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Multi-indexed Meixner and Little q-Jacobi (Laguerre) Polynomials

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Abstract

As the fourth stage of the project *multi-indexed orthogonal polynomials*, we present the multi-indexed Meixner and little q-Jacobi (Laguerre) polynomials in the framework of 'discrete quantum mechanics' with real shifts defined on the semi-infinite lattice in one dimension. They are obtained, in a similar way to the multi-indexed Laguerre and Jacobi polynomials reported earlier, from the quantum mechanical systems corresponding to the original orthogonal polynomials by multiple application of the discrete analogue of the Darboux transformations or the Crum-Krein-Adler deletion of virtual state vectors. The virtual state vectors are the solutions of the matrix Schrödinger equation on all the lattice points having negative energies and infinite norm. This is in good contrast to the (q-)Racah systems defined on a finite lattice, in which the 'virtual state' vectors satisfy the matrix Schrödinger equation except for one of the two boundary points.

1 Introduction

Theory of exactly solvable quantum mechanics (QM) has seen a great surge of interests in recent years aroused by the discovery of the *exceptional* [1, 2, 3] and *multi-indexed orthogonal polynomials* [4, 5]. They are new types of orthogonal polynomials satisfying second order differential equations and forming complete sets of orthogonal bases in appropriate Hilbert spaces. They are distinguished from the *classical orthogonal polynomials* [6], by the fact that there are 'holes' in their degrees and the breakdown of the three term recurrence relations, which is essential to avoid the so-called Bochner's theorem [7, 8].

The concept of the exceptional and multi-indexed orthogonal polynomials has been generalised to the other classical orthogonal polynomials in the Askey scheme [9], which satisfy second order *difference* equations. The exceptional and multi-indexed Wilson, Askey-Wilson,

(q-)Racah polynomials have been constructed [10, 11] as the main parts of the eigenfunctions (vectors) of difference Schrödinger equations within the program of '*discrete quantum mechanics*' [12, 13, 14].

In this paper we report the construction of the multi-indexed Meixner, little q-Jacobi (Laguerre) polynomials. The base polynomials, the Meixner (M), little q-Jacobi (lqJ) and little q-Laguerre (lqL) polynomials, satisfy second order difference equations on *semi-infinite* integer lattices and they form complete orthogonal bases in the corresponding ℓ^2 Hilbert spaces. These polynomials are also called 'orthogonal polynomials of a discrete variable' [15] and they belong to the subclass of discrete quantum mechanics with real shifts. The Meixner case was studied by Durán [16]. In our language his polynomials correspond to the eigenstates and/or virtual states deletion.

The first stage of the multi-indexed orthogonal polynomials project dealt with the Laguerre and the Jacobi polynomials [5] belonging to the ordinary QM. The multi-indexed Wilson and Askey-Wilson polynomials were constructed in the second stage [10]. They belong to discrete QM with pure imaginary shifts [12, 14]. The multi-indexed (q-)Racah polynomials were constructed in the third stage [11]. They belong to the discrete QM with real shifts [13, 17] defined on *finite* integer lattices. The multi-indexed orthogonal polynomials are obtained as the main parts of the eigenfunctions of the deformed quantum systems, which are generated by multiple application of Darboux-Crum transformations [18, 19] on the base systems. The Darboux-Crum transformations for discrete QM with real shifts were developed in [20]. The seed solutions for the Darboux-Crum transformations are called the *virtual state functions (vectors)*, which are obtained from the eigenfunctions by the discrete symmetry transformations of the base Hamiltonians.

The present paper is organised as follows. In section two, the general setting of the discrete QM with real shifts is briefly recapitulated. The main features of the base systems, the Meixner, little q-Jacobi (Laguerre) polynomials are collected in §2.1. The method of constructing virtual state vectors is explained in §2.2 together with their explicit forms. The general procedure of modifying exactly solvable QM by multiple Darboux-Crum transformations in terms of the virtual state vectors is explained in some detail in section 3. Various formulas of the obtained multi-indexed M, lqJ, lqL polynomials are collected in section 4. Certain limits of these multi-indexed polynomials are presented in § 4.1. The final section is for a summary and comments.

2 Discrete QM with Real Shifts on a Semi-infinite Lattice

Let us recapitulate the discrete quantum mechanics with real shifts on a semi-infinite lattice. There are two types: one component systems (Meixner, little q-Jacobi, etc.) [13] and two component systems (q-Meixner, big q-Jacobi, etc.) [17]. We discuss one component systems in this paper.

The Hamiltonian $\mathcal{H} = (\mathcal{H}_{x,y})$ is a *tri-diagonal real symmetric* (Jacobi) matrix and its rows and columns are indexed by integers x and y, which take values in $\mathbb{Z}_{\geq 0}$ (semi-infinite). By adding a scalar matrix to the Hamiltonian, the lowest eigenvalue is adjusted to be zero. This makes the Hamiltonian *positive semi-definite*. Since the eigenvector corresponding to the zero eigenvalue has definite sign, *i.e.* all the components are positive or negative, the Hamiltonian $\mathcal{H}$ has the following form ($x, y \in \mathbb{Z}_{\geq 0}$)

$$\mathcal{H}_{x,y} \stackrel{\text{def}}{=} -\sqrt{B(x)D(x+1)}\,\delta_{x+1,y} - \sqrt{B(x-1)D(x)}\,\delta_{x-1,y} + \big(B(x)+D(x)\big)\delta_{x,y}, \tag{2.1}$$

in which the potential functions $B(x)$ and $D(x)$ are real and positive but vanish at the boundary

$$B(x) > 0 \ \ (x \in \mathbb{Z}_{\geq 0}), \quad D(x) > 0 \ \ (x \in \mathbb{Z}_{\geq 1}), \quad D(0) = 0. \tag{2.2}$$

The potential functions $B(x)$ and $D(x)$ are rational functions of x or q^x (q is a positive constant, $0 < q < 1$). See §2.1 for the explicit forms of these functions. The Hamiltonian (2.1) is real symmetric, $\mathcal{H}_{x,y} = \mathcal{H}_{y,x}$. Reflecting the positive semi-definiteness, the Hamiltonian (2.1) can be expressed in a factorised form:

$$\mathcal{H} = \mathcal{A}^\dagger\mathcal{A}, \qquad \mathcal{A} = (\mathcal{A}_{x,y}), \quad \mathcal{A}^\dagger = ((\mathcal{A}^\dagger)_{x,y}) = (\mathcal{A}_{y,x}), \tag{2.3}$$

$$\mathcal{A}_{x,y} \stackrel{\text{def}}{=} \sqrt{B(x)}\,\delta_{x,y} - \sqrt{D(x+1)}\,\delta_{x+1,y}, \quad (\mathcal{A}^\dagger)_{x,y} = \sqrt{B(x)}\,\delta_{x,y} - \sqrt{D(x)}\,\delta_{x-1,y}. \tag{2.4}$$

Here $\mathcal{A}$ ($\mathcal{A}^\dagger$) is an upper (lower) triangular matrix with the diagonal and the super(sub)-diagonal entries only. For simplicity in notation, we write $\mathcal{H}$, $\mathcal{A}$ and $\mathcal{A}^\dagger$ as follows:

$$e^{\pm\partial} = ((e^{\pm\partial})_{x,y}), \quad (e^{\pm\partial})_{x,y} \stackrel{\text{def}}{=} \delta_{x\pm1,y}, \quad (e^{\partial})^\dagger = e^{-\partial}, \tag{2.5}$$

$$\begin{aligned}\mathcal{H} &= -\sqrt{B(x)D(x+1)}\,e^{\partial} - \sqrt{B(x-1)D(x)}\,e^{-\partial} + B(x) + D(x)\\ &= -\sqrt{B(x)}\,e^{\partial}\sqrt{D(x)} - \sqrt{D(x)}\,e^{-\partial}\sqrt{B(x)} + B(x) + D(x),\end{aligned} \tag{2.6}$$

$$\mathcal{A} = \sqrt{B(x)} - e^{\partial}\sqrt{D(x)}, \quad \mathcal{A}^\dagger = \sqrt{B(x)} - \sqrt{D(x)}\,e^{-\partial}. \tag{2.7}$$

We suppress the unit matrix $\mathbf{1}=(\delta_{x,y})$: $(B(x)+D(x))\mathbf{1}$ in (2.6), $\sqrt{B(x)}\,\mathbf{1}$ in (2.7). Note that the product of e^{∂} and $e^{-\partial}$ is $e^{\partial}e^{-\partial}=\mathbf{1}$ but the reversed order is $e^{-\partial}e^{\partial}=\mathbf{1}-\text{diag}(1,0,0,\ldots)$.

The Hamiltonian (2.1) is a linear operator on the real ℓ^2 Hilbert space with the inner product of two real vectors $f=(f(x))$ and $g=(g(x))$ defined by

$$(f,g)\stackrel{\text{def}}{=}\lim_{N\to\infty}(f,g)_N,\quad (f,g)_N\stackrel{\text{def}}{=}\sum_{x=0}^{N}f(x)g(x),\quad \|f\|^2\stackrel{\text{def}}{=}(f,f)<\infty. \tag{2.8}$$

The Schrödinger equation is the eigenvalue problem for the hermitian matrix $\mathcal{H}$,

$$\mathcal{H}\phi_n(x)=\mathcal{E}_n\phi_n(x)\quad (n=0,1,\ldots),\quad 0=\mathcal{E}_0<\mathcal{E}_1<\cdots, \tag{2.9}$$

where the eigenvector $\phi_n=(\phi_n(x))$ is, by definition, of finite norm, $\|\phi_n\|<\infty$. Let us recall the fact that the spectrum of a Jacobi matrix is simple. The ground state eigenvector, which is chosen positive $\phi_0(x)>0$ ($x\in\mathbb{Z}_{\geq 0}$), satisfies the zero mode equation:

$$\mathcal{A}\phi_0=0\ \Rightarrow \mathcal{H}\phi_0=0,\quad \sqrt{B(x)}\,\phi_0(x)=\sqrt{D(x+1)}\,\phi_0(x+1), \tag{2.10}$$

and it is easily obtained with the normalisation $\phi_0(0)=1$ (convention: $\prod_{k=n}^{n-1}*=1$):

$$\phi_0(x)=\prod_{y=0}^{x-1}\sqrt{\frac{B(y)}{D(y+1)}}\,. \tag{2.11}$$

The self-adjointness of the Hamiltonian and the non-degeneracy of the spectrum (2.9) imply that the eigenvectors are mutually orthogonal:

$$(\phi_n,\phi_m)=\frac{\delta_{nm}}{d_n^2}\quad (n,m\in\mathbb{Z}_{\geq 0}), \tag{2.12}$$

where d_n^2 ($d_n>0$) is the normalisation constant.

For the systems considered in this paper, the eigenvectors have the following factorised form

$$\phi_n(x)=\phi_0(x)\check{P}_n(x),\quad \check{P}_n(x)\stackrel{\text{def}}{=}P_n\bigl(\eta(x)\bigr), \tag{2.13}$$

where $\eta(x)$ is the *sinusoidal coordinate* satisfying the boundary condition $\eta(0)=0$. The other function $P_n(\eta)$ is a degree n polynomial in η and we adopt the universal normalisation condition [13, 17] as

$$P_n(0)=1\ \bigl(\Leftrightarrow \check{P}_n(0)=1\bigr). \tag{2.14}$$

This is important for the proper definition of the *dual polynomials*, in which the roles of x and n are interchanged [13]. The sinusoidal coordinate has a special dynamical meaning [21, 13]. The Heisenberg operator solution for $\eta(x)$ can be expressed in a closed form, which is a consequence of the closure relation [13]. This means that its time evolution is a sinusoidal motion. The similarity transformed Hamiltonian $\widetilde{\mathcal{H}}$ in terms of the ground state wavefunction $\phi_0(x)$ (2.11) is

$$\widetilde{\mathcal{H}} \stackrel{\text{def}}{=} \phi_0(x)^{-1} \circ \mathcal{H} \circ \phi_0(x) = B(x)(1 - e^{\partial}) + D(x)(1 - e^{-\partial}), \tag{2.15}$$

and (2.9) becomes square root free

$$\widetilde{\mathcal{H}} \check{P}_n(x) = \mathcal{E}_n \check{P}_n(x), \tag{2.16}$$

namely

$$B(x)\big(\check{P}_n(x) - \check{P}_n(x+1)\big) + D(x)\big(\check{P}_n(x) - \check{P}_n(x-1)\big) = \mathcal{E}_n \check{P}_n(x), \tag{2.17}$$

which are valid for any $x(\in \mathbb{C})$, as $B(x)$ and $D(x)$ are rational functions of x or q^x.

2.1 Original systems

Let us consider the Meixner (M), little q-Jacobi (lqJ) and little q-Laguerre (lqL) cases. We follow the notation of [13]. Various quantities depend on a set of parameters $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots)$ and their dependence is expressed like, $\mathcal{H} = \mathcal{H}(\boldsymbol{\lambda})$, $\mathcal{A} = \mathcal{A}(\boldsymbol{\lambda})$, $\mathcal{E}_n = \mathcal{E}_n(\boldsymbol{\lambda})$, $B(x) = B(x;\boldsymbol{\lambda})$, $\phi_n(x) = \phi_n(x;\boldsymbol{\lambda})$, $\check{P}_n(x) = \check{P}_n(x;\boldsymbol{\lambda})$, etc. The parameter q is $0 < q < 1$ and $q^{\boldsymbol{\lambda}}$ stands for $q^{(\lambda_1,\lambda_2,\ldots)} = (q^{\lambda_1}, q^{\lambda_2}, \ldots)$. For these three systems the sinusoidal coordinate $\eta(x)$, the auxiliary function $\varphi(x)$ (2.20), (2.25) and (2.31) and the potential function $D(x)$ do not depend on the parameters $\boldsymbol{\lambda}$. The normalisation (boundary condition) of $\eta(x)$ is chosen as $\eta(0) = 0$. The shift parameter $\boldsymbol{\delta}$ and the scale parameter κ appear in the defining formula of shape invariance (2.35).

