{\lambda\nu}\overline{d}_{\nu\mu} \neq 0$, then we have $\lambda \geq \nu \geq \mu$. Thus, if $\zeta(\lambda) = \zeta(\mu)$, we have

$$\sum_{\nu \in \Lambda^+_{n,r}(\mathbf{m})} \beta_{\lambda\nu}\overline{d}_{\nu\mu} = \sum_{\substack{\nu \in \Lambda^+_{n,r}(\mathbf{m}) \\ \zeta(\lambda)=\zeta(\nu)=\zeta(\mu)}} \beta_{\lambda\nu}\overline{d}_{\nu\mu} = \overline{d}_{\lambda\mu},$$

where the last equation follows from Lemma 2.6 (i) and (iii). Similarly, by using Lemma 5.2, we see that $\sum_{\nu \in \Lambda^+_{n,r}(\mathbf{m})} d_{\lambda\nu} x_{\nu\mu} = d_{\lambda\mu}$. Hence, Theorem 5.5 implies the claim of the corollary. $\square$

Remark 5.7. In [**SW**], we also obtained the product formulae for decomposition numbers of ${}_F\mathscr{S}_{n,r}$ which are natural generalization of one in [**Saw**] as follows. Take $\mathbf{p} = (r_1, \cdots, r_g) \in \mathbb{Z}^g_{>0}$ such that $r_1 + \cdots + r_g = r$ as in 3.5. Then, for $\lambda, \mu \in \Lambda^+_{n,r}(\mathbf{m})$ such that $\zeta^{\mathbf{p}}(\lambda) = \zeta^{\mathbf{p}}(\mu)$, we have

$$d_{\lambda\mu} = \prod_{k=1}^{g} d_{\lambda^{[k]_{\mathbf{p}}}\mu^{[k]_{\mathbf{p}}}} \tag{5.7.1}$$

by [**SW**, Theorem 4.17], where $d_{\lambda^{[k]_{\mathbf{p}}}\mu^{[k]_{\mathbf{p}}}}$ is the decomposition number of ${}_F\mathscr{S}_{n_k,r_k}$ ($n_k = |\lambda^{[k]_{\mathbf{p}}}|$) with parameters $q, Q_{p_k+1}, \cdots, Q_{p_k+r_k}$. However, the formula (5.7.1) for general $\mathbf{p}$ ($\neq (1, \cdots, 1)$) is not obtained in a similar way as in Corollary 5.6 since ${}_F\overline{\mathscr{S}}^{\mathbf{p}}_{n,r}$ does not realize as a subalgebra of ${}_F\mathscr{S}_{n,r}$ in a similar way as in Lemma 2.2, where ${}_F\overline{\mathscr{S}}^{\mathbf{p}}_{n,r}$ is a subquotient algebra of ${}_F\mathscr{S}_{n,r}$ defined in [**SW**, 2.12]. (Note that ${}_F\overline{\mathscr{S}}^{\mathbf{p}}_{n,r} \cong \bigoplus_{\substack{(n_1,\cdots,n_g) \\ n_1+\cdots+n_g=n}} {}_F\mathscr{S}_{n_1,r_1} \otimes \cdots \otimes {}_F\mathscr{S}_{n_g,r_g}$ by [**SW**, Theorem 4.15]. Thus, if $\mathbf{p} = (1, \cdots, 1)$, ${}_F\overline{\mathscr{S}}^{\mathbf{p}}_{n,r}$ coincides with the right-hand side of the isomorphism in Lemma 2.2.) Hence, in order to obtain the formula (5.7.1) for general $\mathbf{p}$, it is essential to take the subquotient algebra ${}_F\overline{\mathscr{S}}^{\mathbf{p}}_{n,r}$ as in [**SW**].

For special parameters, we see that the matrix X becomes the identity matrix as the following corollary.

Corollary 5.8.

(i) If $Q_1 = Q_2 = \cdots = Q_r = 0$, the matrix X is the identity matrix. In particular, we have $D = B \cdot \overline{D}$.

(ii) If $q = 1$, $Q_1 = Q_2 = \cdots = Q_r$ (not necessary to be 0), the matrix X is the identity matrix. Moreover, we have $D = B$ if $\operatorname{char} F = 0$.

