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Orthogonal Polynomials from Hermitian Matrices II

Satoru Odake and Ryu Sasaki

Faculty of Science, Shinshu University,
Matsumoto 390-8621, Japan

Abstract

This is the second part of the project 'unified theory of classical orthogonal polynomials of a discrete variable derived from the eigenvalue problems of hermitian matrices.' In a previous paper, orthogonal polynomials having Jackson integral measures were not included, since such measures cannot be obtained from single infinite dimensional hermitian matrices. Here we show that Jackson integral measures for the polynomials of the big q-Jacobi family are the consequence of the recovery of self-adjointness of the unbounded Jacobi matrices governing the difference equations of these polynomials. The recovery of self-adjointness is achieved in an extended ℓ^2 Hilbert space on which a direct sum of two unbounded Jacobi matrices acts as a Hamiltonian or a difference Schrödinger operator for an infinite dimensional eigenvalue problem. The polynomial appearing in the upper/lower end of Jackson integral constitutes the eigenvector of each of the two unbounded Jacobi matrix of the direct sum. We also point out that the orthogonal vectors involving the q-Meixner (q-Charlier) polynomials do not form a *complete basis* of the ℓ^2 Hilbert space, based on the fact that the dual q-Meixner polynomials introduced in a previous paper fail to satisfy the orthogonality relation. The complete set of eigenvectors involving the q-Meixner polynomials is obtained by constructing the duals of the dual q-Meixner polynomials which require the two component Hamiltonian formulation. An alternative solution method based on the closure relation, the Heisenberg operator solution, is applied to the polynomials of the big q-Jacobi family and their duals and q-Meixner (q-Charlier) polynomials.

Contents

1 Introduction

In a previous paper [1], to be referred to I and its equation as (I.2.3) etc. hereafter, a unified theory of classical orthogonal polynomials of a discrete variable has been presented. The classical orthogonal polynomials are polynomials satisfying the three term recurrence relation and second order differential/difference equations. On top of the well known Hermite, Laguerre, Jacobi and Bessel polynomials [2] which satisfy second order differential equations, the rest of about 40 classical orthogonal polynomials satisfying second order difference equations are classified according to Askey scheme of hypergeometric orthogonal polynomials [3, 4, 5, 6, 7]. In contrast to the ordinary orthogonal polynomials, which satisfy the three term recurrence relation only, unified understanding of various properties of the classical orthogonal polynomials is possible by considering them as the main part of the eigenfunctions (eigenvectors) of self-adjoint second order differential/difference operators which are exactly solvable. The Schrödinger operators in quantum mechanics are the typical examples of such second order self-adjoint differential operators. Thus we called the second order self-adjoint difference operators 'discrete Schrödinger operators' or the Hamiltonians in 'discrete quantum mechanics' [8, 9, 10]. Various concepts and methods accumulated since the birth of quantum mechanics are now available for unified understanding of classical orthogonal polynomials [11, 12, 13, 14]. For example, the orthogonality is the consequence of the self-adjointness and the orthogonality measures are provided by the square of the lowest (the ground state) eigenfunctions (eigenvectors) which have no zeros due to the oscillation theorem.

Orthogonal polynomials of a discrete variable [15, 6] have orthogonality measures concentrated on discrete points, either finite or infinite in number. The eigenvalue problems governing the classical orthogonal polynomials of a discrete variable are based on a special class of hermitian matrices, which are real symmetric tri-diagonal (Jacobi) matrices (2.1) of finite or infinite dimensions. Since the spectrum of a Jacobi matrix is simple, the orthogonality of eigenvectors (eigenpolynomials) is guaranteed except for unbounded infinite dimensional ones, for which the self-adjointness could be broken. The lowest eigenvectors (ground state vectors) of Jacobi matrices (to be called the Hamiltonian hereafter) satisfy zero mode equations (2.11), which can be solved easily. By similarity transforming the Hamiltonians in terms of the ground state vectors, one obtains difference operators which are *upper triangular* in certain bases called the *sinusoidal coordinates* $\eta(x)$ [1, 9, 14]. The eigenvalues $\mathcal{E}(n)$ can be easily read off as the coefficients of the highest degree term in the sinusoidal coordinates. By solving the same equations in a different way, one obtains the *dual polynomials* in the eigenvalue $\mathcal{E}(n)$ [16]–[19]. All the Jacobi matrices corresponding to the classical orthogonal polynomials are shown to have symmetries called *shape invariance* [1] and the *closure relations* [14] which in turn provide universal Rodrigues formula (2.40) and the coefficients of the three term recurrence relations through the solutions of the Heisenberg equations [14] for the sinusoidal coordinates (I.4.52)-(I.4.53). These are the main results reported in I.

