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Collision Probability in an In-Line Equipment Model under Erlang Distribution*

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SUMMARY Flat Panel Displays (FPDs) are manufactured using many pieces of different processing equipment arranged sequentially in a line. Although the constant inter-arrival time (i.e., the tact time) of glass substrates in the line should be kept as short as possible, the collision probability between glass substrates increases as tact time decreases. Since the glass substrate is expensive and fragile, collisions should be avoided. In this paper, we derive a closed form formula of the approximate collision probability for a model, in which the processing time on each piece of equipment is assumed to follow Erlang distribution. We also compare some numerical results of the closed form and computer simulation results of the collision probability.

key words: *stochastic model, collision probability, Erlang distribution, closed form, approximation*

1. Introduction

Reflecting the increasing demand for Flat Panel Displays (FPDs) such as LCDs, plasma display panels, etc., more effective methods for their manufacture are required. The production rate improves with technological advancements such as the rapid enlargement of glass substrates and the miniaturization of patterns. Accordingly, production lines have to be modified to accommodate such advancements, and new optimization problems to be solved continue to arise. Lately, an advanced system called *Crystal Flow* [9] has been introduced in the production line of FPDs. It targets a higher level of line control in the next-generation production processes as well as in existing lines.

The main flow of the FPD process is shown in Fig. 1 [11]. Each piece of processing equipment in Fig. 1 has a specialized role, such as cleaning, coating, proximity exposing, developing, etching, resist stripping, etc., and those pieces of equipment are connected in-line. Most production lines adopt a simple strategy to feed glass substrate into the first piece of equipment at a constant inter-arrival time, which is called the *tact time*. This strategy is simple and enables us to accurately predict the number of products.

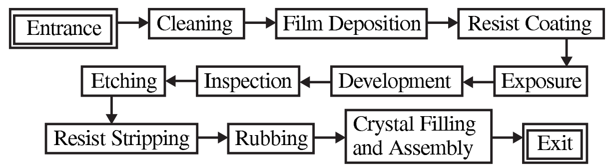

Fig. 1 FPD process flow.

Due to solution foaming, chemicals, heat treating, etc., the processing time at each piece of equipment is uncertain and may vary according to the conditions at that time. If a sheet of substrate is sent to a piece of equipment that is still processing a previously sent sheet of substrate, that newly sent sheet cannot be processed on the piece of equipment. This phenomenon is called a *collision* between substrates. Since the glass substrate is expensive and fragile, collisions should be avoided as much as possible.

A collision-like phenomenon is called a *blocking* in a flow shop model in the field of scheduling theory, and is studied as an important factor to determine line efficiency. If the processing time is deterministic, many study results on blocking exist (see extensive survey in [3]). If the processing time is stochastic, study results are somewhat limited in comparison with their deterministic counterparts. For example, see [7] and [8], where the purpose is to minimize the expected makespan. On the other hand, in queueing theory, depending on the rule for processing blockings (blocked calls cleared, blocked calls delayed, etc.), previous work mainly focused on performance measures in the steady state. We can find, in fact, that a wide range of literature in the field of queuing theory has been investigated, for example in [4], [5] and [6]. In contrast, in this paper, given the number of jobs to be processed in the prescribed tact time span, we focus on the measure recently presented in [1], which is the probability that there will be at least one collision, called the collision probability. In a comparatively new manufacturing system such as one manufacturing FPDs, the evaluation item (i.e., collision probability) is the new focus of observation.

The tact time (i.e., the inter-arrival time of substrates at the first piece of equipment) should be minimized in order to maximize the production rate. This, however, increases the collision probability. Thus there is a trade-off between

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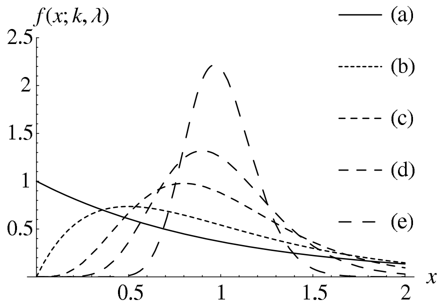

Fig. 2 The pdfs of the Erlang distribution for (a) $k = 1$, $\lambda = 1$ (exponential distribution), (b) $k = 2$, $\lambda = 2$, (c) $k = 5$, $\lambda = 5$, (d) $k = 10$, $\lambda = 10$, (e) $k = 30$, $\lambda = 30$.

the tact time and the collision probability. When considering this trade-off, it is important to evaluate the collision probability under a given tact time.

For the simple model discussed in queuing theory, the evaluation item (i.e., collision probability) was first presented in [1]. It was shown in [1] that the collision probability can be approximately expressed by a multiple integration, assuming that the processing time of each piece of equipment follows general distribution. Thus, the approximate collision probability can be obtained by numerical integration. However, if the processing time follows exponential distribution, a closed form formula of the approximate collision probability is derived without using any multiple integration. A computer simulation method for computing the collision probability was also presented in [1]. This method can compute the approximate probability for the processing time under general distribution.

A problem with exponential distribution is that it is not flexible enough to represent processing time in real production lines. Under exponential distribution, we cannot set the expectation and the variance independently of each other. Furthermore, the probability density function (pdf) of exponential distribution is monotone decreasing, as shown in Fig. 2 (a), while actual processing time is often represented by a bell-like curve such as Fig. 2 (e). The pdf of normal distribution is also bell-shaped, but it has a weakness in that the pdf takes positive values in the negative domain, which is not true in the pdf of actual processing time. To overcome this, we generalize in this paper exponential distribution to Erlang distribution, and show that the approximate collision probability can still be given as a closed form formula.