Proof. Assume that $Q_1 = Q_2 = \cdots = Q_r = 0$. We denote by $E^{(c)}_{(i,k)} = 1 \otimes E^c_{(i,k)}/[c]!$ (resp. $F^{(c)}_{(i,k)} = 1 \otimes F^c_{(i,k)}/[c]!$) $\in F \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathscr{S}_{n,r} \cong {}_F\mathscr{S}_{n,r}$. By the triangular decomposition of ${}_F\mathscr{S}_{n,r}$, we have

$$\sigma^{\lambda}_{(i,k)} = \sum r^{(i'_1,k'_1,c'_1),\cdots,(i'_{l'},k'_{l'},c'_{l'})}_{(i_1,k_1,c_1),\cdots,(i_l,k_l,c_l)} F^{(c'_1)}_{(i'_1,k'_1)} \cdots F^{(c'_{l'})}_{(i'_{l'},k'_{l'})} E^{(c_1)}_{(i_1,k_1)} \cdots E^{(c_l)}_{(i_l,k_l)} 1_\lambda,$$

for some $r^{(i'_1,k'_1,c'_1),\cdots,(i'_{l'},k'_{l'},c'_{l'})}_{(i_1,k_1,c_1),\cdots,(i_l,k_l,c_l)} \in F$. First, we prove the following claim.

Claim A : If

$$r^{(i'_1,k'_1,c'_1),\cdots,(i'_{l'},k'_{l'},c'_{l'})}_{(i_1,k_1,c_1),\cdots,(i_l,k_l,c_l)} \neq 0 \text{ and } F^{(c'_1)}_{(i'_1,k'_1)} \cdots F^{(c'_{l'})}_{(i'_{l'},k'_{l'})} E^{(c_1)}_{(i_1,k_1)} \cdots E^{(c_l)}_{(i_l,k_l)} 1_\lambda \neq 0,$$

then we have $\zeta(\lambda + c_1\alpha_{(i_1,k_1)} + \cdots + c_l\alpha_{(i_l,k_l)}) \succ \zeta(\lambda)$.

Note that $Q_1 = Q_2 = \cdots = Q_r = 0$, we see easily that ${}_F\mathscr{H}_{n,r}$ is a $\mathbb{Z}/r\mathbb{Z}$-graded algebra with $\deg(T_0) = \overline{1}$ and $\deg(T_i) = \overline{0}$, where we put $\overline{k} = k + r\mathbb{Z} \in \mathbb{Z}/r\mathbb{Z}$ for $k \in \mathbb{Z}$. We can also check that m_λ ($\lambda \in \Lambda_{n,r}(\mathbf{m})$) is a homogeneous element of ${}_F\mathscr{H}_{n,r}$. Since $\sigma^\lambda_{(i,k)}(m_\lambda) = m_\lambda \cdot (L_{N+1} + \cdots + L_{N+\lambda^{(k)}_i})$ ($N = \sum_{l=1}^{k-1} |\lambda^{(l)}| + \sum_{j=1}^{i-1} \lambda^{(k)}_j$), we have $\sigma^\lambda_{(i,k)}(m_\lambda)$ is homogeneous and $\deg(\sigma^\lambda_{(i,k)}(m_\lambda)) = \deg(m_\lambda) + \overline{1}$. On the other hand, by [**W**, Lemma 6.10], we see that $F^{(c'_1)}_{(i'_1,k'_1)} \cdots F^{(c'_{l'})}_{(i'_{l'},k'_{l'})} E^{(c_1)}_{(i_1,k_1)} \cdots E^{(c_l)}_{(i_l,k_l)} 1_\lambda(m_\lambda)$ is a homogeneous element of ${}_F\mathscr{H}_{n,r}$ with degree $\deg(m_\lambda)$ if $i_j \neq m_{k_j}$ for any $j = 1, \cdots, l$. (Note that $i_j \neq m_{k_j}$ for any $j = 1, \cdots, l$ if and only if $i'_{j'} \neq m_{k'_{j'}}$ for any $j' = 1, \cdots, l'$ since $\sigma^\lambda_{(i,k)} = 1_\lambda \sigma^\lambda_{(i,k)} 1_\lambda$ from the definitions.) Thus, if $r^{(i'_1,k'_1,c'_1),\cdots,(i'_{l'},k'_{l'},c'_{l'})}_{(i_1,k_1,c_1),\cdots,(i_l,k_l,c_l)} \neq 0$ and $F^{(c'_1)}_{(i'_1,k'_1)} \cdots F^{(c'_{l'})}_{(i'_{l'},k'_{l'})} E^{(c_1)}_{(i_1,k_1)} \cdots E^{(c_l)}_{(i_l,k_l)} 1_\lambda \neq 0$, then there exists j such that $i_j = m_{k_j}$, and this implies that

$$\zeta(\lambda + c_1\alpha_{(i_1,k_1)} + \cdots + c_l\alpha_{(i_l,k_l)}) \succ \zeta(\lambda).$$

Now, we proved Claim A.