2.1.1 Meixner (M)

We rescale the overall normalisation of the Hamiltonian in [13]: ($\mathcal{H}$ in [13]) $\times (1 - c) \to \mathcal{H}$. The fundamental data are as follows [13]:

$$\boldsymbol{\lambda} = (\beta, c), \quad \boldsymbol{\delta} = (1, 0), \quad \kappa = 1, \quad \beta > 0, \quad 0 < c < 1, \tag{2.18}$$

$$B(x;\boldsymbol{\lambda}) = c(x + \beta), \quad D(x) = x, \tag{2.19}$$

$$\mathcal{E}_n(\boldsymbol{\lambda}) = (1-c)n, \quad \eta(x) = x, \quad \varphi(x) = 1, \tag{2.20}$$

$$\check{P}_n(x;\boldsymbol{\lambda}) = {}_2F_1\Bigl(\begin{matrix}-n,\ -x\\ \beta\end{matrix}\Bigm| 1-c^{-1}\Bigr) = M_n(x;\beta,c), \tag{2.21}$$

$$\phi_0(x;\boldsymbol{\lambda})^2 = \frac{(\beta)_x\, c^x}{x!}, \quad d_n(\boldsymbol{\lambda})^2 = \frac{(\beta)_n\, c^n}{n!} \times (1-c)^\beta, \tag{2.22}$$

where $M_n(\eta;\beta,c)$ is the Meixner polynomial, which is obviously *self-dual*, $M_n(x;\beta,c) = M_x(n;\beta,c)$ $(x,n \in \mathbb{Z}_{\geq 0})$.

2.1.2 little q-Jacobi (lqJ)

The little q-Jacobi polynomial to be discussed in this paper (2.26) obeys the universal normalisation (2.14), which is different from the conventional normalisation of the little q-Jacobi polynomial p_n as shown explicitly in (2.27). The fundamental data are as follows [13, 17]:

$$q^{\boldsymbol{\lambda}} = (a,b), \quad \boldsymbol{\delta} = (1,1), \quad \kappa = q^{-1}, \quad 0 < a < q^{-1}, \quad b < q^{-1}, \tag{2.23}$$

$$B(x;\boldsymbol{\lambda}) = a(q^{-x} - bq), \quad D(x) = q^{-x} - 1, \tag{2.24}$$

$$\mathcal{E}_n(\boldsymbol{\lambda}) = (q^{-n}-1)(1-abq^{n+1}), \quad \eta(x) = 1-q^x, \quad \varphi(x) = q^x, \tag{2.25}$$

$$\check{P}_n(x;\boldsymbol{\lambda}) = {}_3\phi_1\Bigl(\begin{matrix}q^{-n},\ abq^{n+1},\ q^{-x}\\ bq\end{matrix}\Bigm| q\,;\frac{q^x}{a}\Bigr) \tag{2.26}$$

$$= \frac{(a^{-1}q^{-n};q)_n}{(bq;q)_n}\, {}_2\phi_1\Bigl(\begin{matrix}q^{-n},\ abq^{n+1}\\ aq\end{matrix}\Bigm| q\,;q^{x+1}\Bigr)$$

$$= \frac{(a^{-1}q^{-n};q)_n}{(bq;q)_n}\, p_n\bigl(1-\eta(x);a,b|q\bigr), \tag{2.27}$$

$$\phi_0(x;\boldsymbol{\lambda})^2 = \frac{(bq;q)_x}{(q;q)_x}(aq)^x, \quad d_n(\boldsymbol{\lambda})^2 = \frac{(bq,abq;q)_n\, a^n q^{n^2}}{(q,aq;q)_n}\,\frac{1-abq^{2n+1}}{1-abq} \times \frac{(aq;q)_\infty}{(abq^2;q)_\infty}. \tag{2.28}$$

For the dual lqJ polynomial, see [22] and [13].

2.1.3 little q-Laguerre (lqL)

The little q-Laguerre polynomial to be discussed in this paper (2.32) obeys the universal normalisation (2.14), which is different from the conventional normalisation of the little q-Laguerre polynomial p_n as shown explicitly in (2.33). The fundamental data are as follows [13]:

$$q^{\boldsymbol{\lambda}} = a, \quad \boldsymbol{\delta} = 1, \quad \kappa = q^{-1}, \quad 0 < a < q^{-1}, \tag{2.29}$$

$$B(x;\boldsymbol{\lambda}) = aq^{-x}, \quad D(x) = q^{-x} - 1, \tag{2.30}$$

$$\mathcal{E}_n = q^{-n} - 1, \quad \eta(x) = 1-q^x, \quad \varphi(x) = q^x, \tag{2.31}$$

$$\check{P}_n(x;\boldsymbol{\lambda})={}_2\phi_0\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\,q^{-x}}{-}\Bigm|q;\frac{q^x}{a}\Bigr)=(a^{-1}q^{-n}\,;q)_n\,{}_2\phi_1\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\,0}{aq}\Bigm|q;q^{x+1}\Bigr) \tag{2.32}$$

$$=(a^{-1}q^{-n}\,;q)_n\,p_n\bigl(1-\eta(x);a|q\bigr), \tag{2.33}$$

$$\phi_0(x;\boldsymbol{\lambda})^2=\frac{(aq)^x}{(q;q)_x},\quad d_n(\boldsymbol{\lambda})^2=\frac{a^nq^{n^2}}{(q,aq;q)_n}\times(aq;q)_\infty. \tag{2.34}$$

The lqL system is obtained from lqJ system by setting $b=0$. The dual lqL polynomial is the Al-Salam-Carlitz II polynomial [9, 22, 13].

2.1.4 shape invariance

These three systems are shape invariant [13],

$$\mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^\dagger=\kappa\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta})^\dagger\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta})+\mathcal{E}_1(\boldsymbol{\lambda}), \tag{2.35}$$

in which $\boldsymbol{\delta}$ denotes the shift of the parameters, κ is a positive constant and $\mathcal{E}_1(\boldsymbol{\lambda})$ is the eigenvalue of the first excited state $\mathcal{E}_1(\boldsymbol{\lambda})>0$. It is a sufficient condition for exact solvability and it provides the explicit formulas for the energy eigenvalues $\mathcal{E}_n(\boldsymbol{\lambda})=\sum_{s=0}^{n-1}\kappa^s\mathcal{E}_1(\boldsymbol{\lambda}+s\boldsymbol{\delta})$ and the eigenfunctions, *i.e.* the universal Rodrigues formula [13, 17]. The forward and backward shift relations are

$$\mathcal{F}(\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda})=\mathcal{E}_n(\boldsymbol{\lambda})\check{P}_{n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}),\quad \mathcal{B}(\boldsymbol{\lambda})\check{P}_{n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})=\check{P}_n(x;\boldsymbol{\lambda}), \tag{2.36}$$

where the forward and backward shift operators are

$$\mathcal{F}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}\sqrt{B(0;\boldsymbol{\lambda})}\,\phi_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta})^{-1}\circ\mathcal{A}(\boldsymbol{\lambda})\circ\phi_0(x;\boldsymbol{\lambda})=B(0;\boldsymbol{\lambda})\varphi(x)^{-1}(1-e^{\partial}), \tag{2.37}$$

$$\begin{aligned}\mathcal{B}(\boldsymbol{\lambda})&\stackrel{\text{def}}{=}\frac{1}{\sqrt{B(0;\boldsymbol{\lambda})}}\,\phi_0(x;\boldsymbol{\lambda})^{-1}\circ\mathcal{A}(\boldsymbol{\lambda})^\dagger\circ\phi_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta})\\&=B(0;\boldsymbol{\lambda})^{-1}\bigl(B(x;\boldsymbol{\lambda})-D(x)e^{-\partial}\bigr)\varphi(x)\\&=\phi_0(x;\boldsymbol{\lambda})^{-2}\circ(1-e^{-\partial})\varphi(x)^{-1}\circ\phi_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta})^2.\end{aligned} \tag{2.38}$$

Starting from $\check{P}_0(x)=1$, $\check{P}_n(x;\boldsymbol{\lambda})$ can be written as

$$\begin{aligned}\check{P}_n(x;\boldsymbol{\lambda})=P_n\bigl(\eta(x);\boldsymbol{\lambda}\bigr)&=\mathcal{B}(\boldsymbol{\lambda})\mathcal{B}(\boldsymbol{\lambda}+\boldsymbol{\delta})\cdots\mathcal{B}\bigl(\boldsymbol{\lambda}+(n-1)\boldsymbol{\delta}\bigr)\cdot 1\\&=\phi_0(x;\boldsymbol{\lambda})^{-2}\Bigl((1-e^{-\partial})\varphi(x)^{-1}\Bigr)^n\cdot\phi_0(x;\boldsymbol{\lambda}+n\boldsymbol{\delta})^2.\end{aligned} \tag{2.39}$$

This is the universal Rodrigues formula mentioned above. Compare this with the individual Rodrigues formulas listed in [9]. We note the following relations between the auxiliary function $\varphi(x)$ and the ground state eigenvectors, etc:

$$\varphi(x)=\sqrt{\frac{B(0;\boldsymbol{\lambda})}{B(x;\boldsymbol{\lambda})}}\,\frac{\phi_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta})}{\phi_0(x;\boldsymbol{\lambda})}=\frac{\eta(x+1)-\eta(x)}{\eta(1)}, \tag{2.40}$$

$$\frac{B(x;\boldsymbol{\lambda}+\boldsymbol{\delta})}{B(x+1;\boldsymbol{\lambda})}=\kappa^{-1}\frac{\varphi(x+1)}{\varphi(x)}=\kappa^{-2}. \tag{2.41}$$

2.2 Virtual systems

2.2.1 modified potential functions

Let us assume the existence of modified potential functions $B'(x)$, $D'(x)$ satisfying

$$\alpha^2B'(x)D'(x+1)=B(x)D(x+1),\qquad \alpha>0, \tag{2.42}$$

$$\alpha\big(B'(x)+D'(x)\big)+\alpha'=B(x)+D(x),\qquad \alpha'<0, \tag{2.43}$$

where α and α' are constant. Here $B'(x)$ and $D'(x)$ are rational functions of x or q^x. Note that these two equations are valid for any $x(\in\mathbb{C})$. These functions are required to satisfy the boundary conditions, too:

$$B'(x)>0\ \ (x\in\mathbb{Z}_{\geq 0}),\quad D'(x)>0\ \ (x\in\mathbb{Z}_{\geq 1}),\quad D'(0)=0. \tag{2.44}$$

Then we obtain a linear relation between the original Hamiltonian (2.6) and the virtual Hamiltonian $\mathcal{H}'$ (2.46) constructed by the modified potential functions $B'(x)$ and $D'(x)$, (2.42)–(2.44):

$$\mathcal{H}=\alpha\mathcal{H}'+\alpha', \tag{2.45}$$

$$\mathcal{H}'\stackrel{\text{def}}{=}-\sqrt{B'(x)}\,e^{\partial}\sqrt{D'(x)}-\sqrt{D'(x)}\,e^{-\partial}\sqrt{B'(x)}+B'(x)+D'(x). \tag{2.46}$$

We define a positive function $\phi'_0(x)$ by

$$\phi'_0(x)\stackrel{\text{def}}{=}\prod_{y=0}^{x-1}\sqrt{\frac{B'(y)}{D'(y+1)}}\,, \tag{2.47}$$

which is annihilated by $\mathcal{A}'=\sqrt{B'(x)}-e^{\partial}\sqrt{D'(x)}$:

$$\mathcal{A}'\phi'_0(x)=0\ \Rightarrow\ \sqrt{B'(x)}\,\phi'_0(x)=\sqrt{D'(x+1)}\,\phi'_0(x). \tag{2.48}$$

In other words, $\phi'_0(x)$ is a solution of the original Schrödinger equation of $\mathcal{H}$ (2.9) with a negative energy α'. That is, $\phi'_0(x)$ does not belong to the ℓ^2 Hilbert space spanned by the eigenvectors of $\mathcal{H}$. We introduce another positive function $\nu(x)$ by the ratio $\phi_0(x)/\phi'_0(x)$:

$$\nu(x) \stackrel{\text{def}}{=} \frac{\phi_0(x)}{\phi'_0(x)} = \prod_{y=0}^{x-1} \frac{B(y)}{\alpha B'(y)} = \prod_{y=0}^{x-1} \frac{\alpha D'(y+1)}{D(y+1)}. \tag{2.49}$$

It can be analytically continued into a meromorphic function of x or q^x through the functional relations:

$$\nu(x+1) = \frac{B(x)}{\alpha B'(x)}\nu(x), \quad \nu(x-1) = \frac{D(x)}{\alpha D'(x)}\nu(x). \tag{2.50}$$

As we will show shortly, these modified potential functions $B'(x)$, $D'(x)$ (2.42)-(2.43) are obtained from the original potential functions $B(x)$, $D(x)$ (2.1) by *twisting* the parameters, which is an essential step for introducing the exceptional and multi-indexed orthogonal polynomials [3, 5, 11, 10]. Other solutions of the Schrödinger equation for the virtual Hamiltonian $\mathcal{H}'$ can be obtained in a factorised form:

$$\mathcal{H}'\tilde{\phi}_{\text{v}}(x) = \mathcal{E}'_{\text{v}}\tilde{\phi}_{\text{v}}(x) \ \Big(\Rightarrow \ \mathcal{H}\tilde{\phi}_{\text{v}}(x) = \tilde{\mathcal{E}}_{\text{v}}\tilde{\phi}_{\text{v}}(x), \ \ \tilde{\mathcal{E}}_{\text{v}} \stackrel{\text{def}}{=} \alpha\mathcal{E}'_{\text{v}} + \alpha'\Big), \tag{2.51}$$

$$\tilde{\phi}_{\text{v}}(x) \stackrel{\text{def}}{=} \phi'_0(x)\check{\xi}_{\text{v}}(x), \tag{2.52}$$

in which $\check{\xi}_{\text{v}}(x)$ is a solution of

$$\widetilde{\mathcal{H}}'\check{\xi}_{\text{v}}(x) = \mathcal{E}'_{\text{v}}\check{\xi}_{\text{v}}(x), \tag{2.53}$$

$$\widetilde{\mathcal{H}}' \stackrel{\text{def}}{=} \phi'_0(x)^{-1} \circ \mathcal{H}' \circ \phi'_0(x) = B'(x)(1 - e^{\partial}) + D'(x)(1 - e^{-\partial}), \tag{2.54}$$

namely,

$$B'(x)\big(\check{\xi}_{\text{v}}(x) - \check{\xi}_{\text{v}}(x+1)\big) + D'(x)\big(\check{\xi}_{\text{v}}(x) - \check{\xi}_{\text{v}}(x-1)\big) = \mathcal{E}'_{\text{v}}\check{\xi}_{\text{v}}(x). \tag{2.55}$$

For a non-negative integer v, $\mathcal{E}'_{\text{v}}$ and $\check{\xi}_{\text{v}}(x)$ can be obtained from the original polynomial solution $\check{P}_{\text{v}}(x) = P_{\text{v}}(\eta(x))$ (2.17) by twisting the parameters (2.57). Note that it also satisfies the universal normalisation condition $\check{\xi}_{\text{v}}(0) = 1$. We call a solution $\tilde{\phi}_{\text{v}}(x) = \phi'_0(x)\check{\xi}_{\text{v}}(x)$ *a virtual state vector* if it has negative energy and positive on the entire semi-infinite lattice:

$$\mathcal{E}'_{\text{v}} < 0, \quad \check{\xi}_{\text{v}}(x) > 0 \ \ (x \in \mathbb{Z}_{\geq 0}). \tag{2.56}$$

The positivity is necessary for the well-definedness of the Darboux transformations in terms of these virtual state vectors. See, for example, (3.5). Let us emphasise that the virtual state vector $\tilde{\phi}_{\text{v}}(x)$ is also a solution of the original Schrödinger equation (2.51) with a negative energy $\tilde{\mathcal{E}}_{\text{v}} < 0$ and of infinite norm $\|\tilde{\phi}_{\text{v}}\| = \infty$. The set $\mathcal{V}$ of non-negative integers v, satisfying the above two conditions (2.56) is called the index set of the virtual state vectors.