The pdf of Erlang distribution is defined as follows:

$$f(x; k, \lambda) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} \quad \text{for } x \geq 0, \tag{1}$$

where two parameters λ and k are a positive real number and a positive integer, respectively. Its expectation and variance are given by k/λ and k/λ^2, respectively. Therefore, under Erlang distribution, we can set the expectation and the variance independently of each other by setting parameters λ and k appropriately. In Fig. 2, five different probability density functions are plotted, where expectations of all cases are the same, but their variances decrease through cases (a) – (e). Some of these pdfs are bell-shaped, and the pdf of Erlang distribution takes a value of zero for $x < 0$. Thus, Erlang distribution is flexible enough to represent actual processing times.

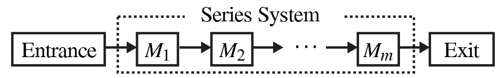

Fig. 3 Model comprising of m pieces of equipment.

2. Model

We describe a formal model of the FPD production line. The following notations will be used:

- $M_1, M_2, \ldots, M_m$: m different pieces of equipment in the line.
- $J_1, J_2, \ldots, J_n$: n jobs to be processed.
- $T_i^{(j)}$ (> 0): Processing time of job J_i on piece of equipment M_j.
- T_{tact} (> 0): Tact time, i.e., the time difference between the start time instants of J_i and J_{i+1} for all $1 \leq i \leq n-1$ at the entrance to the line.

The production model is illustrated in Fig. 3. With the same time interval, T_{tact}, jobs are successively fed into the line at the entrance. Every job is first processed on piece of equipment M_1. It is then automatically transported to the next piece of equipment M_2 after it has been finished on M_1. It is assumed, for simplicity, that the transportation time between pieces of equipment is nil. As soon as M_2 receives the job, it starts processing. In this manner, every job is processed on the pieces of equipment in the order $M_1, M_2, \ldots, M_m$, and then sent to the exit. Moreover, we assume that the processing time $T_i^{(j)}$ on M_j is a random variable that follows Erlang distribution with parameters λ_j and k_j, and all $T_i^{(j)}$ ($1 \leq i \leq n, 1 \leq j \leq m$) are independent of each other.

In the above model, a collision occurs if the next job arrives at M_j while M_j is still processing the current job. The following lemma on the collision condition between jobs was given in [1].

Lemma 1: Suppose that $T_i^{(j)} = t_i^{(j)}$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. For n (≥ 2) jobs, there is no collision in the above production line of m pieces of equipment if and only if

$$\sum_{j=1}^{l} t_i^{(j)} \leq T_{\text{tact}} + \sum_{j=1}^{l-1} t_{i+1}^{(j)}$$

holds for all $1 \leq i \leq n-1$ and $1 \leq l \leq m$.

Lemma 1 is important for the analysis contained in this paper, therefore we explain the physical meaning of the inequality in this lemma. Since a collision can only occur between two consecutive jobs, we pay attention solely to them. For each i and l, we compare the time when job J_i is completed on piece of equipment M_l, with the time when job

J_{i+1}, which is next fed into the system, is completed on the previous piece of equipment M_{l-1}. If the former time value is less than or equal to the latter, then no collision occurs. Otherwise, a collision occurs. Note that the transportation time is assumed to be nil.

3. Collision Probability for $m = 1$

In this section, we derive the collision probability when there is one piece of equipment. By Lemma 1 with $m = 1$, the no-collision probability between consecutive jobs J_i and J_{i+1} becomes as follows:

$$\begin{aligned}
&\Pr\left(T_i^{(1)} \le T_{\text{tact}} : 1 \le i \le n-1\right)\\
&= \left(\Pr\left(T_1^{(1)} \le T_{\text{tact}}\right)\right)^{n-1} \qquad \text{(since the } T_i^{(1)} \text{ are i.i.d.)}\\
&= \left(\int_0^{T_{\text{tact}}} \frac{\lambda_1^{k_1} x^{k_1-1} e^{-\lambda_1 x}}{(k_1-1)!} dx\right)^{n-1}\\
&= \left(1 - \sum_{h=0}^{k_1-1} \frac{(\lambda_1 T_{\text{tact}})^h}{h!} e^{-\lambda_1 T_{\text{tact}}}\right)^{n-1}.
\end{aligned}$$

Note that the last expression is given by iteratively performing integration by parts. Therefore, the collision probability for $m = 1$ is given by the closed form

$$1 - \left(1 - \sum_{h=0}^{k_1-1} \frac{(\lambda_1 T_{\text{tact}})^h}{h!} e^{-\lambda_1 T_{\text{tact}}}\right)^{n-1}. \tag{2}$$

4. Derivation of Collision Probability for $m = 2$

We derive an approximate collision probability for the case of $m = 2$ by considering only two consecutive jobs. The reason why we pay attention to two consecutive jobs is as follows: even if we consider n jobs, a collision is the phenomenon which occurs between only two consecutive jobs. Therefore, we first pay attention to only two consecutive jobs, and then we derive the no-collision probability between them. After that, considering n jobs, as the number of pairs of two consecutive jobs is $n-1$ (J_1 and J_2, J_2 and $J_3, \ldots, J_{n-1}$ and J_n), we approximate the no-collision probability over all n jobs using the $(n-1)$-th power of the above derived probability of two consecutive jobs.