We have already shown that $x_{\lambda\lambda} = 1$, and $x_{\lambda\mu} = 0$ for $\lambda \neq \mu$ such that $\zeta(\lambda) = \zeta(\mu)$ in Lemma 5.2. Thus, it is enough to show that $x_{\lambda\mu} = 0$ for $\lambda, \mu \in \Lambda^+_{n,r}(\mathbf{m})$ such that $\zeta(\lambda) \neq \zeta(\mu)$.

Suppose that $x_{\lambda\mu} \neq 0$ for some $\lambda, \mu \in \Lambda^+_{n,r}(\mathbf{m})$ such that $\zeta(\lambda) \neq \zeta(\mu)$. We recall that $L(\lambda) = \mathscr{S}^-_{n,r} \cdot \overline{v}_\lambda$, where $\overline{v}_\lambda = v_\lambda + \operatorname{rad} W(\lambda) \in W(\lambda)/\operatorname{rad} W(\lambda) \cong L(\lambda)$. Then, it is clear that $L(\lambda)_\mu \neq 0$. This implies the existence of a non-zero element

$$v' = \sum r_{(i_1,k_1),\cdots,(i_c,k_c)} F_{(m_{k'},k')} F_{(i_1,k_1)} \cdots F_{(i_c,k_c)} \cdot \overline{v}_\lambda \in L(\lambda) \quad (r_{(i_1,k_1),\cdots,(i_c,k_c)} \in F)$$

such that $E_{(i,k)} \cdot v' = 0$ for any $(i,k) \in \Gamma'_{\mathfrak{g}}(\mathbf{m})$, where the summation runs

$$\{((i_1,k_1), \cdots, (i_c,k_c)) \in (\Gamma'_{\mathfrak{g}}(\mathbf{m}))^c \mid \alpha_{(i_1,k_1)} + \cdots + \alpha_{(i_c,k_c)} = \alpha\}$$

for some $\alpha \in \bigoplus_{(i,k)\in\Gamma'_{\mathfrak{g}}(\mathbf{m})} \mathbb{Z}\alpha_{(i,k)}$. Namely v' is a ${}_FU_q(\mathfrak{g})$-highest weight vector of highest weight $\mu' = \lambda - \alpha - \alpha_{(m_{k'},k')}$. It is clear that $\zeta(\lambda) = \zeta(\lambda - \alpha)$. Since $E_{(m_k,k)}$ ($k \neq k'$) commute with $F_{(m_{k'},k')}$ and $F_{(j,l)}$ ($(j,l) \in \Gamma'_{\mathfrak{g}}(\mathbf{m})$), we have that $E_{(m_k,k)} \cdot v' = 0$ for any $k \in \{1, \cdots, r-1\} \setminus \{k'\}$. On the other hand, for $((i_1,k_1), \cdots, (i_c,k_c)) \in (\Gamma'_{\mathfrak{g}}(\mathbf{m}))^c$ such that $\alpha_{(i_1,k_1)} + \cdots + \alpha_{(i_c,k_c)} = \alpha$, we have

(5.8.1)

$$\begin{aligned}
&E_{(m_{k'},k')} F_{(m_{k'},k')} F_{(i_1,k_1)} \cdots F_{(i_c,k_c)} \cdot \overline{v}_\lambda \\
&= \Big\{ F_{(m_{k'},k')} E_{(m_{k'},k')} + \Big(q^{(\lambda-\alpha)^{(k')}_{m_{k'}} - (\lambda-\alpha)^{(k'+1)}_1} (q^{-1}\sigma^{\lambda-\alpha}_{(m_{k'},k')} - q\sigma^{\lambda-\alpha}_{(1,k'+1)}) 1_{\lambda-\alpha} \Big) \Big\} \cdot \\
&\quad 1_{\lambda-\alpha} F_{(i_1,k_1)} \cdots F_{(i_c,k_c)} \cdot \overline{v}_\lambda.
\end{aligned}$$