The purpose of the present paper is to rectify two shortcomings of I. The first is that among many proposed dual orthogonal polynomials in I, the dual q-Meixner and the dual q-Charlier polynomials do not satisfy the orthogonality relations. Secondly, those polynomials having the Jackson integral orthogonal measures [3, 5, 6], *i.e.* the big q-Jacobi polynomial (bqJ) and related polynomials, are not included in I. These two points have the same root. The dual orthogonality relation is equivalent to the completeness relation of the original polynomial. That is, the q-Meixner (q-Charlier) polynomials with the orthogonality weight function [6] *do not form a complete set of basis of the corresponding ℓ^2 Hilbert space* [20]. There is no direct path to restore the breakdown of the completeness relation. On the other hand, the failure of the orthogonality relation can be traced to the breakdown of the self-adjointness. Another Jacobi matrix (Hamiltonian) is introduced so that in the extended Hilbert space the direct sum of the two Hamiltonians recover the self-adjointness. The resulting orthogonality relation in the *two component Hamiltonian formalism takes the form of Jackson integral measure.* By constructing the duals of the eigenpolynomials of the two component Hamiltonian system, the remaining part of the complete set of eigenvectors belonging to the q-Meixner (q-Charlier) polynomials is obtained [20]. The problem of identifying various dual polynomials, *e.g.* big q-Jacobi and q-Meixner etc, has been tackled extensively by Atakishiyev and Klimyk [20]–[23] by methods different from ours.

This paper is organised as follows. In section 2.1 the essence of the program 'orthogonal polynomials from hermitian matrices' is reviewed with an eye to the possible breakdown of the self-adjointness. Several basic concepts, notions and notation are introduced and explained in section 2.2. They are the sinusoidal coordinates, shape invariance, universal Rodrigues formula, universal normalisation condition, dual polynomials, etc. The lack of orthogonality of the naive dual q-Meixner polynomial is examined in section 3.1 and its recovery in the framework of two component Hamiltonian formalism is presented in section 3.2. The dual q-Charlier case is treated similarly in section 3.3. The orthogonal polynomials having Jackson integral measures are discussed in section 4. Starting with the general structure of the two component Hamiltonian systems in section 4.1, the most generic case of the big q-Jacobi polynomial is discussed in some detail in section 4.2. The interesting features of the dual big q-Jacobi polynomial are explored in section 4.2.1 [20]–[23]. Restrictions of the parameters of the big q-Jacobi polynomial together with certain limiting procedures produce other polynomials in the same family. The big q-Laguerre and its dual, the Al-Salam-Carlitz I and the discrete q-Hermite I are discussed in section 4.3. In § 4.4 another set of infinitely many orthogonal polynomials related with the q-Meixner (q-Charlier) polynomials is shown to constitute the remaining part of the complete set of orthonormal bases involving the q-Meixner (q-Charlier) polynomials. This set is obtained by constructing the duals of the dual q-Meixner (q-Charlier) polynomials. The discrete q-Hermite II is discussed in section 5, since it has also Jackson integral measure but it is defined on the full integer lattice $x \in \mathbb{Z}$ in contrast to the half line integer lattice $x \in \mathbb{Z}_{\geq 0}$ in other examples. Another polynomial defined on the full integer lattice, the q-Laguerre polynomial is discussed in §5.1 by pursuing the well known connection between the even degree Hermite and the Laguerre polynomials. The universal Rodrigues formulas for those having Jackson integral measures are presented in § 5.2. One interesting application of the orthogonal polynomials of a discrete variable, the stationary stochastic processes called 'Birth and Death (BD) processes,' is discussed in section 6.1. The polynomials discussed in § 3–§ 5 provide new types of exactly solvable birth and death processes. This is a supplementary sequence to [24]. As explained in § 6.1, the exact solvability of BD processes hangs on the completeness of the corresponding eigenpolynomials. The necessary modifications to the solutions of the BD process corresponding to the q-Meixner (q-Charlier) polynomials are mentioned at the end of § 6.1. The issue of the identification of the duals of the classical orthogonal polynomials attracted many researchers [20]–[23]. Most of them are, however, obtained *automatically by the interchange* $x \leftrightarrow n$ (2.49) from the original polynomials *satisfying the universal normalisation condition* (2.26). In section 6.2 we propose new q-hypergeometric expressions for four classical polynomials, the little q-Jacobi, little q-Laguerre, Al-Salam-Carlitz II and the alternative q-Charlier (q-Bessel), which satisfy the universal condition (2.26). The final section is for a summary and comments. In Appendix, an alternative solution method for the classical orthogonal polynomials is applied to those polynomials discussed in the main text. That is an algebraic method based on the symmetry properties of classical orthogonal polynomials called closure relations [14, 1, 9]. For the dual big q-Jacobi family and the q-Meixner (q-Charlier), the same data govern the ordinary sector and the supplementary sector, which is necessary for the completeness.