For this, the following random variables are introduced. $X_1 = T_i^{(1)} - T_{\text{tact}}$, $X_2 = T_i^{(1)} + T_i^{(2)} - T_{i+1}^{(1)} - T_{\text{tact}}$. The random variable X_1 means the time difference between the time when job J_i is completed on piece of equipment M_1 and the tact time. Similarly, the random variable X_2 means the time difference between the time when job J_i is completed on piece of equipment M_2 and the time when job J_{i+1} is completed on the previous piece of equipment M_1. Therefore, each $X_1 \le 0$ and $X_2 \le 0$ is the event that, under the assumption that there are only two consecutive jobs J_i and J_{i+1}, there is no collision between J_i and J_{i+1} on piece of equipment M_1 and M_2, respectively. Next, we introduce the following event E_i for values of i from 1 to $n-1$.

E_i : Event that, under the assumption that there are only two consecutive jobs J_i and J_{i+1}, there is no collision between them.

By Lemma 1 for $m = 2$, the probability of event E_i occurring is given by

$$\begin{aligned}
\Pr(E_i) &= \Pr(T_i^{(1)} \le T_{\text{tact}}, T_i^{(1)} + T_i^{(2)} \le T_{\text{tact}} + T_{i+1}^{(1)})\\
&= \Pr(X_1 \le 0, X_2 \le 0)\\
&= \iint_{S_1} f(x_1, x_2) dx_1 dx_2,\ S_1 : x_1 \le 0, x_2 \le 0,
\end{aligned} \tag{3}$$

where $f(x_1, x_2)$ is the joint probability density function of random variables X_1 and X_2, which are not independent of each other.

Variables are then transformed by $y_1 = x_1, y_2 = x_2 - x_1$. Thus, S_1 is expressed as S_2 in terms of y_1 and y_2:

$$S_2 : y_1 \le 0, y_1 + y_2 \le 0.$$

The Jacobian $\mathscr{J}$ for $x_1 = y_1$ and $x_2 = y_1 + y_2$, which corresponds to S_1, is given by

$$\mathscr{J} = \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1.$$

Therefore,

$$\begin{aligned}
\text{Eq. (3)} &= \iint_{S_2} g(y_1, y_2) \left|\mathscr{J}\right| dy_1 dy_2\\
&= \int_{-\infty}^{0} dy_1 \int_{-\infty}^{-y_1} g_1(y_1) g_2(y_2) dy_2,
\end{aligned} \tag{4}$$

where $g(y_1, y_2)$ is the joint probability density function of random variables

$$Y_1 = T_i^{(1)} - T_{\text{tact}}, \tag{5}$$

$$Y_2 = T_i^{(2)} - T_{i+1}^{(1)}, \tag{6}$$

and $g_1(y_1)$ and $g_2(y_2)$ are the pdfs of Y_1 and Y_2, respectively. The random variable Y_1 means the time difference between the processing time of job J_i on piece of equipment M_1 and the tact time. Similarly, the random variable Y_2 means the time difference between the processing time of job J_i on piece of equipment M_2 and the processing time of job J_{i+1} on the previous piece of equipment M_1. Note that the second equality in Eq. (4) holds since Y_1 and Y_2 are assumed to be independent of each other.

Note that Eq. (4) (i.e. $\Pr(E_i)$) is valid even if the processing time of each piece of equipment follows general distribution. However, Eq. (4) has a closed form formula if Erlang distribution is assumed. In the next section, such a closed form is derived.

Each pdf $g_j(y_j)$ $(j = 1, 2)$ is determined only by parameters related to piece of equipment M_j. Therefore, $\Pr(E_1) = \Pr(E_2) = \cdots = \Pr(E_{n-1})$ holds. Note that two events, E_i and E_j $(i \ne j)$, are not independent of each other. However, we approximate the no-collision probability over all n jobs by $\Pr(E_i)^{n-1}$. In this case, the approximate collision probability for $m = 2$ is given by

$$1 - \Pr(E_i)^{n-1}. \tag{7}$$

5. Closed Form Formula for $m = 2$

In this section, we derive a closed form formula of Eq. (4) (i.e. $\Pr(E_i)$) under Erlang distribution.

By Eq. (5), the pdf $g_1(y_1)$ of Y_1 is obtained by translating the pdf of Erlang distribution with parameters λ_1 and k_1 by $-T_{\text{tact}}$. Therefore, we have

$$g_1(y_1) = \begin{cases} \frac{\lambda_1^{k_1}(y_1+T_{\text{tact}})^{k_1-1}}{(k_1-1)!e^{\lambda_1(y_1+T_{\text{tact}})}} & (y_1 \geq -T_{\text{tact}}), \\ 0 & (y_1 < -T_{\text{tact}}). \end{cases} \quad (8)$$

By Eq. (6), Y_2 is the sum of two independent random variables $T_i^{(2)}$ and $-T_{i+1}^{(1)}$ with the following pdfs, h_1 and h_2, respectively. The pdf of $T_i^{(2)}$ follows Erlang distribution with parameters λ_2 and k_2, i.e.,

$$h_1(x) = \begin{cases} \frac{\lambda_2^{k_2}x^{k_2-1}e^{-\lambda_2 x}}{(k_2-1)!} & (x \geq 0), \\ 0 & (x < 0). \end{cases}$$

The pdf of $-T_{i+1}^{(1)}$ is the reflection of the pdf of $T_{i+1}^{(1)}$ with parameters λ_1 and k_1 with respect to the line $y = 0$,

$$h_2(y) = \begin{cases} 0 & (y > 0), \\ \frac{\lambda_1^{k_1}(-y)^{k_1-1}e^{\lambda_1 y}}{(k_1-1)!} & (y \leq 0). \end{cases}$$

Then, the pdf $g_2(y_2)$ of Y_2 is given by the convolution of h_1 and h_2.