2.2.2 virtual state vectors

In this subsection we present the explicit forms of the modified potential functions $B'(x)$, $D'(x)$ (2.42)-(2.43) and the virtual state vectors $\tilde{\phi}_{\text{v}}(x) = \phi'_0(x)\check{\xi}_{\text{v}}(x)$ (2.52) for M, lqJ and lqL polynomials through twisting.

Let us define the twist operation $\mathfrak{t}$, which is an *involution*,

$$\mathfrak{t}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \left\{ \begin{array}{ll} (\lambda_1, \lambda_2^{-1}) & : \text{M} \\ (-\lambda_1, \lambda_2) & : \text{l}q\text{J} \\ -\lambda_1 & : \text{l}q\text{L} \end{array} \right. , \quad \text{namely} \ \left\{ \begin{array}{ll} \mathfrak{t}(\boldsymbol{\lambda}) = (\beta, c^{-1}) & : \text{M} \\ q^{\mathfrak{t}(\boldsymbol{\lambda})} = (a^{-1}, b) & : \text{l}q\text{J} \\ q^{\mathfrak{t}(\boldsymbol{\lambda})} = a^{-1} & : \text{l}q\text{L} \end{array} \right. , \quad \mathfrak{t}^2 = \text{id}, \tag{2.57}$$

and the two functions $B'(x)$ and $D'(x)$,

$$B'(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} B\bigl(x;\mathfrak{t}(\boldsymbol{\lambda})\bigr), \quad D'(x) \stackrel{\text{def}}{=} D(x), \tag{2.58}$$

namely,

$$B'(x;\boldsymbol{\lambda}) = \left\{ \begin{array}{ll} c^{-1}(x+\beta) & : \text{M} \\ a^{-1}(q^{-x} - bq) & : \text{l}q\text{J} \\ a^{-1}q^{-x} & : \text{l}q\text{L} \end{array} \right. , \quad D'(x) = \left\{ \begin{array}{ll} x & : \text{M} \\ q^{-x} - 1 & : \text{l}q\text{J}, \text{l}q\text{L} \end{array} \right. . \tag{2.59}$$

Then the conditions (2.42)–(2.44) are satisfied with the following α and α' :

$$\alpha(\boldsymbol{\lambda}) = \left\{ \begin{array}{ll} c & : \text{M} \\ a & : \text{l}q\text{J}, \text{l}q\text{L} \end{array} \right. , \quad \alpha'(\boldsymbol{\lambda}) = \left\{ \begin{array}{ll} -(1-c)\beta & : \text{M} \\ -(1-a)(1-bq) & : \text{l}q\text{J} \\ -(1-a) & : \text{l}q\text{L} \end{array} \right. . \tag{2.60}$$

The virtual Hamiltonian $\mathcal{H}'$, $\mathcal{E}'_{\text{v}}$ and the virtual state vector $\tilde{\phi}_{\text{v}}(x)$ are given by $\mathcal{H}(\mathfrak{t}(\boldsymbol{\lambda}))$, $\mathcal{E}_{\text{v}}(\mathfrak{t}(\boldsymbol{\lambda}))$ and $\phi_{\text{v}}(x;\mathfrak{t}(\boldsymbol{\lambda}))$. Namely,

$$\mathcal{H}(\boldsymbol{\lambda}) = \alpha(\boldsymbol{\lambda})\mathcal{H}\bigl(\mathfrak{t}(\boldsymbol{\lambda})\bigr) + \alpha'(\boldsymbol{\lambda}), \tag{2.61}$$

$$\tilde{\phi}_{\text{v}}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \phi_{\text{v}}\bigl(x;\mathfrak{t}(\boldsymbol{\lambda})\bigr) = \tilde{\phi}_0(x;\boldsymbol{\lambda})\check{\xi}_{\text{v}}(x;\boldsymbol{\lambda}) \quad (\text{v} \in \mathcal{V}), \tag{2.62}$$

$$\phi'_0(x;\boldsymbol{\lambda}) = \tilde{\phi}_0(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \phi_0\bigl(x;\mathfrak{t}(\boldsymbol{\lambda})\bigr), \ \check{\xi}_{\text{v}}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \check{P}_{\text{v}}\bigl(x;\mathfrak{t}(\boldsymbol{\lambda})\bigr), \ \xi_{\text{v}}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \xi_{\text{v}}\bigl(\eta(x);\boldsymbol{\lambda}\bigr), \tag{2.63}$$

$$\mathcal{H}(\boldsymbol{\lambda})\tilde{\phi}_{\text{v}}(x;\boldsymbol{\lambda}) = \tilde{\mathcal{E}}_{\text{v}}(\boldsymbol{\lambda})\tilde{\phi}_{\text{v}}(x;\boldsymbol{\lambda}), \quad \mathcal{E}'_{\text{v}}(\boldsymbol{\lambda}) = \mathcal{E}_{\text{v}}\bigl(\mathfrak{t}(\boldsymbol{\lambda})\bigr), \tag{2.64}$$

$$\tilde{\mathcal{E}}_{\text{v}}(\boldsymbol{\lambda}) = \alpha(\boldsymbol{\lambda})\mathcal{E}'_{\text{v}}(\boldsymbol{\lambda}) + \alpha'(\boldsymbol{\lambda}) = \left\{ \begin{array}{ll} -(1-c)(\text{v}+\beta) & : \text{M} \\ -(1-aq^{-\text{v}})(1-bq^{\text{v}+1}) & : \text{l}q\text{J} \\ -(1-aq^{-\text{v}}) & : \text{l}q\text{L} \end{array} \right. , \tag{2.65}$$

$$\nu(x;\boldsymbol{\lambda}) = \left\{ \begin{array}{ll} c^x & : \text{M} \\ a^x & : \text{l}q\text{J}, \text{l}q\text{L} \end{array} \right. . \tag{2.66}$$

Note that $\alpha'(\boldsymbol{\lambda}) = \tilde{\mathcal{E}}_0(\boldsymbol{\lambda}) < 0$. The negative virtual state energy condition (2.56) is automatically satisfied for M case but it restricts the maximal possible degree v_{max} for a given parameter a for lqJ and lqL cases as

$$0<a<q^{\mathrm{v}_{\max}}. \tag{2.67}$$

For the positivity of $\check{\xi}_{\mathrm{v}}(x)$ (2.56), we write down $\check{\xi}_{\mathrm{v}}(x)$ explicitly. For M case, it is

$$\begin{aligned}\check{\xi}_{\mathrm{v}}(x;\boldsymbol{\lambda})&={}_2F_1\Bigl(\genfrac{}{}{0pt}{}{-\mathrm{v},\,-x}{\beta}\Bigm|1-c\Bigr)=\sum_{k=0}^{\min(\mathrm{v},x)}\frac{(-\mathrm{v},-x)_k}{(\beta)_k}\frac{(1-c)^k}{k!}\\&=\sum_{k=0}^{\min(\mathrm{v},x)}\frac{(\mathrm{v}-k+1,x-k+1)_k}{(\beta)_k}\frac{(1-c)^k}{k!},\end{aligned} \tag{2.68}$$

where we have used $(a)_k=(-1)^k(-a-k+1)_k$. Since each k-th term of the sum in the last expression is positive, the positivity of $\check{\xi}_{\mathrm{v}}(x;\boldsymbol{\lambda})$ (2.56) is shown for all non-negative integer v. For lqJ case, by using the identity ((1.13.17) in [9])

$${}_2\phi_1\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\,b}{c}\Bigm|q;z\Bigr)=(bc^{-1}q^{-n}z;q)_n\,{}_3\phi_2\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\,b^{-1}c,\,0}{c,\,b^{-1}cqz^{-1}}\Bigm|q;q\Bigr)\quad(n\in\mathbb{Z}_{\geq0}),$$

$\check{\xi}_{\mathrm{v}}(x)$ is rewritten as

$$\begin{aligned}\check{\xi}_{\mathrm{v}}(x;\boldsymbol{\lambda})&=\frac{(aq^{-\mathrm{v}};q)_{\mathrm{v}}}{(bq;q)_{\mathrm{v}}}\,{}_2\phi_1\Bigl(\genfrac{}{}{0pt}{}{q^{-\mathrm{v}},\,a^{-1}bq^{\mathrm{v}+1}}{a^{-1}q}\Bigm|q;q^{x+1}\Bigr)\\&=\frac{(aq^{-\mathrm{v}};q)_{\mathrm{v}}}{(bq;q)_{\mathrm{v}}}(bq^{x+1};q)_{\mathrm{v}}\,{}_3\phi_2\Bigl(\genfrac{}{}{0pt}{}{q^{-\mathrm{v}},\,b^{-1}q^{-\mathrm{v}},\,0}{a^{-1}q,\,b^{-1}q^{-\mathrm{v}-x}}\Bigm|q;q\Bigr)\\&=\frac{(aq^{-\mathrm{v}},bq^{x+1};q)_{\mathrm{v}}}{(bq;q)_{\mathrm{v}}}\sum_{k=0}^{\mathrm{v}}\frac{(q^{-\mathrm{v}},b^{-1}q^{-\mathrm{v}};q)_k}{(a^{-1}q,b^{-1}q^{-\mathrm{v}-x};q)_k}\frac{q^k}{(q;q)_k}\\&=\frac{(aq^{-\mathrm{v}},bq^{x+1};q)_{\mathrm{v}}}{(bq;q)_{\mathrm{v}}}\sum_{k=0}^{\mathrm{v}}\frac{(q^{\mathrm{v}-k+1},bq^{\mathrm{v}-k+1};q)_k}{(aq^{-k},bq^{\mathrm{v}-k+1+x},q;q)_k}(aq^{-\mathrm{v}+x})^k,\end{aligned} \tag{2.69}$$

where we have used $(a;q)_k=(-a)^kq^{\frac12k(k-1)}(a^{-1}q^{-k+1};q)_k$. Since each k-th term of the sum and the overall factor in the last expression is positive for $0<a<q^{\mathrm{v}}$ (2.67), the positivity of $\check{\xi}_{\mathrm{v}}(x;\boldsymbol{\lambda})$ (2.56) is shown. By setting $b=0$, the same conclusion for lqL case is obtained.

Remark For lqJ, $\check{\xi}_{\mathrm{v}}(x;\boldsymbol{\lambda})$ is generically a degree v polynomial in $\eta(x)=1-q^x$. However, when $a=bq^{m+1}$, $\mathrm{v}\leq m\in\mathbb{Z}_{\geq0}$, $(a^{-1}bq^{\mathrm{v}+1};q)_k=0$ for $k\geq m-\mathrm{v}+1$. That is, the highest degree is $m-\mathrm{v}$. To sum up, if $a=bq^{m+1}$ and $\mathrm{v}\leq m<2\mathrm{v}$, $\check{\xi}_{\mathrm{v}}(x;\boldsymbol{\lambda})$ is not a degree v polynomial. In later discussion we assume to avoid these special configurations of the parameters.

The index set of the virtual state vectors is

$$\mathcal{V}=\left\{\begin{array}{ll}\{1,2,\ldots\} & :\mathrm{M}\\ \{1,2,\ldots,\mathrm{v}_{\max}\} & :\mathrm{l}q\mathrm{J},\mathrm{l}q\mathrm{L}\end{array}\right., \tag{2.70}$$

where $\mathrm{v}_{\max}$ is the greatest integer satisfying $0 < a < q^{\mathrm{v}}$.

The non square-summability of the virtual state vector $\tilde{\phi}_{\mathrm{v}}$ ($\|\tilde{\phi}_{\mathrm{v}}\| = \infty$) can be directly verified through its large x behaviour

$$\tilde{\phi}_0(x;\boldsymbol{\lambda})^2 \simeq \left\{\begin{array}{ll} \Gamma(\beta)^{-1}x^{\beta-1}c^{-x} & : \mathrm{M} \\ (bq;q)_\infty(q;q)_\infty^{-1}(a^{-1}q)^x & : \mathrm{l}q\mathrm{J} \\ (q;q)_\infty^{-1}(a^{-1}q)^x & : \mathrm{l}q\mathrm{L} \end{array}\right., \quad \tilde{\xi}_{\mathrm{v}}(x;\boldsymbol{\lambda}) \simeq \left\{\begin{array}{ll} \text{const} \times x^{\mathrm{v}} & : \mathrm{M} \\ \text{const} & : \mathrm{l}q\mathrm{J}, \mathrm{l}q\mathrm{L} \end{array}\right. .$$

Before closing this section, let us emphasise that the twisting is based on the discrete symmetries of the difference Schrödinger equations (2.9), not of the equations governing the polynomials (2.16)-(2.17), as evidenced by the linear relation between the two Hamiltonians (2.45).

3 Method of Virtual State Deletion

We present the method of virtual states deletion for the discrete quantum mechanics with real shifts (rdQM) in terms of the discrete QM analogue of the multiple Darboux transformations [20, 11]. The concept of the virtual states on a semi-infinite lattice is slightly different from those on finite lattices as introduced in [11]. On semi-infinite lattices, the solutions of the Schrödinger (eigenvalue) equation (2.9) are eigenvectors with finite norm and the other solutions with infinite norms, to which the virtual state vectors belong. In contrast, the solutions of the Schrödinger (eigenvalue) equation on finite lattices are eigenvectors only. This requires a subtle definition of the virtual state vectors as introduced in [11].

In the following discussion, the discrete analogue of the Wronskians, to be called Casoratians, play an important role. Here is a summary of useful properties of Casoratians.