2 Orthogonal Polynomials from Hermitian Matrices

2.1 Formulation

Let us recapitulate the essence of the discrete quantum mechanics with real shifts developed in I. The Hamiltonian $\mathcal{H} = (\mathcal{H}_{x,y})$ is a *tri-diagonal real symmetric* (Jacobi) matrix and its rows and columns are indexed by integers x and y, which take values in $\{0, 1, \ldots, N\}$ (finite) or $\mathbb{Z}_{\geq 0}$ (semi-infinite) or $\mathbb{Z}$ (full infinite). Since the finite dimensional case is fully developed in I, we concentrate here on the semi-infinite ($\mathbb{Z}_{\geq 0}$) case. The full infinite case will be discussed separately in § 5. Let us assume that the spectrum consists of discrete eigenvalues only and is bounded from below. By adding a scalar matrix to the Hamiltonian, the lowest eigenvalue is adjusted to be zero. This makes the Hamiltonian *positive semi-definite*. Since the eigenvector corresponding to the zero eigenvalue has definite sign, *i.e.* all the components are positive or negative, the Hamiltonian $\mathcal{H}$ has the following tri-diagonal form $(x, y \in \mathbb{Z}_{\geq 0})$

$$\mathcal{H}_{x,y} \stackrel{\text{def}}{=} -\sqrt{B(x)D(x+1)}\,\delta_{x+1,y} - \sqrt{B(x-1)D(x)}\,\delta_{x-1,y} + \big(B(x) + D(x)\big)\delta_{x,y}, \tag{2.1}$$

in which the potential functions $B(x)$ and $D(x)$ are real and positive but vanish at the boundary

$$B(x) > 0 \quad (x \geq 0), \quad D(x) > 0 \quad (x \geq 1), \quad D(0) = 0, \tag{2.2}$$

and the Hamiltonian (2.1) is real symmetric, $\mathcal{H}_{x,y} = \mathcal{H}_{y,x}$. Reflecting the positive semi-definiteness, the Hamiltonian (2.1) can be expressed in a factorised form:

$$\mathcal{H} = \mathcal{A}^\dagger \mathcal{A}, \qquad \mathcal{A} = (\mathcal{A}_{x,y}), \quad \mathcal{A}^\dagger = ((\mathcal{A}^\dagger)_{x,y}) = (\mathcal{A}_{y,x}), \tag{2.3}$$

$$\mathcal{A}_{x,y} \stackrel{\text{def}}{=} \sqrt{B(x)}\,\delta_{x,y} - \sqrt{D(x+1)}\,\delta_{x+1,y}, \quad (\mathcal{A}^\dagger)_{x,y} = \sqrt{B(x)}\,\delta_{x,y} - \sqrt{D(x)}\,\delta_{x-1,y}. \tag{2.4}$$