Case 1: $y_2 \geq 0$

$$\begin{aligned} g_2(y_2) &= \int_{-\infty}^{\infty} h_1(x)\,h_2(y_2 - x)\,dx \\ &= \int_{y_2}^{\infty} \frac{\lambda_2^{k_2}x^{k_2-1}e^{-\lambda_2 x}}{(k_2-1)!} \cdot \frac{\lambda_1^{k_1}(x-y_2)^{k_1-1}e^{\lambda_1(y_2-x)}}{(k_1-1)!}dx \\ &= \frac{\lambda_1^{k_1}\lambda_2^{k_2}e^{-\lambda_2 y_2}}{(k_2-1)!(\lambda_1+\lambda_2)^{k_1+k_2-1}} \\ &\quad \cdot \sum_{l=0}^{k_1-1} \frac{(k_1+k_2-2-l)!}{l!(k_1-1-l)!}(-1)^l y_2^l(\lambda_1+\lambda_2)^l \\ &\quad \cdot \sum_{h=0}^{k_1+k_2-2-l} \frac{y_2^h(\lambda_1+\lambda_2)^h}{h!}, \end{aligned} \quad (9)$$

Case 2: $y_2 < 0$

$$\begin{aligned} g_2(y_2) &= \int_{-\infty}^{\infty} h_1(x)\,h_2(y_2 - x)\,dx \\ &= \int_{0}^{\infty} \frac{\lambda_2^{k_2}x^{k_2-1}e^{-\lambda_2 x}}{(k_2-1)!} \cdot \frac{\lambda_1^{k_1}(x-y_2)^{k_1-1}e^{\lambda_1(y_2-x)}}{(k_1-1)!}dx \\ &= \frac{\lambda_1^{k_1}\lambda_2^{k_2}e^{\lambda_1 y_2}}{(k_2-1)!(\lambda_1+\lambda_2)^{k_1+k_2-1}} \\ &\quad \cdot \sum_{h=0}^{k_1-1} \frac{(k_1+k_2-2-h)!}{h!(k_1-1-h)!}(-1)^h y_2^h(\lambda_1+\lambda_2)^h. \end{aligned} \quad (10)$$

Using Eqs. (8) – (10) we simplify Eq. (4) (i.e. $\Pr(E_i)$) in the following equation. Since $y_1 \leq 0$ holds in Eq. (4), we have

$$\begin{aligned} &\int_{-\infty}^{0} g_2(y_2)dy_2 + \int_{0}^{-y_1} g_2(y_2)dy_2 \\ &= \frac{\lambda_1^{k_1}\lambda_2^{k_2}}{(k_2-1)!(\lambda_1+\lambda_2)^{k_1+k_2-1}} \cdot \sum_{h=0}^{k_1-1} \frac{(k_1+k_2-2-h)!(\lambda_1+\lambda_2)^h}{(k_1-1-h)!\lambda_1^{h+1}} \\ &\quad + \frac{\lambda_1^{k_1}\lambda_2^{k_2}}{(k_2-1)!(\lambda_1+\lambda_2)^{k_1+k_2-1}} \cdot \sum_{l=0}^{k_1-1} \frac{(k_1+k_2-2-l)!(\lambda_1+\lambda_2)^l(-1)^l}{(k_1-1-l)!\lambda_2^{l+1}l!} \\ &\quad \cdot \sum_{h=0}^{k_1+k_2-2-l} \frac{(l+h)!(\lambda_1+\lambda_2)^h}{h!\lambda_2^h} \\ &\quad - \frac{\lambda_1^{k_1}\lambda_2^{k_2}e^{\lambda_2 y_1}}{(k_2-1)!(\lambda_1+\lambda_2)^{k_1+k_2-1}} \cdot \sum_{l=0}^{k_1-1} \frac{(k_1+k_2-2-l)!(\lambda_1+\lambda_2)^l(-1)^l}{(k_1-1-l)!\lambda_2^{l+1}l!} \\ &\quad \cdot \sum_{h=0}^{k_1+k_2-2-l} \frac{(l+h)!(\lambda_1+\lambda_2)^h}{h!\lambda_2^h} \cdot \sum_{q=0}^{l+h} \frac{\lambda_2^q y_1^q(-1)^q}{q!}. \end{aligned}$$

Therefore, by denoting the above expression as I, we obtain

$$\begin{aligned} \Pr(E_i) &= \int_{-\infty}^{0} g_1(y_1) \cdot I dy_1 \\ &= \int_{-\infty}^{-T_{\text{tact}}} g_1(y_1) \cdot I dy_1 + \int_{-T_{\text{tact}}}^{0} g_1(y_1) \cdot I dy_1 \\ &= \int_{-T_{\text{tact}}}^{0} \frac{\lambda_1^{k_1}(y_1+T_{\text{tact}})^{k_1-1}e^{-\lambda_1(y_1+T_{\text{tact}})}}{(k_1-1)!} \cdot I dy_1 \\ &= \begin{cases} A - FAD + B - FBD + FC - GE, & \text{if } \lambda_1 \neq \lambda_2, \\ A' - E'A'C' + B' - E'B'C' + E'F'D', & \text{if } \lambda = \lambda_1 = \lambda_2, \end{cases} \end{aligned} \quad (11)$$

where

$$A = \frac{\lambda_1^{k_1}\lambda_2^{k_2}}{(k_2-1)!(\lambda_1+\lambda_2)^{k_1+k_2-1}} \cdot \sum_{h=0}^{k_1-1} \frac{(k_1+k_2-2-h)!(\lambda_1+\lambda_2)^h}{(k_1-1-h)!\lambda_1^{h+1}},$$