3.1 Casoratian

The Casorati determinant of a set of n functions $\{f_j(x)\}$ is defined by

$$\mathrm{W}_{\mathrm{C}}[f_1,\ldots,f_n](x) \stackrel{\text{def}}{=} \det\Bigl(f_k(x+j-1)\Bigr)_{1\le j,k\le n}, \tag{3.1}$$

(for $n=0$, we set $\mathrm{W}_{\mathrm{C}}[\cdot](x)=1$). It satisfies identities ($n \ge 0$)

$$\mathrm{W}_{\mathrm{C}}[gf_1,gf_2,\ldots,gf_n](x) = \prod_{k=0}^{n-1} g(x+k)\cdot \mathrm{W}_{\mathrm{C}}[f_1,f_2,\ldots,f_n](x), \tag{3.2}$$

$$\mathrm{W}_{\mathrm{C}}\bigl[\mathrm{W}_{\mathrm{C}}[f_1,f_2,\ldots,f_n,g],\mathrm{W}_{\mathrm{C}}[f_1,f_2,\ldots,f_n,h]\bigr](x)$$

$$= \mathrm{W_C}[f_1, f_2, \ldots, f_n](x+1)\, \mathrm{W_C}[f_1, f_2, \ldots, f_n, g, h](x), \tag{3.3}$$

$$\mathrm{W_C}[F_1, F_2, \ldots, F_n](x) = (-1)^{\frac{1}{2}n(n-1)} \prod_{k=0}^{n-2} \mathrm{W_C}[f_1, f_2, \ldots, f_n](x+k),$$
$$\text{where } F_j(x) = \mathrm{W_C}[f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_n](x). \tag{3.4}$$

The first and second identities will be used in the next subsection. The third one is not explicitly used in this paper but it is used to express the step backward shift operator in a determinant form [23].

3.2 Virtual states deletion

We delete virtual state vectors $\tilde{\phi}_{d_1}(x), \tilde{\phi}_{d_2}(x), \ldots$ (d_j : mutually distinct)[1] by the multiple application of Darboux transformations.

3.2.1 one virtual state vector deletion

First we rewrite the original Hamiltonian $\mathcal{H}$ (2.1) in a factorised form in which the first virtual state vector $\tilde{\phi}_{d_1}(x)$ ($d_1 \in \mathcal{V}$) is annihilated by $\hat{\mathcal{A}}_{d_1}$, $\hat{\mathcal{A}}_{d_1}\tilde{\phi}_{d_1}(x) = 0$:

$$\mathcal{H} = \hat{\mathcal{A}}_{d_1}^{\dagger} \hat{\mathcal{A}}_{d_1} + \tilde{\mathcal{E}}_{d_1},$$
$$\hat{\mathcal{A}}_{d_1} \stackrel{\text{def}}{=} \sqrt{\hat{B}_{d_1}(x)} - e^{\partial}\sqrt{\hat{D}_{d_1}(x)}, \quad \hat{\mathcal{A}}_{d_1}^{\dagger} = \sqrt{\hat{B}_{d_1}(x)} - \sqrt{\hat{D}_{d_1}(x)}\, e^{-\partial}.$$

Here the potential functions $\hat{B}_{d_1}(x)$ and $\hat{D}_{d_1}(x)$ are determined by $B'(x)$, $D'(x)$ and the virtual state polynomial $\check{\xi}_{d_1}(x)$:

$$\hat{B}_{d_1}(x) \stackrel{\text{def}}{=} \alpha B'(x) \frac{\check{\xi}_{d_1}(x+1)}{\check{\xi}_{d_1}(x)}, \quad \hat{D}_{d_1}(x) \stackrel{\text{def}}{=} \alpha D'(x) \frac{\check{\xi}_{d_1}(x-1)}{\check{\xi}_{d_1}(x)}. \tag{3.5}$$

We have $\hat{B}_{d_1}(x) > 0$ $(x \in \mathbb{Z}_{\geq 0})$, $\hat{D}_{d_1}(x) > 0$ $(x \in \mathbb{Z}_{\geq 1})$, $\hat{D}_{d_1}(0) = 0$ and

$$\hat{B}_{d_1}(x)\hat{D}_{d_1}(x+1) = B(x)D(x+1),$$
$$\hat{B}_{d_1}(x) + \hat{D}_{d_1}(x) + \tilde{\mathcal{E}}_{d_1} = B(x) + D(x),$$

where use is made of (2.55) in the second equation.

Next let us define a new Hamiltonian $\mathcal{H}_{d_1}$ by changing the order of the two matrices $\hat{\mathcal{A}}_{d_1}^{\dagger}$ and $\hat{\mathcal{A}}_{d_1}$ together with the sets of new eigenvectors $\phi_{d_1 n}(x)$ and new virtual state vectors

[1] Although this notation d_j conflicts with the notation of the normalisation constant d_n in (2.12), we think this does not cause any confusion because the latter appears as $\frac{1}{d_n^2}\,\delta_{nm}$.

$\tilde{\phi}_{d_1\mathrm{v}}(x)$:

$$\mathcal{H}_{d_1} \stackrel{\text{def}}{=} \hat{\mathcal{A}}_{d_1}\hat{\mathcal{A}}_{d_1}^{\dagger} + \tilde{\mathcal{E}}_{d_1}, \quad \mathcal{H}_{d_1} = (\mathcal{H}_{d_1\,x,y}) \ \ (x, y \in \mathbb{Z}_{\geq 0}), \tag{3.6}$$

$$\phi_{d_1 n}(x) \stackrel{\text{def}}{=} \hat{\mathcal{A}}_{d_1}\phi_n(x) \ \ (n \in \mathbb{Z}_{\geq 0}), \tag{3.7}$$

$$\tilde{\phi}_{d_1\mathrm{v}}(x) \stackrel{\text{def}}{=} \hat{\mathcal{A}}_{d_1}\tilde{\phi}_{\mathrm{v}}(x) \ \ (\mathrm{v} \in \mathcal{V}\backslash\{d_1\}). \tag{3.8}$$

It is easy to verify that $\phi_{d_1 n}(x)$ is an eigenvector and that $\tilde{\phi}_{d_1\mathrm{v}}(x)$ is a virtual state vector

$$\mathcal{H}_{d_1}\phi_{d_1 n}(x) = \mathcal{E}_n\phi_{d_1 n}(x) \ \ (n \in \mathbb{Z}_{\geq 0}), \quad \mathcal{H}_{d_1}\tilde{\phi}_{d_1\mathrm{v}}(x) = \tilde{\mathcal{E}}_{\mathrm{v}}\tilde{\phi}_{d_1\mathrm{v}}(x) \ \ (\mathrm{v} \in \mathcal{V}\backslash\{d_1\}).$$

For example,

$$\begin{aligned}\mathcal{H}_{d_1}\phi_{d_1 n} &= (\hat{\mathcal{A}}_{d_1}\hat{\mathcal{A}}_{d_1}^{\dagger} + \tilde{\mathcal{E}}_{d_1})\hat{\mathcal{A}}_{d_1}\phi_n = \hat{\mathcal{A}}_{d_1}(\hat{\mathcal{A}}_{d_1}^{\dagger}\hat{\mathcal{A}}_{d_1} + \tilde{\mathcal{E}}_{d_1})\phi_n \\ &= \hat{\mathcal{A}}_{d_1}\mathcal{H}\phi_n = \hat{\mathcal{A}}_{d_1}\mathcal{E}_n\phi_n = \mathcal{E}_n\hat{\mathcal{A}}_{d_1}\phi_n = \mathcal{E}_n\phi_{d_1 n}.\end{aligned}$$

The two Hamiltonians $\mathcal{H}$ and $\mathcal{H}_{d_1}$ are *exactly iso-spectral*. If the original system is exactly solvable, this new system is also exactly solvable. The orthogonality relation for the new eigenvectors is

$$\begin{aligned}(\phi_{d_1 n}, \phi_{d_1 m}) &= (\hat{\mathcal{A}}_{d_1}\phi_n, \hat{\mathcal{A}}_{d_1}\phi_m) = (\hat{\mathcal{A}}_{d_1}^{\dagger}\hat{\mathcal{A}}_{d_1}\phi_n, \phi_m) = \big((\mathcal{H} - \tilde{\mathcal{E}}_{d_1})\phi_n, \phi_m\big) \\ &= (\mathcal{E}_n - \tilde{\mathcal{E}}_{d_1})(\phi_n, \phi_m) = (\mathcal{E}_n - \tilde{\mathcal{E}}_{d_1})\frac{\delta_{nm}}{d_n^2} \ \ (n, m \in \mathbb{Z}_{\geq 0}).\end{aligned}$$

The second equality requires the condition $\lim_{N\to\infty}\sqrt{\hat{D}_{d_1}(N+1)}\,\phi_{d_1\,n}(N)\phi_m(N+1) = 0$, which is easily verified for the three systems (M, lqJ and lqL). This shows clearly that the *negative* virtual state energy ($\tilde{\mathcal{E}}_{\mathrm{v}} < 0$) is necessary for the positivity of the inner products.

The new eigenvector $\phi_{d_1 n}(x)$ (3.7) and the virtual state vector $\tilde{\phi}_{d_1\mathrm{v}}(x)$ (3.8) are expressed neatly in terms of the Casoratians:

$$\phi_{d_1 n}(x) = \frac{-\sqrt{\alpha B'(x)}\,\phi_0'(x)\mathrm{W_C}[\check{\xi}_{d_1}, \nu\check{P}_n](x)}{\sqrt{\check{\xi}_{d_1}(x)\check{\xi}_{d_1}(x+1)}}, \quad \tilde{\phi}_{d_1\mathrm{v}}(x) = \frac{-\sqrt{\alpha B'(x)}\,\phi_0'(x)\mathrm{W_C}[\check{\xi}_{d_1}, \check{\xi}_{\mathrm{v}}](x)}{\sqrt{\check{\xi}_{d_1}(x)\check{\xi}_{d_1}(x+1)}}. \tag{3.9}$$

The positivity of the virtual state vectors $\tilde{\phi}_{\mathrm{v}}(x)$ is inherited by the new virtual state vectors $\tilde{\phi}_{d_1\mathrm{v}}(x)$ (3.8). The Casoratian $\mathrm{W_C}[\check{\xi}_{d_1}, \check{\xi}_{\mathrm{v}}](x)$ has definite sign for $x \in \mathbb{Z}_{\geq 0}$, namely all positive or all negative. By using (2.55) we have

$$\alpha B'(x)\mathrm{W_C}[\check{\xi}_{d_1}, \check{\xi}_{\mathrm{v}}](x) = \alpha D'(x)\mathrm{W_C}[\check{\xi}_{d_1}, \check{\xi}_{\mathrm{v}}](x-1) + (\tilde{\mathcal{E}}_{d_1} - \tilde{\mathcal{E}}_{\mathrm{v}})\check{\xi}_{d_1}(x)\check{\xi}_{\mathrm{v}}(x).$$

At $x=0$, $\mathrm{W_C}[\check{\xi}_{d_1},\check{\xi}_{\rm v}](0)$ has the same sign as $(\tilde{\mathcal{E}}_{d_1}-\tilde{\mathcal{E}}_{\rm v})$, as $D'(0)=0$, $\alpha B'(0)>0$ and $\check{\xi}_{d_1}(0)\check{\xi}_{\rm v}(0)=1$. By setting $x=1,2,\ldots$ in turn, we obtain

$$\mathrm{sgn}(\tilde{\mathcal{E}}_{d_1}-\tilde{\mathcal{E}}_{\rm v})\mathrm{W_C}[\check{\xi}_{d_1},\check{\xi}_{\rm v}](x)>0 \quad (x\in\mathbb{Z}_{\geq 0}).$$

The new ground state eigenvector $\phi_{d_10}(x)$ is of definite sign as the original one $\phi_0(x)$ (2.11). We show that the Casoratian $\mathrm{W_C}[\check{\xi}_{d_1},\nu](x)$ has definite sign for $x\in\mathbb{Z}_{\geq 0}$. By writing down the equation $\mathcal{H}\phi_n(x)=\mathcal{E}_n\phi_n(x)$ with $\mathcal{H}=\hat{\mathcal{A}}_{d_1}^{\dagger}\hat{\mathcal{A}}_{d_1}+\tilde{\mathcal{E}}_{d_1}$, we obtain

$$\alpha B'(x)\bigl(\nu\check{P}_n\bigr)(x+1)+\alpha D'(x)\bigl(\nu\check{P}_n\bigr)(x-1)=\bigl(\alpha B'(x)+\alpha D'(x)+\alpha'-\mathcal{E}_n\bigr)\bigl(\nu\check{P}_n\bigr)(x), \tag{3.10}$$

where $\bigl(fg\bigr)(x)\stackrel{\text{def}}{=}f(x)g(x)$. By using this, we can show

$$\alpha B'(x)\mathrm{W_C}[\check{\xi}_{d_1},\nu\check{P}_n](x)=\alpha D'(x)\mathrm{W_C}[\check{\xi}_{d_1},\nu\check{P}_n](x-1)+(\tilde{\mathcal{E}}_{d_1}-\mathcal{E}_n)\check{\xi}_{d_1}(x)\nu(x)\check{P}_n(x).$$

By setting $n=0$ and $x=0,1,2,\ldots$ in turn, we obtain

$$-\mathrm{W_C}[\check{\xi}_{d_1},\nu](x)>0 \quad (x\in\mathbb{Z}_{\geq 0}).$$

Let us rewrite the deformed Hamiltonian $\mathcal{H}_{d_1}$ in the standard form based on the ground state eigenvector $\phi_{d_10}(x)$. Let us introduce the potential functions $B_{d_1}(x)$ and $D_{d_1}(x)$ by

$$B_{d_1}(x)\stackrel{\text{def}}{=}\alpha B'(x+1)\frac{\check{\xi}_{d_1}(x)}{\check{\xi}_{d_1}(x+1)}\frac{\mathrm{W_C}[\check{\xi}_{d_1},\nu](x+1)}{\mathrm{W_C}[\check{\xi}_{d_1},\nu](x)}, \tag{3.11}$$

$$D_{d_1}(x)\stackrel{\text{def}}{=}\alpha D'(x)\frac{\check{\xi}_{d_1}(x+1)}{\check{\xi}_{d_1}(x)}\frac{\mathrm{W_C}[\check{\xi}_{d_1},\nu](x-1)}{\mathrm{W_C}[\check{\xi}_{d_1},\nu](x)}. \tag{3.12}$$

The positivity of $B_{d_1}(x)$ and $D_{d_1}(x)$ is obvious and the boundary condition $D_{d_1}(0)=0$ is satisfied. They satisfy the relations

$$B_{d_1}(x)D_{d_1}(x+1)=\hat{B}_{d_1}(x+1)\hat{D}_{d_1}(x+1),$$
$$B_{d_1}(x)+D_{d_1}(x)=\hat{B}_{d_1}(x)+\hat{D}_{d_1}(x+1)+\tilde{\mathcal{E}}_{d_1}.$$

The standard form Hamiltonian $\mathcal{H}_{d_1}$ is obtained:

$$\mathcal{H}_{d_1}=\mathcal{A}_{d_1}^{\dagger}\mathcal{A}_{d_1}, \tag{3.13}$$

$$\mathcal{A}_{d_1}\stackrel{\text{def}}{=}\sqrt{B_{d_1}(x)}-e^{\partial}\sqrt{D_{d_1}(x)},\quad \mathcal{A}_{d_1}^{\dagger}=\sqrt{B_{d_1}(x)}-\sqrt{D_{d_1}(x)}\,e^{-\partial}, \tag{3.14}$$

in which $\mathcal{A}_{d_1}$ annihilates the ground state eigenvector, $\mathcal{A}_{d_1}\phi_{d_10}(x)=0$.