Note that $\zeta(\lambda - \alpha) = \zeta(\lambda)$, (5.8.1) together with Claim A implies that

$$E_{(m_{k'},k')} F_{(m_{k'},k')} F_{(i_1,k_1)} \cdots F_{(i_c,k_c)} \cdot \overline{v}_\lambda = 0.$$

Thus, we have

$$E_{(m_{k'},k')} \cdot v' = \sum r_{(i_1,k_1),\cdots,(i_c,k_c)} E_{(m_{k'},k')} F_{(m_{k'},k')} F_{(i_1,k_1)} \cdots F_{(i_c,k_c)} \cdot \overline{v}_\lambda = 0.$$

As a consequence, we have that $E_{(i,k)} \cdot v' = 0$ for any $(i,k) \in \Gamma'(\mathbf{m})$, and this implies that ${}_F\mathscr{S}_{n,r} \cdot v'$ is a proper ${}_F\mathscr{S}_{n,r}$-submodule of $L(\lambda)$. However, this contradicts to the irreducibility of $L(\lambda)$ as ${}_F\mathscr{S}_{n,r}$-module. Thus, we have that $x_{\lambda\mu} = 0$ for $\lambda, \mu \in \Lambda^+_{n,r}(\mathbf{m})$ such that $\zeta(\lambda) \neq \zeta(\mu)$. Now we proved (i).

Next we prove (ii). Let ${}_F\mathscr{H}_{n,r}$ (resp. ${}_F\mathscr{H}'_{n,r}$) be the Ariki-Koike algebra over F with parameters $q = 1, Q_1 = \cdots = Q_r = 0$ (resp. $q = 1, Q'_1 = \cdots = Q'_r = Q' \neq 0$), and ${}_F\mathscr{S}_{n,r}$ (resp. ${}_F\mathscr{S}'_{n,r}$) be the cyclotomic q-Schur algebra associated to ${}_F\mathscr{H}_{n,r}$ (resp. ${}_F\mathscr{H}'_{n,r}$). We denote by $T_0, T_1, \cdots, T_{n-1}$ (resp. $T'_0, T'_1, \cdots, T'_{n-1}$) the generators of ${}_F\mathscr{H}_{n,r}$ (resp. ${}_F\mathscr{H}'_{n,r}$) as in 1.1. Then we can check that there exists an isomorphism $\phi : {}_F\mathscr{H}_{n,r} \to {}_F\mathscr{H}'_{n,r}$ such that $\phi(T_0) = T'_0 - Q'$ and $\phi(T_i) = T'_i$ $(1 \leq i \leq n-1)$. We can also check that $M^\mu \cong M'^\mu$ for $\mu \in \Lambda_{n,r}(\mathbf{m})$ under the isomorphism ϕ, where M^μ (resp. M'^μ) is the right ${}_F\mathscr{H}_{n,r}$-module (resp. ${}_F\mathscr{H}'_{n,r}$-module) defined in 1.3. Thus, we have ${}_F\mathscr{S}_{n,r} \cong {}_F\mathscr{S}'_{n,r}$ as algebras. Then (i) implies (ii) since $\overline{D}$ is the identity matrix when $q = 1$ if $\operatorname{char} F = 0$. □

Remark 5.9. (i) In Theorem 5.5, the matrix $B \cdot \overline{D}$ does not depend on the choice of parameters $Q_1, \cdots, Q_r$.

(ii) If ${}_F\mathscr{S}_{n,r}$ is semi-simple, both of D and $\overline{D}$ are identity matrices. Thus, we have $B = X$.

(iii) By Theorem 5.5, for $\lambda, \mu \in \Lambda^+_{n,r}$, we have

$$d_{\lambda\mu} + x_{\lambda\mu} = \sum_{\nu \in \Lambda^+_{n,r}} \beta_{\lambda\nu} \bar{d}_{\nu\mu} - \sum_{\substack{\nu \in \Lambda^+_{n,r} \\ \lambda > \nu > \mu}} d_{\lambda\nu} x_{\nu\mu}.$$

Thus, we see that the matrix $B \cdot \overline{D}$ gives an upper bound of both $d_{\lambda\mu}$ and $x_{\lambda\mu}$.