$$B = \frac{\lambda_1^{k_1}\lambda_2^{k_2}}{(k_2-1)!(\lambda_1+\lambda_2)^{k_1+k_2-1}} \cdot \sum_{l=0}^{k_1-1} \frac{(k_1+k_2-2-l)!(\lambda_1+\lambda_2)^l(-1)^l}{(k_1-1-l)!\lambda_2^{l+1}l!} \cdot \sum_{h=0}^{k_1+k_2-2-l} \frac{(l+h)!(\lambda_1+\lambda_2)^h}{h!\lambda_2^h},$$

$$\begin{aligned} C &= \frac{\lambda_1^{2k_1}\lambda_2^{k_2}}{(\lambda_1-\lambda_2)^{k_1}(k_2-1)!(\lambda_1+\lambda_2)^{k_1+k_2-1}} \\ &\quad \cdot \sum_{l=0}^{k_1-1} \frac{(k_1+k_2-2-l)!(\lambda_1+\lambda_2)^l(-1)^l}{(k_1-1-l)!\lambda_2^{l+1}l!} \cdot \sum_{h=0}^{k_1+k_2-2-l} \frac{(l+h)!(\lambda_1+\lambda_2)^h}{h!\lambda_2^h} \\ &\quad \cdot \sum_{q=0}^{l+h} \frac{\lambda_2^q(-1)^q}{q!(\lambda_1-\lambda_2)^q} \cdot \sum_{v=0}^{k_1-1} T_{\text{tact}}^v \frac{(q+k_1-1-v)!(\lambda_1-\lambda_2)^v}{v!(k_1-1-v)!}, \end{aligned}$$

$$D = \sum_{u=0}^{k_1-1} \frac{T_{\text{tact}}^u\lambda_1^u}{u!},$$

$$E = \frac{\lambda_1^{2k_1}\lambda_2^{k_2}}{(\lambda_1-\lambda_2)^{k_1}(k_2-1)!(\lambda_1+\lambda_2)^{k_1+k_2-1}}$$

$$\cdot \sum_{l=0}^{k_1-1} \frac{(k_1+k_2-2-l)!(\lambda_1+\lambda_2)^l(-1)^l}{(k_1-1-l)!\lambda_2^{l+1}l!} \cdot \sum_{h=0}^{k_1+k_2-2-l} \frac{(l+h)!(\lambda_1+\lambda_2)^h}{h!\lambda_2^h}$$
$$\cdot \sum_{q=0}^{l+h} \frac{\lambda_2^q(-1)^q}{q!(\lambda_1-\lambda_2)^q} \cdot \sum_{v=0}^{k_1-1} T_{\mathrm{tact}}^v \frac{(q+k_1-1-v)!(\lambda_1-\lambda_2)^v}{v!(k_1-1-v)!}$$
$$\cdot \sum_{w=0}^{q+k_1-1-v} \frac{(-1)^w T_{\mathrm{tact}}^w (\lambda_1-\lambda_2)^w}{w!},$$
$$F = e^{-\lambda_1 T_{\mathrm{tact}}}, \quad G = e^{-\lambda_2 T_{\mathrm{tact}}},$$
$$A' = \frac{1}{(k_2-1)!2^{k_1+k_2-1}} \cdot \sum_{h=0}^{k_1-1} \frac{(k_1+k_2-2-h)!2^h}{(k_1-1-h)!},$$
$$B' = \frac{1}{(k_2-1)!2^{k_1+k_2-1}} \cdot \sum_{l=0}^{k_1-1} \frac{(k_1+k_2-2-l)!2^l(-1)^l}{(k_1-1-l)!l!}$$
$$\cdot \sum_{h=0}^{k_1+k_2-2-l} \frac{(l+h)!2^h}{h!},$$
$$C' = \sum_{u=0}^{k_1-1} \frac{T_{\mathrm{tact}}^u \lambda_1^u}{u!},$$
$$D' = \frac{1}{(k_2-1)!2^{k_1+k_2-1}} \cdot \sum_{l=0}^{k_1-1} \frac{(k_1+k_2-2-l)!2^l(-1)^l}{(k_1-1-l)!l!}$$
$$\cdot \sum_{h=0}^{k_1+k_2-2-l} \frac{(l+h)!2^h}{h!} \cdot \sum_{q=0}^{l+h} \frac{\lambda^q T_{\mathrm{tact}}^q}{q!}$$
$$\cdot \sum_{v=0}^{k_1-1} \frac{1}{(q+k_1-v)(-1)^v v!(k_1-1-v)!},$$
$$E' = e^{-\lambda T_{\mathrm{tact}}}, \quad F' = \lambda^{k_1} T_{\mathrm{tact}}^{k_1}(-1)^{k_1}.$$

This is a closed form formula of $\Pr(E_i)$, from which the approximate collision probability $1-\Pr(E_i)^{n-1}$ is also obtained in a closed form formula. These formulae contain parameters n, T_{tact}, and parameters of the Erlang distributions.

6. Derivation of Collision Probability for the General Model for *m* Pieces of Equipment

It is possible to extend the above derivation for $m = 2$ to the general model for m pieces of equipment in a straightforward manner. In this section, we sketch its derivation.