It is worthwhile to emphasise that the new Hamiltonian $\mathcal{H}_{d_1}$ is constructed by using the virtual state vector $\tilde{\phi}_{d_1}(x)$, which is now deleted from the new index set of available virtual state vectors $\mathcal{V}\backslash\{d_1\}$.

3.2.2 multi virtual state vectors deletion

We repeat the above procedure and obtain the modified systems. The number of deleted virtual state vectors should be less than or equal $|\mathcal{V}|$.

For simplicity in notation, we introduce the following quantities ($s \geq 0$):

$$w_s(x) \stackrel{\text{def}}{=} \mathrm{W_C}[\check{\xi}_{d_1}, \ldots, \check{\xi}_{d_s}](x), \tag{3.15}$$

$$w'_{s,\mathrm{v}}(x) \stackrel{\text{def}}{=} \mathrm{W_C}[\check{\xi}_{d_1}, \ldots, \check{\xi}_{d_s}, \check{\xi}_{\mathrm{v}}](x), \tag{3.16}$$

$$w''_{s,n}(x) \stackrel{\text{def}}{=} \mathrm{W_C}[\check{\xi}_{d_1}, \ldots, \check{\xi}_{d_s}, \nu\check{P}_n](x). \tag{3.17}$$

Note that $w'_{s,d_{s+1}}(x) = w_{s+1}(x)$.

Let us assume that we have already deleted s virtual state vectors ($s \geq 1$), which are labeled by $\{d_1, \ldots, d_s\}$ ($d_j \in \mathcal{V}$: mutually distinct). Namely we have

$$\mathcal{H}_{d_1\ldots d_s} \stackrel{\text{def}}{=} \hat{\mathcal{A}}_{d_1\ldots d_s}\hat{\mathcal{A}}^{\dagger}_{d_1\ldots d_s} + \tilde{\mathcal{E}}_{d_s}, \quad \mathcal{H}_{d_1\ldots d_s} = (\mathcal{H}_{d_1\ldots d_s\,x,y}) \quad (x, y \in \mathbb{Z}_{\geq 0}), \tag{3.18}$$

$$\hat{\mathcal{A}}_{d_1\ldots d_s} \stackrel{\text{def}}{=} \sqrt{\hat{B}_{d_1\ldots d_s}(x)} - e^{\partial}\sqrt{\hat{D}_{d_1\ldots d_s}(x)}, \quad \hat{\mathcal{A}}^{\dagger}_{d_1\ldots d_s} = \sqrt{\hat{B}_{d_1\ldots d_s}(x)} - \sqrt{\hat{D}_{d_1\ldots d_s}(x)}\, e^{-\partial}, \tag{3.19}$$

$$\hat{B}_{d_1\ldots d_s}(x) \stackrel{\text{def}}{=} \alpha B'(x+s-1)\frac{w_{s-1}(x)}{w_{s-1}(x+1)}\frac{w_s(x+1)}{w_s(x)}, \tag{3.20}$$

$$\hat{D}_{d_1\ldots d_s}(x) \stackrel{\text{def}}{=} \alpha D'(x)\frac{w_{s-1}(x+1)}{w_{s-1}(x)}\frac{w_s(x-1)}{w_s(x)}, \tag{3.21}$$

$$\phi_{d_1\ldots d_s\,n}(x) \stackrel{\text{def}}{=} \hat{\mathcal{A}}_{d_1\ldots d_s}\phi_{d_1\ldots d_{s-1}\,n}(x) \quad (n \in \mathbb{Z}_{\geq 0}), \tag{3.22}$$

$$\tilde{\phi}_{d_1\ldots d_s\,\mathrm{v}}(x) \stackrel{\text{def}}{=} \hat{\mathcal{A}}_{d_1\ldots d_s}\tilde{\phi}_{d_1\ldots d_{s-1}\,\mathrm{v}}(x) \quad (\mathrm{v} \in \mathcal{V}\backslash\{d_1, \ldots, d_s\}), \tag{3.23}$$

$$\mathcal{H}_{d_1\ldots d_s}\phi_{d_1\ldots d_s\,n}(x) = \mathcal{E}_n\phi_{d_1\ldots d_s\,n}(x) \quad (n \in \mathbb{Z}_{\geq 0}), \tag{3.24}$$

$$\mathcal{H}_{d_1\ldots d_s}\tilde{\phi}_{d_1\ldots d_s\,\mathrm{v}}(x) = \tilde{\mathcal{E}}_{\mathrm{v}}\tilde{\phi}_{d_1\ldots d_s\,\mathrm{v}}(x) \quad (\mathrm{v} \in \mathcal{V}\backslash\{d_1, \ldots, d_s\}), \tag{3.25}$$

$$(\phi_{d_1\ldots d_s\,n}, \phi_{d_1\ldots d_s\,m}) = \prod_{j=1}^{s}(\mathcal{E}_n - \tilde{\mathcal{E}}_{d_j}) \cdot \frac{\delta_{nm}}{d_n^2} \quad (n, m \in \mathbb{Z}_{\geq 0}). \tag{3.26}$$

The eigenvectors and the virtual state vectors have Casoratian expressions:

$$\phi_{d_1\ldots d_s\,n}(x) = \mathcal{S}_{d_1\ldots d_s}\frac{\sqrt{\prod_{j=1}^{s}\alpha B'(x+j-1)}\,\phi'_0(x)}{\sqrt{w_s(x)w_s(x+1)}}\,w''_{s,n}(x), \tag{3.27}$$

$$\tilde{\phi}_{d_1\ldots d_s\,\mathrm{v}}(x) = \mathcal{S}_{d_1\ldots d_s}\frac{\sqrt{\prod_{j=1}^{s}\alpha B'(x+j-1)}\,\phi'_0(x)}{\sqrt{w_s(x)w_s(x+1)}}\,w'_{s,\mathrm{v}}(x), \tag{3.28}$$

where the sign factor $\mathcal{S}_{d_1\ldots d_s}$ is

$$\mathcal{S}_{d_1\ldots d_s} = (-1)^s\prod_{1\leq i<j\leq s}\mathrm{sgn}(\tilde{\mathcal{E}}_{d_i} - \tilde{\mathcal{E}}_{d_j}). \tag{3.29}$$

The Casoratians of multiple virtual state vectors $w_s(x)$, $w'_{s,\text{v}}(x)$ and with the ground state eigenvector $w''_{s,0}(x)$ (3.15)–(3.17) are of definite sign

$$\left.\begin{array}{r}\displaystyle\prod_{1\le i<j\le s}\text{sgn}(\tilde{\mathcal{E}}_{d_i}-\tilde{\mathcal{E}}_{d_j})\cdot w_s(x)>0\\ \displaystyle\prod_{1\le i<j\le s}\text{sgn}(\tilde{\mathcal{E}}_{d_i}-\tilde{\mathcal{E}}_{d_j})\cdot\prod_{i=1}^{s}\text{sgn}(\tilde{\mathcal{E}}_{d_i}-\tilde{\mathcal{E}}_{\text{v}})\cdot w'_{s,\text{v}}(x)>0\\ \displaystyle\prod_{1\le i<j\le s}\text{sgn}(\tilde{\mathcal{E}}_{d_i}-\tilde{\mathcal{E}}_{d_j})\cdot(-1)^s w''_{s,0}(x)>0\end{array}\right\}\quad(x\in\mathbb{Z}_{\ge0}). \tag{3.30}$$

So we have $\hat{B}_{d_1\ldots d_s}(x)>0$ $(x\in\mathbb{Z}_{\ge0})$, $\hat{D}_{d_1\ldots d_s}(x)>0$ $(x\in\mathbb{Z}_{\ge1})$ and $\hat{D}_{d_1\ldots d_s}(0)=0$. For (3.26), the asymptotic condition

$$\lim_{N\to\infty}\sqrt{\hat{D}_{d_1\ldots d_s}(N+1)}\,\phi_{d_1\ldots d_s\,n}(N)\phi_{d_1\ldots d_{s-1}\,m}(N+1)=0, \tag{3.31}$$

is necessary. This can be easily verified for M, lqJ and lqL systems.

By writing down (3.25) and (3.24) explicitly, we obtain the following identities

$$\begin{aligned}&\hat{B}_{d_1\ldots d_s}(x)+\hat{D}_{d_1\ldots d_s}(x+1)+\tilde{\mathcal{E}}_{d_s}-\tilde{\mathcal{E}}_{\text{v}}\\&=\alpha B'(x+s)\frac{w_s(x)}{w_s(x+1)}\frac{w'_{s,\text{v}}(x+1)}{w'_{s,\text{v}}(x)}+\alpha D'(x)\frac{w_s(x+1)}{w_s(x)}\frac{w'_{s,\text{v}}(x-1)}{w'_{s,\text{v}}(x)}, &(3.32)\\&\hat{B}_{d_1\ldots d_s}(x)+\hat{D}_{d_1\ldots d_s}(x+1)+\tilde{\mathcal{E}}_{d_s}-\mathcal{E}_n\\&=\alpha B'(x+s)\frac{w_s(x)}{w_s(x+1)}\frac{w''_{s,n}(x+1)}{w''_{s,n}(x)}+\alpha D'(x)\frac{w_s(x+1)}{w_s(x)}\frac{w''_{s,n}(x-1)}{w''_{s,n}(x)}, &(3.33)\end{aligned}$$

which are valid for any $x(\in\mathbb{C})$. Note that (3.32)-(3.33) are valid for $s=0$ by replacing $\hat{B}_{d_1\ldots d_s}(x)+\hat{D}_{d_1\ldots d_s}(x+1)+\tilde{\mathcal{E}}_{d_s}$ with $\alpha B'(x)+\alpha D'(x)+\alpha'$. The identity (3.3) gives

$$w_s(x+1)w''_{s+1,n}(x)=\text{W}_{\text{C}}[w_{s+1},w''_{s,n}](x),\quad w_s(x+1)w'_{s+1,\text{v}}(x)=\text{W}_{\text{C}}[w_{s+1},w'_{s,\text{v}}](x). \tag{3.34}$$

By using (3.32)–(3.34), we obtain the following identities $(s\ge0)$

$$\begin{aligned}&\alpha B'(x+s)w_s(x)w'_{s+1,\text{v}}(x)\\&=\alpha D'(x)w_s(x+1)w'_{s+1,\text{v}}(x-1)+(\tilde{\mathcal{E}}_{d_{s+1}}-\tilde{\mathcal{E}}_{\text{v}})w_{s+1}(x)w'_{s,\text{v}}(x), &(3.35)\\&\alpha B'(x+s)w_s(x)w''_{s+1,n}(x)\\&=\alpha D'(x)w_s(x+1)w'_{s+1,n}(x-1)+(\tilde{\mathcal{E}}_{d_{s+1}}-\mathcal{E}_n)w_{s+1}(x)w''_{s,n}(x), &(3.36)\end{aligned}$$

which are valid for any $x(\in\mathbb{C})$.

The next step begins with rewriting the Hamiltonian $\mathcal{H}_{d_1\ldots d_s}$ by choosing the next virtual state to be deleted $d_{s+1}\in\mathcal{V}\backslash\{d_1,\ldots,d_s\}$. The new potential functions $\hat{B}_{d_1\ldots d_{s+1}}(x)$ and

$\hat{D}_{d_1\ldots d_{s+1}}(x)$ are defined as in (3.20)–(3.21) by $s\to s+1$, and they contain the Casoratians (3.15)–(3.16) as ratios (recall $w'_{s,d_{s+1}}(x)=w_{s+1}(x)$). These Casoratians are of the same sign as shown in (3.30), the new potential functions are positive $\hat{B}_{d_1\ldots d_{s+1}}(x)>0$ ($x\in\mathbb{Z}_{\geq 0}$), $\hat{D}_{d_1\ldots d_{s+1}}(x)>0$ ($x\in\mathbb{Z}_{\geq 1}$) and $\hat{D}_{d_1\ldots d_{s+1}}(0)=0$. By using (3.32), we can show the relations

$$\hat{B}_{d_1\ldots d_{s+1}}(x)\hat{D}_{d_1\ldots d_{s+1}}(x+1)=\hat{B}_{d_1\ldots d_s}(x+1)\hat{D}_{d_1\ldots d_s}(x+1),$$
$$\hat{B}_{d_1\ldots d_{s+1}}(x)+\hat{D}_{d_1\ldots d_{s+1}}(x)+\tilde{\mathcal{E}}_{d_{s+1}}=\hat{B}_{d_1\ldots d_s}(x)+\hat{D}_{d_1\ldots d_s}(x+1)+\tilde{\mathcal{E}}_{d_s}.$$

Therefore the Hamiltonian $\mathcal{H}_{d_1\ldots d_s}$ is rewritten as:

$$\mathcal{H}_{d_1\ldots d_s}=\hat{\mathcal{A}}^{\dagger}_{d_1\ldots d_{s+1}}\hat{\mathcal{A}}_{d_1\ldots d_{s+1}}+\tilde{\mathcal{E}}_{d_{s+1}},$$
$$\hat{\mathcal{A}}_{d_1\ldots d_{s+1}}\stackrel{\text{def}}{=}\sqrt{\hat{B}_{d_1\ldots d_{s+1}}(x)}-e^{\partial}\sqrt{\hat{D}_{d_1\ldots d_{s+1}}(x)},\ \ \hat{\mathcal{A}}^{\dagger}_{d_1\ldots d_{s+1}}=\sqrt{\hat{B}_{d_1\ldots d_{s+1}}(x)}-\sqrt{\hat{D}_{d_1\ldots d_{s+1}}(x)}\,e^{-\partial}.$$