Here $\mathcal{A}$ ($\mathcal{A}^\dagger$) is an upper (lower) triangular matrix with the diagonal and the super(sub)-diagonal entries only. For simplicity in notation, we write $\mathcal{H}$, $\mathcal{A}$ and $\mathcal{A}^\dagger$ as follows:

$$e^{\pm\partial} = ((e^{\pm\partial})_{x,y}), \quad (e^{\pm\partial})_{x,y} \stackrel{\text{def}}{=} \delta_{x\pm1,y}, \quad (e^{\partial})^\dagger = e^{-\partial}, \tag{2.5}$$

$$\begin{aligned} \mathcal{H} &= -\sqrt{B(x)D(x+1)}\,e^{\partial} - \sqrt{B(x-1)D(x)}\,e^{-\partial} + B(x) + D(x) \\ &= -\sqrt{B(x)}\,e^{\partial}\sqrt{D(x)} - \sqrt{D(x)}\,e^{-\partial}\sqrt{B(x)} + B(x) + D(x), \end{aligned} \tag{2.6}$$

$$\mathcal{A} = \sqrt{B(x)} - e^{\partial}\sqrt{D(x)}, \quad \mathcal{A}^\dagger = \sqrt{B(x)} - \sqrt{D(x)}\,e^{-\partial}. \tag{2.7}$$

(We suppress the unit matrix $\mathbf{1} = (\delta_{x,y})$: $(B(x) + D(x))\mathbf{1}$ in (2.6), $\sqrt{B(x)}\,\mathbf{1}$ in (2.7).) Note that the self-adjointness of the Hamiltonian (2.1) is trivial for finite systems, but non-trivial for infinite systems.

The Hamiltonian (2.1) is a linear operator on the real ℓ^2 Hilbert space with the inner product of two real vectors $f = (f(x))$ and $g = (g(x))$ defined by

$$(f, g) = \sum_{x=0}^{\infty} f(x)g(x), \quad \|f\|^2 \stackrel{\text{def}}{=} (f, f) < \infty, \tag{2.8}$$

where the infinite sum is defined by the limit of N-truncated inner product $(f, g)_N$:

$$(f, g) \stackrel{\text{def}}{=} \lim_{N\to\infty} (f, g)_N, \quad (f, g)_N \stackrel{\text{def}}{=} \sum_{x=0}^{N} f(x)g(x). \tag{2.9}$$

The Schrödinger equation is the eigenvalue problem for the hermitian matrix $\mathcal{H}$,

$$\mathcal{H}\phi_n(x) = \mathcal{E}(n)\phi_n(x) \quad (n = 0, 1, \ldots), \quad 0 = \mathcal{E}(0) < \mathcal{E}(1) < \cdots, \tag{2.10}$$

where the eigenvector $\phi_n = (\phi_n(x))$ is, by definition, of finite norm, $\|\phi_n\| < \infty$. Let us recall the fact that the spectrum of a Jacobi matrix is simple. The ground state eigenvector, which is chosen positive $\phi_0(x) > 0$ ($x \in \mathbb{Z}_{\geq 0}$), satisfies the zero mode equation:

$$\mathcal{A}\phi_0 = 0 \;\Rightarrow \mathcal{H}\phi_0 = 0, \quad \sqrt{B(x)}\,\phi_0(x) = \sqrt{D(x+1)}\,\phi_0(x+1) \quad (x \in \mathbb{Z}_{\geq 0}), \tag{2.11}$$

and it is easily obtained (convention: $\prod_{k=n}^{n-1} * = 1$):

$$\phi_0(x) = \phi_0(0) \prod_{y=0}^{x-1} \sqrt{\frac{B(y)}{D(y+1)}} \quad (x \in \mathbb{Z}_{\geq 0}). \tag{2.12}$$