We first introduce the following random variables:

$$X_l = \sum_{j=1}^{l} T_i^{(j)} - \sum_{j=1}^{l-1} T_{i+1}^{(j)} - T_{\mathrm{tact}} \ \text{ for all } \ 1 \le l \le m.$$

The random variable X_l means the time difference between the time when job J_i is completed on piece of equipment M_l and the time when job J_{i+1} is completed on the previous piece of equipment M_{l-1}. Therefore, $X_l \le 0$ is the event that, under the assumption that there are only two consecutive jobs J_i and J_{i+1}, there is no collision between J_i and J_{i+1} on piece of equipment M_l. Then, by Lemma 1, the probability of event E_i occurring is given as follows:

$$\begin{aligned}\Pr(E_i) &= \Pr\left(\sum_{j=1}^{l} T_i^{(j)} \le T_{\mathrm{tact}} + \sum_{j=1}^{l-1} T_{i+1}^{(j)} : 1 \le l \le m\right)\\ &= \Pr(X_l \le 0 : 1 \le l \le m)\\ &= \iint \cdots \int_{S_1'} f(x_1, x_2, \ldots, x_m) dx_1 dx_2 \cdots dx_m,\\ &\quad S_1' : x_l \le 0 \ \text{ for all } \ 1 \le l \le m, \end{aligned} \tag{12}$$

where $f(x_1, x_2, \ldots, x_m)$ is the joint probability density function of random variables X_l for all $1 \le l \le m$.

These variables are transformed by $y_1 = x_1$, $y_j = x_j - x_{j-1}$, and S_1' is expressed as S_2':

$$S_2' : \sum_{i=1}^{j} y_i \le 0 \quad \text{for all } 1 \le j \le m.$$

The Jacobian $\mathscr{J}$ for $x_i = \sum_{j=1}^{i} y_j$ for all $1 \le i \le m$, which corresponds to S_1', is given by

$$\mathscr{J} = \frac{\partial(x_1, x_2, \ldots, x_m)}{\partial(y_1, y_2, \ldots, y_m)} = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{vmatrix} = 1.$$

Therefore, Eq. (12) becomes as follows:

$$\begin{aligned}&\iint \cdots \int_{S_2} g(y_1, y_2, \ldots, y_m) \left|\mathscr{J}\right| dy_1 dy_2 \cdots dy_m\\ &= \int_{-\infty}^{0} dy_1 \int_{-\infty}^{-y_1} dy_2 \cdots \int_{-\infty}^{-\sum_{i=1}^{m-1} y_i} g_1(y_1) g_2(y_2) \cdots g_m(y_m) dy_m, \end{aligned} \tag{13}$$

where $g(y_1, y_2, \ldots, y_m)$ is the joint probability density function of random variables

$$Y_1 = T_i^{(1)} - T_{\mathrm{tact}},$$
$$Y_j = T_i^{(j)} - T_{i+1}^{(j-1)} \quad \text{for } 2 \le j \le m,$$

and each $g_j(y_j)$ is the pdf of Y_j. The random variable Y_j means the time difference between the processing time of job J_i on piece of equipment M_j and the processing time of job J_{i+1} on the previous piece of equipment M_{j-1}. Note that the equality in Eq. (13) holds because we assume all Y_j are independent of each other.

The $g_1(y_1)$ is given by Eq. (8), and $g_j(y_j)$ for all $2 \le j \le m$ are given by the convolution of two pdfs of $T_i^{(j)}$ and $-T_{i+1}^{(j-1)}$. Since the derivation of Eq. (13) into a closed form formula is similar to the one shown in the previous section, we have omitted the details. Therefore, the approximate collision probability $1-\Pr(E_i)^{n-1}$ is also given in a closed form formula.

We close this section by the following simple observation that holds for the general distribution of processing time. From Eq. (12),

$$\Pr(E_i) = \Pr\left(\sum_{j=1}^{l} T_i^{(j)} \leq T_{\text{tact}} + \sum_{j=1}^{l-1} T_{i+1}^{(j)} : 1 \leq l \leq m\right)$$

we know that $\Pr(E_i)$ is an increasing function of T_{tact}. This naturally implies that $1 - \Pr(E_i)^{n-1}$ is a decreasing function of T_{tact}.

Proposition 1: The approximate collision probability $1 - \Pr(E_i)^{n-1}$ is monotonically decreasing with tact time for the general distribution of processing time.

7. Numerical Results and Computer Simulation

Based on the above formulae, we obtain some numerical results by using MATHEMATICA [12]. For our computation in this section, the number of jobs is set to $n = 1,000$, and the parameters of the Erlang distributions are set so that the expectation of the processing time on each piece of equipment becomes equal to one (i.e., $k = \lambda$ in Eq. (1)).

Firstly, we compute the collision probability for $m = 1$, i.e. Eq. (2), as shown in Fig. 4, where the horizontal axis denotes the tact time T_{tact}. It shows that, as the tact time passes a certain threshold, the collision probability decreases rapidly, clearly exhibiting the trade-off between the tact time and the collision probability.

We consider the optimization problem presented in [1], which minimizes the tact time by including the collision probability as part of the input. Therefore, we denote by T^*_{tact} the minimum tact time under the condition that the collision probability is less than or equal to α % ($0 \leq \alpha \leq 100$). In this section, α is set to one. In Fig. 4, T^*_{tact} is about 1.49, which is 1.49 times the expectation of the processing time.

Note that the variance of the Erlang distribution for Fig. 4 is $k_1/\lambda_1^2 = 0.01$. We also test smaller variances, 0.001 and 0.0001, and the results are shown in Fig. 5 and Fig. 6, respectively. They show that T^*_{tact} for these variances are about 1.15 and 1.05, respectively, indicating that T^*_{tact} decreases as the variance of processing time decreases.