Now let us define a new Hamiltonian $\mathcal{H}_{d_1\ldots d_{s+1}}$ by changing the orders of $\hat{\mathcal{A}}^{\dagger}_{d_1\ldots d_{s+1}}$ and $\hat{\mathcal{A}}_{d_1\ldots d_{s+1}}$ together with the eigenvectors $\phi_{d_1\ldots d_{s+1}\,n}(x)$ and the virtual state vectors $\tilde{\phi}_{d_1\ldots d_{s+1}\,\mathrm{v}}(x)$:

$$\mathcal{H}_{d_1\ldots d_{s+1}}\stackrel{\text{def}}{=}\hat{\mathcal{A}}_{d_1\ldots d_{s+1}}\hat{\mathcal{A}}^{\dagger}_{d_1\ldots d_{s+1}}+\tilde{\mathcal{E}}_{d_{s+1}},\quad \mathcal{H}_{d_1\ldots d_{s+1}}=(\mathcal{H}_{d_1\ldots d_{s+1}\,x,y})\ \ (x,y\in\mathbb{Z}_{\geq 0}),$$
$$\phi_{d_1\ldots d_{s+1}\,n}(x)\stackrel{\text{def}}{=}\hat{\mathcal{A}}_{d_1\ldots d_{s+1}}\phi_{d_1\ldots d_s\,n}(x)\ \ (n\in\mathbb{Z}_{\geq 0}),$$
$$\tilde{\phi}_{d_1\ldots d_{s+1}\,\mathrm{v}}(x)\stackrel{\text{def}}{=}\hat{\mathcal{A}}_{d_1\ldots d_{s+1}}\tilde{\phi}_{d_1\ldots d_s\,\mathrm{v}}(x)\ \ (\mathrm{v}\in\mathcal{V}\backslash\{d_1,\ldots,d_{s+1}\}).$$

It is easy to show that they satisfy (3.24)–(3.26) with $s\to s+1$. By using (3.34), we can show that the functions $\phi_{d_1\ldots d_{s+1}\,n}(x)$ and $\tilde{\phi}_{d_1\ldots d_{s+1}\,\mathrm{v}}(x)$ are expressed as (3.27)–(3.28) with $s\to s+1$ and $\mathcal{S}_{d_1\ldots d_{s+1}}$ satisfies

$$\mathcal{S}_{d_1\ldots d_{s+1}}=-\mathcal{S}_{d_1\ldots d_s}\prod_{i=1}^{s}\mathrm{sgn}(\tilde{\mathcal{E}}_{d_i}-\tilde{\mathcal{E}}_{d_{s+1}}),\tag{3.37}$$

which is consistent with (3.29) and the initial value $\mathcal{S}_{d_1}=-1$, see (3.9).

The identities (3.35)–(3.36) will be used to show that the Casoratians $w'_{s+1,\mathrm{v}}(x)$ and $w''_{s+1,0}(x)$ do not change sign for $x\in\mathbb{Z}_{\geq 0}$, in the same manner as below (3.9). These establish the $s+1$ case.

At the end of this subsection we present this deformed Hamiltonian $\mathcal{H}_{d_1\ldots d_s}$ in the standard form, in which the $\mathcal{A}$ operator annihilates the ground state eigenvector:

$$\mathcal{H}_{d_1\ldots d_s}=\mathcal{A}^{\dagger}_{d_1\ldots d_s}\mathcal{A}_{d_1\ldots d_s},\tag{3.38}$$
$$\mathcal{A}_{d_1\ldots d_s}\stackrel{\text{def}}{=}\sqrt{B_{d_1\ldots d_s}(x)}-e^{\partial}\sqrt{D_{d_1\ldots d_s}(x)},\quad \mathcal{A}^{\dagger}_{d_1\ldots d_s}=\sqrt{B_{d_1\ldots d_s}(x)}-\sqrt{D_{d_1\ldots d_s}(x)}\,e^{-\partial},\tag{3.39}$$

which satisfies $\mathcal{A}_{d_1\ldots d_s}\phi_{d_1\ldots d_s\,0}(x)=0$. The potential functions $B_{d_1\ldots d_s}(x)$ and $D_{d_1\ldots d_s}(x)$ are:

$$B_{d_1\ldots d_s}(x)\stackrel{\text{def}}{=}\alpha B'(x+s)\frac{w_s(x)}{w_s(x+1)}\frac{w''_{s,0}(x+1)}{w''_{s,0}(x)},\tag{3.40}$$

$$D_{d_1\ldots d_s}(x)\stackrel{\text{def}}{=}\alpha D'(x)\frac{w_s(x+1)}{w_s(x)}\frac{w''_{s,0}(x-1)}{w''_{s,0}(x)}.\tag{3.41}$$

We have $B_{d_1\ldots d_s}(x)>0$ ($x\in\mathbb{Z}_{\geq 0}$), $D_{d_1\ldots d_s}(x)>0$ ($x\in\mathbb{Z}_{\geq 1}$) and $D_{d_1\ldots d_s}(0)=0$. By using (3.33) with $n=0$, we can show the relations

$$B_{d_1\ldots d_s}(x)D_{d_1\ldots d_s}(x+1)=\hat{B}_{d_1\ldots d_s}(x+1)\hat{D}_{d_1\ldots d_s}(x+1),$$
$$B_{d_1\ldots d_s}(x)+D_{d_1\ldots d_s}(x)=\hat{B}_{d_1\ldots d_s}(x)+\hat{D}_{d_1\ldots d_s}(x+1)+\tilde{\mathcal{E}}_{d_s}.$$

It should be stressed that the above results after s-deletions are independent of the orders of deletions ($\phi_{d_1\ldots d_s\,n}(x)$ and $\tilde{\phi}_{d_1\ldots d_s\,\mathrm{v}}(x)$ may change sign).

4 Multi-indexed Orthogonal Polynomials

In this section we apply the method of virtual state deletions to the exactly solvable systems whose eigenstates are described by the Meixner, little q-Jacobi, and little q-Laguerre polynomials. We delete M virtual state vectors labeled by

$$\mathcal{D}=\{d_1,d_2,\ldots,d_M\}\quad(d_j\in\mathcal{V}:\text{mutually distinct}),\tag{4.1}$$

and denote $\mathcal{H}_{d_1\ldots d_M}$, $\phi_{d_1\ldots d_M\,n}$, $\mathcal{A}_{d_1\ldots d_M}$, etc. by $\mathcal{H}_{\mathcal{D}}$, $\phi_{\mathcal{D}\,n}$, $\mathcal{A}_{\mathcal{D}}$, etc. The Schrödinger equation for the multi-indexed Meixner, little q-Jacobi, and little q-Laguerre systems reads

$$\mathcal{H}_{\mathcal{D}}(\boldsymbol{\lambda})\phi_{\mathcal{D}\,n}(x;\boldsymbol{\lambda})=\mathcal{E}_n(\boldsymbol{\lambda})\phi_{\mathcal{D}\,n}(x;\boldsymbol{\lambda})\quad(n=0,1,\ldots).\tag{4.2}$$

Without loss of generality, we assume $1\leq d_1<d_2<\cdots<d_M$. As with the ordinary orthogonal polynomials, the index n gives the *number of nodes* of $\phi_{\mathcal{D}\,n}(x;\boldsymbol{\lambda})$ in $x\in(0,\infty)$.

Let us denote the eigenvector $\phi_{\mathcal{D}\,n}(x;\boldsymbol{\lambda})$ in (3.27) after M-deletions ($s=M$) by $\phi^{\text{gen}}_{\mathcal{D}\,n}(x;\boldsymbol{\lambda})$. We define two polynomials $\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda})$ and $\check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda})$, to be called the denominator polynomial and the *multi-indexed orthogonal polynomial*, respectively, from the Casoratians as follows:

$$\mathrm{W}_{\mathrm{C}}[\check{\xi}_{d_1},\ldots,\check{\xi}_{d_M}](x;\boldsymbol{\lambda})=\mathcal{C}_{\mathcal{D}}(\boldsymbol{\lambda})\varphi_M(x)\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda}),\tag{4.3}$$

$$\mathrm{W}_{\mathrm{C}}[\check{\xi}_{d_1},\ldots,\check{\xi}_{d_M},\nu\check{P}_n](x;\boldsymbol{\lambda})=\mathcal{C}_{\mathcal{D},n}(\boldsymbol{\lambda})\varphi_{M+1}(x)\check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda})\nu(x;\boldsymbol{\lambda}+M\tilde{\boldsymbol{\delta}}),\tag{4.4}$$

$$\tilde{\boldsymbol{\delta}} \stackrel{\text{def}}{=} \begin{cases} (1,0) & :\text{M} \\ (-1,1) & :\text{lqJ} \\ -1 & :\text{lqL} \end{cases}, \qquad \mathfrak{t}(\boldsymbol{\lambda}) + u\boldsymbol{\delta} = \mathfrak{t}(\boldsymbol{\lambda} + u\tilde{\boldsymbol{\delta}}) \quad (\forall u \in \mathbb{R}). \tag{4.5}$$

The constants $\mathcal{C}_{\mathcal{D}}(\boldsymbol{\lambda})$ and $\mathcal{C}_{\mathcal{D},n}(\boldsymbol{\lambda})$ are specified later. The auxiliary function $\varphi_M(x)$ is defined by [20]:

$$\begin{aligned} \varphi_M(x) &\stackrel{\text{def}}{=} \prod_{1\leq j<k\leq M} \frac{\eta(x+k-1)-\eta(x+j-1)}{\eta(k-j)} \qquad \big(\varphi_0(x)=\varphi_1(x)=1\big) \\ &= \prod_{1\leq j<k\leq M} \varphi(x+j-1) = \begin{cases} 1 & :\text{M} \\ q^{\frac{1}{2}M(M-1)x+\frac{1}{6}M(M-1)(M-2)} & :\text{lqJ, lqL} \end{cases}. \end{aligned} \tag{4.6}$$

The eigenvector (3.27) is rewritten as

$$\begin{aligned} \phi^{\text{gen}}_{\mathcal{D}\,n}(x;\boldsymbol{\lambda}) &= (-1)^M \kappa^{\frac{1}{4}M(M-1)} \frac{\mathcal{C}_{\mathcal{D},n}(\boldsymbol{\lambda})}{\mathcal{C}_{\mathcal{D}}(\boldsymbol{\lambda})} \sqrt{\prod_{j=1}^{M} \alpha(\boldsymbol{\lambda}) B'\big(0;\boldsymbol{\lambda}+(j-1)\tilde{\boldsymbol{\delta}}\big)} \\ &\quad \times \frac{\phi_0(x;\boldsymbol{\lambda}+M\tilde{\boldsymbol{\delta}})}{\sqrt{\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda})\check{\Xi}_{\mathcal{D}}(x+1;\boldsymbol{\lambda})}} \check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda}). \end{aligned} \tag{4.7}$$

The denominator polynomial $\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda})$ (4.3) and the multi-indexed orthogonal polynomial $\check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda})$ (4.4) are polynomials in η

$$\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \Xi_{\mathcal{D}}\big(\eta(x);\boldsymbol{\lambda}\big), \quad \check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_{\mathcal{D},n}\big(\eta(x);\boldsymbol{\lambda}\big), \tag{4.8}$$

and their *degrees* are generically $\ell_{\mathcal{D}}$ and $\ell_{\mathcal{D}}+n$, respectively. Here $\ell_{\mathcal{D}}$ is

$$\ell_{\mathcal{D}} \stackrel{\text{def}}{=} \sum_{j=1}^{M} d_j - \tfrac{1}{2}M(M-1). \tag{4.9}$$

We adopt the universal normalisation for $\check{\Xi}_{\mathcal{D}}$ and $\check{P}_{\mathcal{D},n}$: $\check{\Xi}_{\mathcal{D}}(0;\boldsymbol{\lambda})=1$, $\check{P}_{\mathcal{D},n}(0;\boldsymbol{\lambda})=1$, which determine the constants $\mathcal{C}_{\mathcal{D}}(\boldsymbol{\lambda})$ and $\mathcal{C}_{\mathcal{D},n}(\boldsymbol{\lambda})$ (convention: $\prod_{1\leq j<k\leq M} * = 1$ for $M=1$),

$$\mathcal{C}_{\mathcal{D}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{1}{\varphi_M(0)} \prod_{1\leq j<k\leq M} \frac{\tilde{\mathcal{E}}_{d_j}(\boldsymbol{\lambda})-\tilde{\mathcal{E}}_{d_k}(\boldsymbol{\lambda})}{\alpha(\boldsymbol{\lambda})B'(j-1;\boldsymbol{\lambda})}, \tag{4.10}$$

$$\mathcal{C}_{\mathcal{D},n}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (-1)^M \mathcal{C}_{\mathcal{D}}(\boldsymbol{\lambda}) \tilde{d}_{\mathcal{D},n}(\boldsymbol{\lambda})^2, \quad \tilde{d}_{\mathcal{D},n}(\boldsymbol{\lambda})^2 \stackrel{\text{def}}{=} \frac{\varphi_M(0)}{\varphi_{M+1}(0)} \prod_{j=1}^{M} \frac{\mathcal{E}_n(\boldsymbol{\lambda})-\tilde{\mathcal{E}}_{d_j}(\boldsymbol{\lambda})}{\alpha(\boldsymbol{\lambda})B'(j-1;\boldsymbol{\lambda})}. \tag{4.11}$$

The denominator polynomial $\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda})$ is positive for $x \in \mathbb{Z}_{\geq 0}$. The lowest degree multi-indexed orthogonal polynomial $\check{P}_{\mathcal{D},0}(x;\boldsymbol{\lambda})$ is related to $\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda})$ by the parameter shift $\boldsymbol{\lambda} \to \boldsymbol{\lambda}+\boldsymbol{\delta}$:

$$\check{P}_{\mathcal{D},0}(x;\boldsymbol{\lambda}) = \check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}). \tag{4.12}$$

The coefficients of the highest degree term of the polynomials $\check{P}_n$, $\check{\Xi}_{\mathcal{D}}$ and $\check{P}_{\mathcal{D},n}$,

$$\begin{aligned}\check{P}_n(x;\boldsymbol{\lambda})&=c_n(\boldsymbol{\lambda})\eta(x)^n+\text{lower degree terms},\\ \check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda})&=c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda})\eta(x)^{\ell_{\mathcal{D}}}+\text{lower degree terms},\\ \check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda})&=c_{\mathcal{D},n}^{P}(\boldsymbol{\lambda})\eta(x)^{\ell_{\mathcal{D}}+n}+\text{lower degree terms},\end{aligned}\tag{4.13}$$

are

$$c_n(\boldsymbol{\lambda})=\begin{cases}(1-c^{-1})^n(\beta)_n^{-1} & :\text{M}\\ (-a)^{-n}q^{-n^2}(abq^{n+1};q)_n(bq;q)_n^{-1} & :\text{l}q\text{J}\\ (-a)^{-n}q^{-n^2} & :\text{l}q\text{L}\end{cases},\tag{4.14}$$

$$c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda})=\prod_{j=1}^{M}\frac{c_{d_j}\big(\mathfrak{t}(\boldsymbol{\lambda})\big)}{c_{j-1}\big(\mathfrak{t}(\boldsymbol{\lambda})\big)}\times\begin{cases}1 & :\text{M}\\ \displaystyle\prod_{1\le j<k\le M}\frac{aq^{-(j-1+k-1)}-bq}{aq^{-(d_j+d_k)}-bq} & :\text{l}q\text{J}\\ \displaystyle\prod_{1\le j<k\le M}q^{d_j+d_k-(j-1+k-1)} & :\text{l}q\text{L}\end{cases},\tag{4.15}$$

$$c_{\mathcal{D},n}^{P}(\boldsymbol{\lambda})=c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda})c_n(\boldsymbol{\lambda})\times\begin{cases}\displaystyle\prod_{j=1}^{M}\frac{\beta+j-1}{\beta+d_j+n} & :\text{M}\\ \displaystyle\prod_{j=1}^{M}\frac{q^{-(j-1)}-bq}{q^{-(d_j+n)}-bq} & :\text{l}q\text{J}\\ \displaystyle\prod_{j=1}^{M}q^{d_j+n-(j-1)} & :\text{l}q\text{L}\end{cases}.\tag{4.16}$$

Among these coefficients, $c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda})$ and $c_{\mathcal{D},n}^{P}(\boldsymbol{\lambda})$ for the lqJ can vanish for certain ratios of the parameters, that is $a=bq^{m+1}$, $m\in\mathbb{Z}_{\geq 0}$ as remarked below (2.69). These special configurations of parameters require separate treatment.