The self-adjointness of the Hamiltonian and the non-degeneracy of the spectrum (2.10) imply that the eigenvectors are mutually orthogonal:

$$(\phi_n, \phi_m) = \frac{\delta_{nm}}{d_n^2} \quad (n, m = 0, 1, \ldots), \tag{2.13}$$

where d_n^2 ($d_n > 0$) is a normalisation constant. It should be emphasised that $\phi_0(x)^2$ can be analytically continued to the entire complex x-plane as a meromorphic function. Likewise d_n^2 can also be analytically continued to the entire complex n-plane as a meromorphic function. They vanish on the negative integer lattice

$$\phi_0(x)^2 = 0 \quad (x \in \mathbb{Z}_{<0}), \quad d_n^2 = 0 \quad (n \in \mathbb{Z}_{<0}). \tag{2.14}$$

Now let us consider the self-adjointness of the above tri-diagonal infinite dimensional Hamiltonian (2.1). In general the self-adjointness of the Hamiltonian $\mathcal{H}$ means the equality of the following two quantities,

$$(f, \mathcal{H}g) = (\mathcal{H}f, g) \quad \Big(\Leftrightarrow \lim_{N\to\infty}\big((f, \mathcal{H}g)_N - (\mathcal{H}f, g)_N\big) = 0\Big), \tag{2.15}$$

for an arbitrary choice of two vectors f and g, $\|f\|, \|g\| < \infty$. Of course the summations of both sides should be absolutely convergent.

The tri-diagonality provides a simple criterion of the self-adjointness by considering the action of the Hamiltonian $\mathcal{H}$ on the vector $f(x)$, which has a factorised form $f(x) = \phi_0(x)\check{\mathcal{P}}(x)$:

$$\begin{aligned}\mathcal{H}f(x) &= \sum_{y=0}^{\infty} \mathcal{H}_{x,y} f(y)\\ &= \big(B(x) + D(x)\big) f(x) - \sqrt{B(x)D(x+1)}\, f(x+1) - \sqrt{B(x-1)D(x)}\, f(x-1)\end{aligned}$$

$$= \phi_0(x)\Big(B(x)\big(\check{\mathcal{P}}(x) - \check{\mathcal{P}}(x+1)\big) + D(x)\big(\check{\mathcal{P}}(x) - \check{\mathcal{P}}(x-1)\big)\Big) = \phi_0(x)\widetilde{\mathcal{H}}\check{\mathcal{P}}(x). \tag{2.16}$$

Here the similarity transformed Hamiltonian $\widetilde{\mathcal{H}}$ is

$$\widetilde{\mathcal{H}} \stackrel{\text{def}}{=} \phi_0(x)^{-1} \circ \mathcal{H} \circ \phi_0(x) = B(x)(1 - e^{\partial}) + D(x)(1 - e^{-\partial}). \tag{2.17}$$

To verify the equality (2.15), we use the N-truncated inner product (2.9). For two vectors $f(x) = \phi_0(x)\check{\mathcal{P}}(x)$ and $g(x) = \phi_0(x)\check{\mathcal{Q}}(x)$, we have

$$\begin{aligned}
(f, \mathcal{H}g)_N &= \sum_{x=0}^{N} \phi_0(x)\check{\mathcal{P}}(x)\phi_0(x)\Big(B(x)\big(\check{\mathcal{Q}}(x) - \check{\mathcal{Q}}(x+1)\big) + D(x)\big(\check{\mathcal{Q}}(x) - \check{\mathcal{Q}}(x-1)\big)\Big)\\
&= \sum_{x=0}^{N} \phi_0(x)^2\big(B(x) + D(x)\big)\check{\mathcal{P}}(x)\check{\mathcal{Q}}(x) - \sum_{x=0}^{N} \phi_0(x)^2 B(x)\check{\mathcal{P}}(x)\check{\mathcal{Q}}(x+1)\\
&\quad - \sum_{x=0}^{N} \phi_0(x)^2 D(x)\check{\mathcal{P}}(x)\check{\mathcal{Q}}(x-1),
\end{aligned} \tag{2.18}$$