Next, using Eq. (11) we compute the approximate collision probability Eq. (7) for $m = 2$. The numerical results are shown in Fig. 7, Fig. 8, and Fig. 9, for variances 0.2, 0.1 and 0.0667, respectively. The resulting T^*_{tact} are about

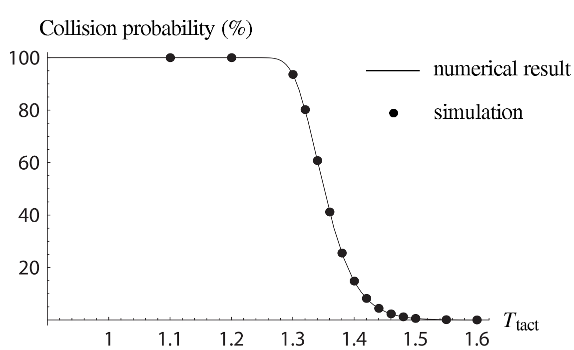

Fig. 4 Collision probability when $m = 1$, $k_1 = 100$, $\lambda_1 = 100$ (variance = 0.01), and $n = 1,000$.

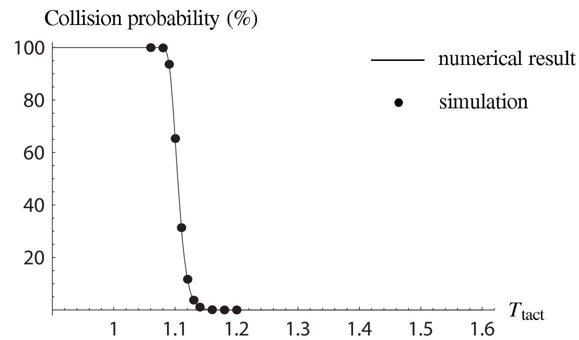

Fig. 5 Collision probability when $m = 1$, $k_1 = 1,000$, $\lambda_1 = 1,000$ (variance = 0.001), and $n = 1,000$.

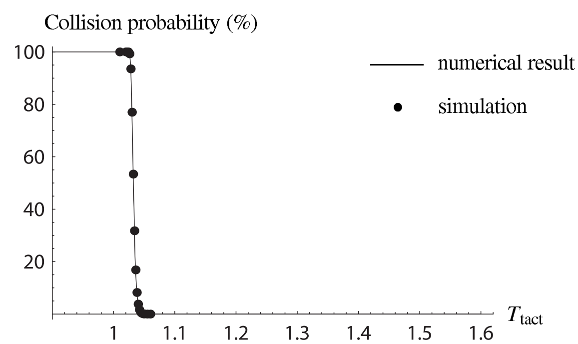

Fig. 6 Collision probability when $m = 1$, $k_1 = 10,000$, $\lambda_1 = 10,000$ (variance = 0.0001), and $n = 1,000$.

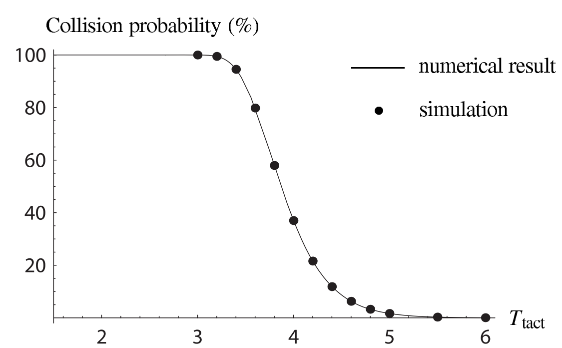

Fig. 7 Collision probability when $m = 2$, $k_1 = 5$, $k_2 = 5$, $\lambda_1 = 5$, $\lambda_2 = 5$ (variance = 0.2), and $n = 1,000$.

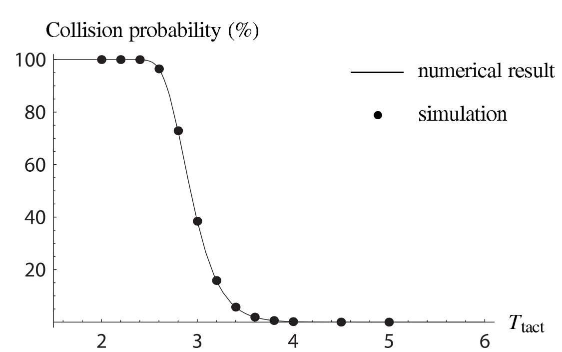

Fig. 8 Collision probability when $m = 2$, $k_1 = 10$, $k_2 = 10$, $\lambda_1 = 10$, $\lambda_2 = 10$ (variance = 0.1), and $n = 1,000$.

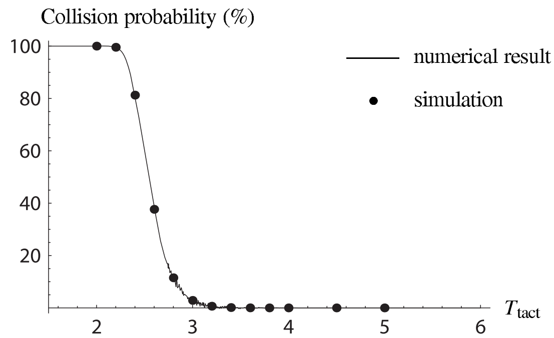

Fig. 9 Collision probability when $m = 2$, $k_1 = 15$, $k_2 = 15$, $\lambda_1 = 15$, $\lambda_2 = 15$ (variance = 0.0667), and $n = 1,000$.