6. The Ariki-Koike algebra as a subalgebra of $\mathscr{S}_{n,r}$

In this section, we consider the algebras over an commutative ring R with parameters $q, Q_1, \cdots, Q_r \in R$ such that q is invertible in R.

6.1. For $\mu \in \Lambda_{n,r}(\mathbf{m})$, put

$$X^\mu_{\mu+\alpha_{(i,k)}} = \big\{1, s_{N+1}, s_{N+1}s_{N+2}, \cdots, s_{N+1}s_{N+2}\cdots s_{N+\mu^{(k)}_{i+1}-1}\big\},$$
$$X^\mu_{\mu-\alpha_{(i,k)}} = \big\{1, s_{N-1}, s_{N-1}s_{N-2}, \cdots, s_{N-1}s_{N-2}\cdots s_{N-\mu^{(k)}_{i}+1}\big\},$$

where $s_j = (j, j+1) \in \mathfrak{S}_n$ is the adjacent transposition, and $N = \sum_{l=1}^{k-1} |\mu^{(l)}| + \sum_{j=1}^{i} \mu^{(k)}_j$. Then, by [**W**, Lemma 6.10, Proposition 7.7 and Theorem 7.16 (i)], we have

$$1_\nu(m_\mu) = \delta_{\mu,\nu} m_\mu, \tag{6.1.1}$$

$$e_{(i,k)}(m_\mu) = q^{-\mu^{(k)}_{i+1}+1} m_{\mu+\alpha_{(i,k)}} \Big(\sum_{y \in X^\mu_{\mu+\alpha_{(i,k)}}} q^{\ell(y)} T_y \Big), \tag{6.1.2}$$

$$f_{(i,k)}(m_\mu) = q^{-\mu^{(k)}_{i}+1} m_{\mu-\alpha_{(i,k)}} h^\mu_{-(i,k)} \Big(\sum_{x \in X^\mu_{\mu-\alpha_{(i,k)}}} q^{\ell(x)} T_x \Big), \tag{6.1.3}$$

where $h^{\mu}_{-(i,k)} = \begin{cases} 1 & (i \neq m_k), \\ L_N - Q_{k+1} & (i = m_k)\ (N := |\mu^{(1)}| + \cdots + |\mu^{(k)}|). \end{cases}$

6.2. Put $\omega = (\emptyset, \cdots, \emptyset, (1^n)) \in \Lambda^+_{n,r}(\mathbf{m})$. Then, it is clear that $M^{\omega} \cong {}_R\mathscr{H}_{n,r}$ as right ${}_R\mathscr{H}_{n,r}$-modules, and that $1_{\omega}\, {}_R\mathscr{S}_{n,r} 1_{\omega} = \mathrm{End}_{{}_R\mathscr{H}_{n,r}}(M^{\omega}, M^{\omega}) \cong {}_R\mathscr{H}_{n,r}$ as R-algebras. Put $C_0 = 1_{\omega} f_{(m_{r-1}, r-1)} e_{(m_{r-1}, r-1)} 1_{\omega}$, $C_i = 1_{\omega} f_{(i,r)} e_{(i,r)} 1_{\omega} \in {}_R\mathscr{S}_{n,r}$ for $i = 1, \cdots, n-1$. Then, we can realize ${}_R\mathscr{H}_{n,r}$ as a subalgebra of ${}_R\mathscr{S}_{n,r}$ as the following proposition.

Proposition 6.3.

(i) The subalgebra of ${}_R\mathscr{S}_{n,r}$ generated by $C_0, C_1, \cdots, C_{n-1}$ is isomorphic to the Ariki-Koike algebra ${}_R\mathscr{H}_{n,r}$. Moreover, the subalgebra of ${}_R\mathscr{S}_{n,r}$ generated by $C_1, \cdots, C_{n-1}$ is isomorphic to the Iwahori-Hecke algebra ${}_R\mathscr{H}_n$ of the symmetric group $\mathfrak{S}_n$.

(ii) Under the isomorphism $1_{\omega}\, {}_R\mathscr{S}_{n,r} 1_{\omega} \cong {}_R\mathscr{H}_{n,r}$, we have $T_0 = C_0 + Q_r 1_{\omega}$, $T_i = C_i - q^{-1} 1_{\omega}$.