The potential functions $B_{\mathcal{D}}(x)$ and $D_{\mathcal{D}}(x)$ (3.40)–(3.41) after M-deletion ($s=M$) can be expressed neatly in terms of the denominator polynomial:

$$B_{\mathcal{D}}(x;\boldsymbol{\lambda})=B(x;\boldsymbol{\lambda}+M\tilde{\boldsymbol{\delta}})\,\frac{\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x+1;\boldsymbol{\lambda})}\frac{\check{\Xi}_{\mathcal{D}}(x+1;\boldsymbol{\lambda}+\boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})},\tag{4.17}$$

$$D_{\mathcal{D}}(x;\boldsymbol{\lambda})=D(x)\,\frac{\check{\Xi}_{\mathcal{D}}(x+1;\boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda})}\frac{\check{\Xi}_{\mathcal{D}}(x-1;\boldsymbol{\lambda}+\boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})},\tag{4.18}$$

$$\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})=\sqrt{B_{\mathcal{D}}(x;\boldsymbol{\lambda})}-e^{\partial}\sqrt{D_{\mathcal{D}}(x;\boldsymbol{\lambda})},\quad \mathcal{A}_{\mathcal{D}}^{\dagger}(\boldsymbol{\lambda})=\sqrt{B_{\mathcal{D}}(x;\boldsymbol{\lambda})}-\sqrt{D_{\mathcal{D}}(x;\boldsymbol{\lambda})}\,e^{-\partial}.\tag{4.19}$$

The ground state eigenvector $\phi_{\mathcal{D}\,0}(x)$ is expressed by $\phi_0(x)$ (2.11) and $\check{\Xi}_{\mathcal{D}}(x)$:

$$\phi_{\mathcal{D}\,0}(x;\boldsymbol{\lambda})=\prod_{y=0}^{x-1}\sqrt{\frac{B_{\mathcal{D}}(y;\boldsymbol{\lambda})}{D_{\mathcal{D}}(y+1;\boldsymbol{\lambda})}}=\phi_0(x;\boldsymbol{\lambda}+M\tilde{\boldsymbol{\delta}})\sqrt{\frac{\check{\Xi}_{\mathcal{D}}(1;\boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda})\check{\Xi}_{\mathcal{D}}(x+1;\boldsymbol{\lambda})}}\,\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})$$

$$= \psi_{\mathcal{D}}(x;\boldsymbol{\lambda})\check{P}_{\mathcal{D},0}(x;\boldsymbol{\lambda}) \propto \phi^{\text{gen}}_{\mathcal{D}\,0}(x;\boldsymbol{\lambda}), \tag{4.20}$$

$$\psi_{\mathcal{D}}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \sqrt{\check{\Xi}_{\mathcal{D}}(1;\boldsymbol{\lambda})}\,\frac{\phi_0(x;\boldsymbol{\lambda}+M\tilde{\boldsymbol{\delta}})}{\sqrt{\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda})\,\check{\Xi}_{\mathcal{D}}(x+1;\boldsymbol{\lambda})}}, \quad \psi_{\mathcal{D}}(0;\boldsymbol{\lambda}) = 1. \tag{4.21}$$

We arrive at the normalised eigenvector $\phi_{\mathcal{D}\,n}(x;\boldsymbol{\lambda})$ with the orthogonality relation,

$$\phi_{\mathcal{D}\,n}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \psi_{\mathcal{D}}(x;\boldsymbol{\lambda})\check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda}) \propto \phi^{\text{gen}}_{\mathcal{D}\,n}(x;\boldsymbol{\lambda}), \quad \phi_{\mathcal{D}\,n}(0;\boldsymbol{\lambda}) = 1, \tag{4.22}$$

$$\sum_{x=0}^{\infty}\frac{\psi_{\mathcal{D}}(x;\boldsymbol{\lambda})^2}{\check{\Xi}_{\mathcal{D}}(1;\boldsymbol{\lambda})}\check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda})\check{P}_{\mathcal{D},m}(x;\boldsymbol{\lambda}) = \frac{\delta_{nm}}{d_n(\boldsymbol{\lambda})^2\tilde{d}_{\mathcal{D},n}(\boldsymbol{\lambda})^2} \quad (n,m\in\mathbb{Z}_{\geq 0}). \tag{4.23}$$

It is worthwhile to emphasise that the above orthogonality relation is a rational equation of $\boldsymbol{\lambda}$ or $q^{\boldsymbol{\lambda}}$, and it is valid for any value of $\boldsymbol{\lambda}$ (except for the zeros of denominators and those parameter ranges giving divergent sums) but the weight function may not be positive definite.

The shape invariance of the original system is inherited by the deformed systems:

$$\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})^{\dagger} = \kappa\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda}+\boldsymbol{\delta})^{\dagger}\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda}+\boldsymbol{\delta}) + \mathcal{E}_1(\boldsymbol{\lambda}). \tag{4.24}$$

As a consequence of the shape invariance and the normalisation, the actions of $\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})$ and $\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})^{\dagger}$ on the eigenvectors $\phi_{\mathcal{D}\,n}(x;\boldsymbol{\lambda})$ are

$$\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})\phi_{\mathcal{D}\,n}(x;\boldsymbol{\lambda}) = \frac{\mathcal{E}_n(\boldsymbol{\lambda})}{\sqrt{B_{\mathcal{D}}(0;\boldsymbol{\lambda})}}\,\phi_{\mathcal{D}\,n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}), \tag{4.25}$$

$$\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})^{\dagger}\phi_{\mathcal{D}\,n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}) = \sqrt{B_{\mathcal{D}}(0;\boldsymbol{\lambda})}\,\phi_{\mathcal{D}\,n}(x;\boldsymbol{\lambda}). \tag{4.26}$$

The forward and backward shift operators are defined by

$$\begin{aligned}\mathcal{F}_{\mathcal{D}}(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} \sqrt{B_{\mathcal{D}}(0;\boldsymbol{\lambda})}\,\psi_{\mathcal{D}}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})^{-1}\circ\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})\circ\psi_{\mathcal{D}}(x;\boldsymbol{\lambda})\\ &= \frac{B(0;\boldsymbol{\lambda}+M\tilde{\boldsymbol{\delta}})}{\varphi(x)\check{\Xi}_{\mathcal{D}}(x+1;\boldsymbol{\lambda})}\Big(\check{\Xi}_{\mathcal{D}}(x+1;\boldsymbol{\lambda}+\boldsymbol{\delta}) - \check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})e^{\partial}\Big),\end{aligned} \tag{4.27}$$

$$\begin{aligned}\mathcal{B}_{\mathcal{D}}(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} \frac{1}{\sqrt{B_{\mathcal{D}}(0;\boldsymbol{\lambda})}}\,\psi_{\mathcal{D}}(x;\boldsymbol{\lambda})^{-1}\circ\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})^{\dagger}\circ\psi_{\mathcal{D}}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})\\ &= \frac{1}{B(0;\boldsymbol{\lambda}+M\tilde{\boldsymbol{\delta}})\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})}\Big(B(x;\boldsymbol{\lambda}+M\tilde{\boldsymbol{\delta}})\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda}) - D(x)\check{\Xi}_{\mathcal{D}}(x+1;\boldsymbol{\lambda})e^{-\partial}\Big)\varphi(x)\\ &= \psi_{\mathcal{D}}(x;\boldsymbol{\lambda})^{-2}\circ\frac{\check{\Xi}_{\mathcal{D}}(1;\boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(1;\boldsymbol{\lambda}+\boldsymbol{\delta})}\Big(\check{\Xi}_{\mathcal{D}}(x+1;\boldsymbol{\lambda}+\boldsymbol{\delta}) - \check{\Xi}_{\mathcal{D}}(x-1;\boldsymbol{\lambda}+\boldsymbol{\delta})e^{-\partial}\Big)\\ &\qquad\qquad\times\frac{1}{\varphi(x)\check{\Xi}_{\mathcal{D}}(x+1;\boldsymbol{\lambda})}\circ\psi_{\mathcal{D}}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})^2,\end{aligned} \tag{4.28}$$

and their actions on $\check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda})$ are

$$\mathcal{F}_{\mathcal{D}}(\boldsymbol{\lambda})\check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda})=\mathcal{E}_n(\boldsymbol{\lambda})\check{P}_{\mathcal{D},n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}),\quad \mathcal{B}_{\mathcal{D}}(\boldsymbol{\lambda})\check{P}_{\mathcal{D},n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})=\check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda}). \tag{4.29}$$

The similarity transformed Hamiltonian is

$$\begin{aligned}\widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda})&\stackrel{\text{def}}{=}\psi_{\mathcal{D}}(x;\boldsymbol{\lambda})^{-1}\circ\mathcal{H}_{\mathcal{D}}(\boldsymbol{\lambda})\circ\psi_{\mathcal{D}}(x;\boldsymbol{\lambda})=\mathcal{B}_{\mathcal{D}}(\boldsymbol{\lambda})\mathcal{F}_{\mathcal{D}}(\boldsymbol{\lambda})\\&=B(x;\boldsymbol{\lambda}+M\tilde{\boldsymbol{\delta}})\,\frac{\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x+1;\boldsymbol{\lambda})}\biggl(\frac{\check{\Xi}_{\mathcal{D}}(x+1;\boldsymbol{\lambda}+\boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})}-e^{\partial}\biggr)\\&\quad+D(x)\,\frac{\check{\Xi}_{\mathcal{D}}(x+1;\boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda})}\biggl(\frac{\check{\Xi}_{\mathcal{D}}(x-1;\boldsymbol{\lambda}+\boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})}-e^{-\partial}\biggr),\end{aligned} \tag{4.30}$$

and the multi-indexed orthogonal polynomials $\check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda})$ are its eigenpolynomials:

$$\widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda})\check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda})=\mathcal{E}_n(\boldsymbol{\lambda})\check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda}). \tag{4.31}$$

Including the level 0 deletion corresponds to $M-1$ virtual states deletion:

$$\check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda})\,\Big|_{d_M=0}=\check{P}_{\mathcal{D}',n}(x;\boldsymbol{\lambda}+\tilde{\boldsymbol{\delta}}),\quad \mathcal{D}'=\{d_1-1,\ldots,d_{M-1}-1\}. \tag{4.32}$$

The denominator polynomial $\Xi_{\mathcal{D}}$ behaves similarly. This is why we have restricted $d_j\geq 1$.

4.1 Limits

It is well known [9] that the Meixner (2.21), little q-Jacobi (2.27) and little q-Laguerre (2.33) polynomials reduce to the Laguerre (L) $L_n^{(\alpha)}(\eta)$ or Jacobi (J) $P_n^{(\alpha,\beta)}(\eta)$ polynomials in certain limits:

$$\text{M}:\ \ \lim_{c\to 1}P_n\Bigl(\frac{\eta}{1-c};(\alpha+1,c)\Bigr)=\frac{L_n^{(\alpha)}(\eta)}{L_n^{(\alpha)}(0)},\qquad i.e.\ \ \beta=\alpha+1, \tag{4.33}$$

$$\text{l}q\text{J}:\ \ \lim_{q\to 1}P_n\bigl(1-\eta;(\alpha,\beta)\bigr)=\frac{P_n^{(\alpha,\beta)}(1-2\eta)}{P_n^{(\alpha,\beta)}(-1)},\qquad i.e.\ \ a=q^{\alpha},\ b=q^{\beta}, \tag{4.34}$$

$$\text{l}q\text{L}:\ \ \lim_{q\to 1}\frac{P_n\bigl(1-(1-q)\eta;\alpha\bigr)}{P_n(1;\alpha)}=\frac{L_n^{(\alpha)}(\eta)}{L_n^{(\alpha)}(0)},\qquad i.e.\ \ a=q^{\alpha}. \tag{4.35}$$

The constant factors on r.h.s. reflect the universal normalisation $P_n(0)=1$ of M, lqJ(L) polynomials (2.14). Note that $L_n^{(\alpha)}(0)=\frac{1}{n!}(\alpha+1)_n$ and $P_n^{(\alpha,\beta)}(-1)=(-1)^nP_n^{(\beta,\alpha)}(1)=\frac{(-1)^n}{n!}(\beta+1)_n$.