Table 1 Collision probability (CP) evaluated by simulation when $m = 2$, $k_1 = 100$, $k_2 = 100$, $\lambda_1 = 100$, $\lambda_2 = 100$ (variance = 0.01), and $n = 1,000$. [%]

T_{tact}	1.45	1.50	1.55	1.60	1.65	1.70	1.75	1.80	1.85	1.90
CP	99.52	90.25	61.37	30.59	12.40	4.42	1.44	0.44	0.12	0.03

Table 2 Collision probability (CP) evaluated by simulation when $m = 2$, $k_1 = 1,000$, $k_2 = 1,000$, $\lambda_1 = 1,000$, $\lambda_2 = 1,000$ (variance = 0.001), and $n = 1,000$. [%]

T_{tact}	1.12	1.14	1.16	1.18	1.20	1.22	1.24	1.26	1.28	1.30
CP	100.0	99.60	84.16	42.21	13.48	3.35	0.72	0.13	0.02	0.00

5.15, 3.72, and 3.15, respectively, again showing that T^*_{tact} decreases with the variance.

We also carried out the following simulations to evaluate the exact probabilities. The procedure is stated as follows: given the number of jobs n, the number of pieces of equipment m, the tact time T_{tact}, the parameters of the Erlang distribution, and a positive integer c (specifying the number of iterations, which is related to the accuracy), derive the collision probability by the following algorithm [1].

Simulation Algorithm

Step 1: $loop := 1$.

Step 2: Generate the processing time $t_i^{(j)}$ ($1 \leq i \leq n, 1 \leq j \leq m$) randomly from the Erlang distribution.

Step 3: Based on the condition in Lemma 1, check whether a collision occurs. Let $loop := loop + 1$. If $loop \leq c$, return to Step 2; otherwise go to Step 4.

Step 4: Output the collision probability (the number of collisions observed in Step 3)$/c$.

The computation time is $\Theta(cmn)$. Throughout all simulations, we use Mersenne Twister [10] as the pseudorandom generator, and the number of iterations is set to $c = 1,000,000$.

In Figs. 4–9, the obtained simulation results are indicated by black dots. As numerical results and simulation results are reasonably close in most cases, we may conclude that simulation results are reasonably accurate and our approximation is justified.

We then checked by simulation the probability for smaller variances of processing times. The results are shown in Tables 1 and 2, from which we read that T^*_{tact} are about 1.80 and 1.24, respectively. Finally, we checked the probability in some cases of larger m and larger parameter values of the Erlang distribution. The simulation results are shown in Table 3, where $k_j = 1,000$, $\lambda_j = 1,000$ for all $1 \leq j \leq m$, i.e., the expectation and variance are 1 and 0.001, respectively. From Table 3, we confirmed that the collision probability increases with m, but decreases with T_{tact}. Table 3 also shows that T^*_{tact} increases with m.

Table 3 Collision probability evaluated by simulation when $k_j = 1,000$, $\lambda_j = 1,000$ (variance = 0.001) for all $1 \leq j \leq m$, and $n = 1,000$. [%]

	T_{tact}									
m	1.20	1.25	1.30	1.35	1.40	1.45	1.50	1.55	1.60	1.65
3	91.39	19.27	1.21	0.04	0.00	0.00	0.00	0.00	0.00	0.00
4	99.99	77.83	16.32	1.52	0.09	0.00	0.00	0.00	0.00	0.00
5	100.0	99.16	57.61	11.35	1.29	0.11	0.01	0.00	0.00	0.00
6	100.0	100.0	91.23	37.52	7.19	0.98	0.10	0.01	0.00	0.00
7	100.0	100.0	99.40	71.11	22.36	4.26	0.63	0.07	0.01	0.00
8	100.0	100.0	99.99	92.39	47.06	12.72	2.46	0.39	0.05	0.01
9	100.0	100.0	100.0	98.95	72.81	27.97	6.95	1.37	0.23	0.03

8. Conclusions

In the model presented in [1], it is assumed that the processing time of each piece of equipment is a random variable that follows a continuous probability distribution. In [1], two cases were specifically discussed, in which processing time followed a normal and an exponential distribution. In this paper, we have assumed, considering the features of actual processing times, that processing time follows an Erlang distribution, which is a new focus of observation. Moreover, since our model is simple, it has broad utility and the potential of having a wide domain of applicability in the field of mass production and not just in the manufacture of FPDs and semiconductors.

We presented a closed form formula of the approximate collision probability in the model comprising of m pieces of equipment when the processing time of each piece of equipment follows an Erlang distribution. This closed form is an extension of the one presented in [1], which showed a closed form of the approximate collision probability when the processing time follows an exponential distribution. We have also shown numerical results, as well as computer simulation results. Our approximate formula has reasonable accuracy, as experimentally confirmed.

There are also cases where numerical results and simulation results are different. For example, since the approximate formula is expressed using the $(n-1)$-th power, when the number of jobs n is very small and the value for the collision probability is relatively large, cases of small percentage differences between numerical results and simulation results exist. It could be considered as one possible area of future work to reveal this phenomenon mathematically from the viewpoint of guaranteeing accuracy.

Presenting a closed form for the case of general m can be considered as another area of future work. Since such an expression may be very complicated, it can be assumed that some recursive formulae may be useful.

Additionally, it may become hard to obtain numerical values from the closed form depending on parameter conditions. For example, when comparing such values with simulation results in Table 3, overflow might occur trying to perform numerical computation for the closed form using MATHEMATICA. Solving this is a possible area of further work. Moreover, research related to the computable order for the parameters is also an area of future interest.

Finally, we discuss some differences between the model presented in this paper and an actual FPD production line. In a real FPD production line, in order to avoid a collision between substrates, a substrate will not be transported to the next piece of equipment if it is occupied. In this situation, the substrate may be kept at the current piece of equipment even if the required process on it has been completed. This is referred to as tandem queues with blocking. However, considering chemicals, heat treating, etc., it is not very acceptable to allow waiting on a piece of equipment because FPD quality could decay. As another solution for avoiding collisions, buffers could be made before each piece of equipment. Analyzing the collision probability with buffer space included is also a potential area of future research.

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