Proof. It is clear that $C_0, C_1, \cdots, C_{n-1}$ are elements of $1_{\omega}\, {}_R\mathscr{S}_{n,r} 1_{\omega}$. We remark that the isomorphism $\mathrm{End}_{{}_R\mathscr{H}_{n,r}}(M^{\omega}, M^{\omega}) \cong {}_R\mathscr{H}_{n,r}$ is given by $\varphi \mapsto \varphi(m_{\omega})$ (note that $m_{\omega} = 1$). Moreover, by (6.1.1) - (6.1.3), we have

$$\begin{aligned} C_0(m_{\omega}) &= 1_{\omega} f_{(m_{r-1}, r-1)} e_{(m_{r-1}, r-1)} 1_{\omega}(m_{\omega}) \\ &= m_{\omega}(L_1 - Q_r). \end{aligned}$$

Since $m_{\omega} = 1$ and $L_1 = T_0$, we have $C_0(m_{\omega}) = T_0 - Q_r$. Similarly, we have $C_i(m_{\omega}) = T_i + q^{-1}$ for $i = 1, \cdots, n-1$. Thus, ${}_R\mathscr{H}_{n,r}$ is generated by $C_0, C_1, \cdots, C_{n-1}$ under the isomorphism $1_{\omega}\, {}_R\mathscr{S}_{n,r} 1_{\omega} \cong {}_R\mathscr{H}_{n,r}$, and ${}_R\mathscr{H}_n$ is generated by $C_1, \cdots, C_{n-1}$. Now, (ii) is clear. □

6.4. Let $\mathcal{F} = \mathrm{Hom}_{{}_R\mathscr{S}_{n,r}}({}_R\mathscr{S}_{n,r} 1_{\omega}, -) : {}_R\mathscr{S}_{n,r}\text{-mod} \to {}_R\mathscr{H}_{n,r}\text{-mod}$ be the Schur functor. Then, for $M \in {}_R\mathscr{S}_{n,r}$-mod, we have that $\mathcal{F}(M) = 1_{\omega} M$ under the isomorphism $1_{\omega}\, {}_R\mathscr{S}_{n,r} 1_{\omega} \cong {}_R\mathscr{H}_{n,r}$. It is known that $\{1_{\omega} L(\lambda) \neq 0 \mid \lambda \in \Lambda^+_{n,r}(\mathbf{m})\}$ gives a complete set of non-isomorphic simple ${}_R\mathscr{H}_{n,r}$-modules when R is a field.

Let e be the smallest positive integer such that $1 + (q^2) + (q^2)^2 + \cdots (q^2)^{e-1} = 0$. We say that a partition (not multi-partition) $\lambda = (\lambda_1, \lambda_2, \cdots)$ is e-restricted if $\lambda_i - \lambda_{i+1} < e$ for any $i \geq 1$.

As a corollary of Corollary 5.8, we have the following classification of simple ${}_R\mathscr{H}_{n,r}$-modules for some special parameters. We remark that this classification has already proved by [**AM**, Theorem 1.6] and [**M1**, Theorem 3.7] by the other methods.

Corollary 6.5. Assume that R is a field. If $Q_1 = Q_2 = \cdots = Q_r = 0$ or $q = 1$, $Q_1 = Q_2 = \cdots = Q_r$, then $1_{\omega} L(\lambda) \neq 0$ if and only if $\lambda^{(k)} = \emptyset$ for $k < r$ and $\lambda^{(r)}$ is an e-restricted partition.

Proof. By Corollary 5.8, we have that $1_{\mu} L(\lambda) \neq 0$ only if $\zeta(\mu) = \zeta(\lambda)$. In particular, we have that $\lambda^{(k)} = 0$ for any $k < r$ if $1_{\omega} L(\lambda) \neq 0$. On the other hand, $L(\lambda) \cong L(\lambda^{(1)}) \boxtimes \cdots \boxtimes L(\lambda^{(r)})$ as ${}_R U_q(\mathfrak{g})$-modules by Corollary 5.8. In particular, when $\lambda^{(k)} = \emptyset$ for any $k < r$, we have that $L(\lambda) \cong L(\lambda^{(r)})$ as ${}_R U_q(\mathfrak{gl}_{m_r})$-modules. Moreover, it is well known that $1_{\omega} L(\lambda^{(r)}) \neq 0$ if and only if $\lambda^{(r)}$ is an e-restricted partition ([**DJ**, Theorem 6.3, 6.8]). These results imply the corollary. □

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