These relations (4.33)–(4.35) also mean that the deforming polynomials $\xi(\eta;\boldsymbol{\lambda})$ of multi-indexed M, lqJ(L) polynomials, (2.68), (2.69) reduce to those of the multi-indexed Laguerre and Jacobi polynomials:

$$\text{M}:\ \lim_{c\to1}\xi_{\text{v}}\Bigl(\frac{\eta}{1-c};(\alpha+1,c)\Bigr)=\frac{L_{\text{v}}^{(\alpha)}(-\eta)}{L_{\text{v}}^{(\alpha)}(0)},\tag{4.36}$$

$$\text{l}q\text{J}:\ \lim_{q\to1}\xi_{\text{v}}\bigl(1-\eta;(\alpha,\beta)\bigr)=\frac{P_{\text{v}}^{(-\alpha,\beta)}(1-2\eta)}{P_{\text{v}}^{(-\alpha,\beta)}(-1)},\tag{4.37}$$

$$\text{l}q\text{L}:\ \lim_{q\to1}\frac{\xi_{\text{v}}\bigl(1-(1-q)\eta;\alpha\bigr)}{\xi_{\text{v}}(1;\alpha)}=\frac{L_{\text{v}}^{(-\alpha)}(\eta)}{L_{\text{v}}^{(-\alpha)}(0)}.\tag{4.38}$$

As shown in [5], the deforming polynomial $L_{\text{v}}^{(\alpha)}(-\eta)$ (4.36) will lead to type I multi-indexed Laguerre polynomials, whereas $L_{\text{v}}^{(-\alpha)}(\eta)$ (4.38) to type II Laguerre. The deforming polynomial $P_{\text{v}}^{(-\alpha,\beta)}(\eta)$ will generate type II multi-indexed Jacobi polynomials. Based on these formulas (4.33)–(4.38) and the Casoratian definition (4.4), it is straightforward to demonstrate that the multi-indexed M, lqJ(L) polynomials reduce to the Wronskian definition of the corresponding multi-indexed Laguerre and Jacobi polynomials [5]:

$$\text{M}:\ \lim_{c\to1}P_{\mathcal{D},n}\Bigl(\frac{\eta}{1-c};(\alpha+1,c)\Bigr)=\frac{P_{\mathcal{D},n}^{\text{L}}(\eta;g)}{P_{\mathcal{D},n}^{\text{L}}(0;g)},\qquad g\stackrel{\text{def}}{=}\alpha+\tfrac12,\tag{4.39}$$

$$\text{l}q\text{J}:\ \lim_{q\to1}P_{\mathcal{D},n}\bigl(1-\eta;(\alpha,\beta)\bigr)=\frac{P_{\mathcal{D},n}^{\text{J}}\bigl(1-2\eta;(g,h)\bigr)}{P_{\mathcal{D},n}^{\text{J}}\bigl(-1;(g,h)\bigr)},\qquad g\stackrel{\text{def}}{=}\alpha+\tfrac12,\ h\stackrel{\text{def}}{=}\beta+\tfrac12,\tag{4.40}$$

$$\text{l}q\text{L}:\ \lim_{q\to1}\frac{P_{\mathcal{D},n}\bigl(1-(1-q)\eta;\alpha\bigr)}{P_{\mathcal{D},n}\bigl(1;\alpha\bigr)}=\frac{P_{\mathcal{D},n}^{\text{L}}(\eta;g)}{P_{\mathcal{D},n}^{\text{L}}(0;g)},\qquad g\stackrel{\text{def}}{=}\alpha+\tfrac12.\tag{4.41}$$

Here $P_{\mathcal{D},n}^{\text{L}}(\eta;\boldsymbol{\lambda})$ and $P_{\mathcal{D},n}^{\text{J}}(\eta;\boldsymbol{\lambda})$ are the multi-indexed Laguerre and Jacobi polynomials [5] with the index set $\mathcal{D}=\{d_1,\ldots,d_M\}$, which are all type I for (4.39) and all type II for (4.40)–(4.41). Again the constant factors on r.h.s. reflect the universal normalisation of the multi-indexed M, lqJ(L) polynomials.

5 Summary and Comments

Following the general procedure [5], *i.e.* i) generating virtual state vectors by twisting parameters, ii) applying multiple Darboux transformations by using the above virtual state vectors as seed solutions, the multi-indexed Meixner, little q-Jacobi and little q-Laguerre polynomials are constructed. As with the other types of multi-indexed orthogonal polynomials, they are shape invariant and form a complete basis in the ℓ^2 Hilbert space, although they lack certain lower degrees.

For the quantum mechanical systems described by the multi-index Laguerre and Jacobi polynomials, the eigenfunctions have the following form [5]

$$\phi_{\mathcal{D}\,n}(x;\boldsymbol{\lambda}) \propto \frac{\phi_0(x;\boldsymbol{\lambda}')}{\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda})}\check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda}) \ \Bigl(= \phi_0(x;\boldsymbol{\lambda}')\frac{\check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda})}\Bigr),$$

where $\boldsymbol{\lambda}' = \boldsymbol{\lambda}+(\text{shift})$. Usual interpretation is that the polynomials $\check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda}) = P_{\mathcal{D},n}(\eta(x);\boldsymbol{\lambda})$ are orthogonal with respect to the weight function $\bigl(\phi_0(x;\boldsymbol{\lambda}')/\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda})\bigr)^2$ and form a complete set. Another interpretation is possible; the rational functions $\check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda})/\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda})$ are orthogonal with respect to the original weight function (with shifted parameters) $\bigl(\phi_0(x;\boldsymbol{\lambda}')\bigr)^2$ and form a complete set. The lowest degree element is almost constant $\check{P}_{\mathcal{D},0}(x;\boldsymbol{\lambda})/\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda}) \propto \check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})/\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda})$ as (4.12). The $(n+1)$-th element has n zeros in $x \in (0,\infty)$ having the structure (degree $\ell_{\mathcal{D}}+n$ polynomial)/(degree $\ell_{\mathcal{D}}$ polynomial). For the discrete quantum mechanics with real shifts described by the multi-index polynomials, e.g. (q-)Racah [11] and M, lqJ and lqL in this paper, the above rational functions are replaced as (4.7)

$$\frac{\check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda})} \ \to \ \frac{\check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda})}{\sqrt{\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda})\check{\Xi}_{\mathcal{D}}(x+1;\boldsymbol{\lambda})}}.$$

These functions $\check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda})/\sqrt{\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda})\check{\Xi}_{\mathcal{D}}(x+1;\boldsymbol{\lambda})}$ are no longer rational functions but they are orthogonal with respect to the original weight function (with shifted parameters) $\bigl(\phi_0(x;\boldsymbol{\lambda}+M\tilde{\boldsymbol{\delta}})\bigr)^2$ and form a complete set.

These multi-indexed orthogonal polynomials reduce to the multi-indexed Laguerre or Jacobi polynomials in certain limits: M $\to$ L, lqJ $\to$ J, lqL $\to$ L. It is an interesting question whether the other types of multi-indexed orthogonal polynomials, *i.e.* type II for M and type I for lqJ(L), could be constructed for the M, lqJ and lqL systems. The existence of such multi-indexed orthogonal polynomials is not obvious at all, as the discrete symmetries of the reduced systems do not imply those of the original difference Schrödinger systems.

From exactly solvable models of discrete QM with real shifts, exactly solvable birth and death processes can be constructed [24, 17]. The exactly solvable models described by the multi-indexed M, lqJ, lqL polynomials provide new exactly solvable birth and death processes.

Durán [16] reported multi-indexed Meixner polynomials, in which he used both eigenvectors and virtual state vectors as seed solutions. The method of construction consists in dualizing Krall discrete orthogonal polynomials. Krall discrete orthogonal polynomials are sequences $(p_n)_{n\in\mathbb{N}}$ of orthogonal polynomials which in addition are eigenfunctions of a higher order difference operator. Durán uses suitable instances of Christoffel transforms for constructing Krall discrete orthogonal polynomials [27, 28, 29]. Using eigenvectors as seed solutions for constructing exactly solvable QM and multi-indexed orthogonal polynomials was established by Krein [25] and Adler [26] more than 20 years ago, and for discrete QM with real shifts it was done in [20]. Durán suggested an interesting possibility that the parameter ranges for positive definite orthogonality measure of the multi-indexed Meixner polynomials could be enlarged when the eigenvectors and virtual state vectors are used simultaneously. In this paper we have not pursued its feasibility, as we have not adopted eigenvectors as seed solutions.

There are 39 classical orthogonal polynomials satisfying second order difference equations in the Askey scheme of hypergeometric orthogonal polynomials and its q-version. The aim of our multi-indexed orthogonal polynomial project is to construct the *shape invariant deformation* of all possible examples in the (q-)Askey scheme. Obviously, the continuous q-Hermite polynomial is excluded as it has no twistable parameter, like the Hermite polynomial. We have reported the shape invariant multi-indexed deformations of the Wilson, Askey-Wilson [10], Racah and q-Racah [11] polynomials on top of those reported in the present paper. All the rest of the polynomials in the (q-)Askey scheme are obtained by parameter restriction and/or certain limiting procedures from the above four polynomials. For some of them, the multi-indexed deformation is expected to be compatible with the parameter restriction/limiting procedures, but the actual verification is yet to be done for each case. Among these, the (dual) big q-Jacobi (Laguerre), (dual) Al-Salam-Carltz I and the discrete q-Hermite polynomials provide a big challenge. The orthogonality measures of these polynomials are of the Jackson integral [6] type and their quantum mechanical formulation requires two-component Hamiltonians [17]. The parameter twisting is rather complicated in these cases. It seems brand new thinking is required to tackle this problem.

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References

[1] D. Gómez-Ullate, N. Kamran and R. Milson, "An extension of Bochner's problem: exceptional invariant subspaces," J. Approx. Theory **162** (2010) 987-1006, `arXiv:0805.3376[math-ph]`; "An extended class of orthogonal polynomials defined by a Sturm-Liouville problem," J. Math. Anal. Appl. **359** (2009) 352-367, `arXiv:0807.3939[math-ph]`.

[2] C. Quesne, "Exceptional orthogonal polynomials, exactly solvable potentials and supersymmetry," J. Phys. **A41** (2008) 392001 (6pp), `arXiv:0807.4087[quant-ph]`; B. Bagchi, C. Quesne and R. Roychoudhury, "Isospectrality of conventional and new extended potentials, second-order supersymmetry and role of PT symmetry," Pramana J. Phys. **73** (2009) 337-347, `arXiv:0812.1488[quant-ph]`.

[3] S. Odake and R. Sasaki, "Infinitely many shape invariant potentials and new orthogonal polynomials," Phys. Lett. **B679** (2009) 414-417, `arXiv:0906.0142[math-ph]`.

[4] D. Gómez-Ullate, N. Kamran and R. Milson, "Two-step Darboux transformations and exceptional Laguerre polynomials," J. Math. Anal. Appl. **387** (2012) 410-418, `arXiv:1103.5724[math-ph]`.

[5] S. Odake and R. Sasaki, "Exactly solvable quantum mechanics and infinite families of multi-indexed orthogonal polynomials," Phys. Lett. **B702** (2011) 164-170, `arXiv:1105.0508[math-ph]`.

[6] G. E. Andrews, R. Askey and R. Roy, *Special Functions*, vol. 71 of Encyclopedia of mathematics and its applications, Cambridge Univ. Press, Cambridge, (1999).

[7] E. Routh, "On some properties of certain solutions of a differential equation of the second order," Proc. London Math. Soc. **16** (1884) 245-261.

[8] S. Bochner, "Über Sturm-Liouvillesche Polynomsysteme," Math. Zeit. **29** (1929) 730-736.

[9] R. Koekoek, P. A. Lesky and R. F. Swarttouw, *Hypergeometric orthogonal polynomials and their q-analogues,* Springer-Verlag Berlin-Heidelberg (2010).

[10] S. Odake and R. Sasaki, "Multi-indexed Wilson and Askey-Wilson polynomials," J. Phys. **A46** (2013) 045204 (22pp), arXiv:1207.5584[math-ph].

[11] S. Odake and R. Sasaki, "Multi-indexed (q-)Racah polynomials," J. Phys. **A 45** (2012) 385201 (21pp), arXiv:1203.5868[math-ph].

[12] S. Odake and R. Sasaki, "Discrete quantum mechanics," (Topical Review) J. Phys. **A44** (2011) 353001 (47pp), arXiv:1104.0473[math-ph].

[13] S. Odake and R. Sasaki, "Orthogonal Polynomials from Hermitian Matrices," J. Math. Phys. **49** (2008) 053503 (43pp), arXiv:0712.4106[math.CA].

[14] S. Odake and R. Sasaki, "Exactly solvable 'discrete' quantum mechanics; shape invariance, Heisenberg solutions, annihilation-creation operators and coherent states," Prog. Theor. Phys. **119** (2008) 663-700, arXiv:0802.1075[quant-ph].

[15] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable*, Springer-Verlag, Berlin (1991).

[16] A. J. Durán, "Exceptional Meixner and Laguerre orthogonal polynomials," J. Approx. Theory **184** (2014) 176-208, arXiv:1310.4658[math.CA].

[17] S. Odake and R. Sasaki, "Orthogonal Polynomials from Hermitian Matrices II," arXiv:1604.00714[math.CA].

[18] G. Darboux, "Sur une proposition relative aux équations linéaires," *C. R. Acad. Paris* **94** (1882) 1456-1459.

[19] M. M. Crum, "Associated Sturm-Liouville systems," Quart. J. Math. Oxford Ser. (2) **6** (1955) 121-127, arXiv:physics/9908019[physics.hist-ph].

[20] S. Odake and R. Sasaki, "Dual Christoffel transformations," Prog. Theor. Phys. **126** (2011) 1-34, arXiv:1101.5468[math-ph].

[21] S. Odake and R. Sasaki, "Unified theory of annihilation-creation operators for solvable ('discrete') quantum mechanics," J. Math. Phys. **47** (2006) 102102 (33pp), arXiv:quant-ph/0605215; "Exact solution in the Heisenberg picture and annihilation-creation operators," Phys. Lett. **B641** (2006) 112-117, arXiv:quant-ph/0605221.

[22] N. M. Atakishiyev and A. U. Klimyk, "On q-orthogonal polynomials, dual to little and big q-Jacobi polynomials," J. Math. Anal. Appl. **294** (2004) 246-257, `arXiv:math/0307250[math.CA]`.

[23] S. Odake, "Recurrence Relations of the Multi-Indexed Orthogonal Polynomials : III," J. Math. Phys. **57** (2016) 023514 (24pp), `arXiv:1509.08213[math-ph]`.

[24] R. Sasaki, "Exactly solvable birth and death processes," J. Math. Phys. **50** (2009) 103509 (18pp), `arXiv:0903.3097[math-ph]`.

[25] M. G. Krein, "On a continual analogue of a Christoffel formula from the theory of orthogonal polynomials," (Russian) Doklady Acad. Nauk. CCCP, **113** (1957) 970-973.

[26] V. É. Adler, "A modification of Crum's method," Theor. Math. Phys. **101** (1994) 1381-1386.

[27] A. J. Durán, "Orthogonal polynomials satisfying higher-order difference equations," Constr. Approx. **36** (2012) 459-486.

[28] A. J. Durán, "Using $\mathcal{D}$-operators to construct orthogonal polynomials satisfying higher order difference or differential equations," J. Approx. Theory **174** (2013) 10-53. `arXiv:1302.0881[math.CA]`.

[29] A. J. Durán and M. D. de la Iglesia, "Constructing bispectral orthogonal polynomials from the classical discrete families of Charlier, Meixner and Krawtchouk," Constr. Approx. **41** (2015) 49-91, `arXiv:1307.1326[math.CA]`.