and the last term is rewritten with the help of the zero mode equation (2.11)

$$\begin{aligned}
&\sum_{x=0}^{N} \phi_0(x)^2 D(x)\check{\mathcal{P}}(x)\check{\mathcal{Q}}(x-1) = \sum_{x=1}^{N} \phi_0(x)^2 D(x)\check{\mathcal{P}}(x)\check{\mathcal{Q}}(x-1)\\
&= \sum_{x=0}^{N-1} \phi_0(x+1)^2 D(x+1)\check{\mathcal{P}}(x+1)\check{\mathcal{Q}}(x) = \sum_{x=0}^{N-1} \phi_0(x)^2 B(x)\check{\mathcal{P}}(x+1)\check{\mathcal{Q}}(x).
\end{aligned} \tag{2.19}$$

The expression of $(\mathcal{H}f, g)_N$ can be obtained by exchanging $\check{\mathcal{P}}$ and $\check{\mathcal{Q}}$. The criterion of the self-adjointness of the Hamiltonian with respect to the inner product $(\cdot,\cdot)$ (2.8)–(2.9) is the vanishing of the difference in $N \to \infty$ limit,

$$\begin{aligned}
0 &= \lim_{N\to\infty}\big((f, \mathcal{H}g)_N - (\mathcal{H}f, g)_N\big)\\
&= \lim_{N\to\infty} \phi_0(N)^2 B(N)\big(\check{\mathcal{P}}(N+1)\check{\mathcal{Q}}(N) - \check{\mathcal{P}}(N)\check{\mathcal{Q}}(N+1)\big).
\end{aligned} \tag{2.20}$$

2.2 Hypergeometric orthogonal polynomials and their duals

Each specific theory to be discussed hereafter depends on a certain set of parameters, to be denoted symbolically by $\boldsymbol{\lambda}$. Various quantities are expressed like, $\mathcal{H}(\boldsymbol{\lambda}) = \mathcal{A}(\boldsymbol{\lambda})^\dagger\mathcal{A}(\boldsymbol{\lambda})$, $\mathcal{E}(n;\boldsymbol{\lambda})$, $\phi_n(x;\boldsymbol{\lambda})$, and $P_n(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda})$, etc. The parameter q is $0 < q < 1$ and $q^{\boldsymbol{\lambda}}$ stands for $q^{(\lambda_1,\lambda_2,\ldots)} = (q^{\lambda_1}, q^{\lambda_2}, \ldots)$. As shown in I, for special choices of the potential functions $B(x;\boldsymbol{\lambda})$ and $D(x;\boldsymbol{\lambda})$, the eigenvalue problem (2.10) is exactly solvable and various hypergeometric orthogonal polynomials $\{P_n\}$, the (q-)Meixner, (dual) (q-)Hahn, (q-)Racah, etc. are obtained as the main part of the eigenvector

$$\phi_n(x;\boldsymbol{\lambda}) = \phi_0(x;\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda}), \quad \check{P}_n(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_n\bigl(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda}\bigr), \tag{2.21}$$

$$\widetilde{\mathcal{H}}(\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda}) = \mathcal{E}(n;\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda}) \quad (n = 0, 1, \ldots), \tag{2.22}$$

in which $\phi_0(x;\boldsymbol{\lambda})$ is the ground state eigenvector (2.12) and $\eta(x;\boldsymbol{\lambda})$ is a certain function of x, called the *sinusoidal coordinate*. In other words, the similarity transformed Hamiltonian $\widetilde{\mathcal{H}}$ is *triangular* in the basis spanned by $1, \eta(x;\boldsymbol{\lambda}), \ldots, \eta(x;\boldsymbol{\lambda})^n, \ldots$,

$$\widetilde{\mathcal{H}}(\boldsymbol{\lambda})\eta(x;\boldsymbol{\lambda})^n = \mathcal{E}(n;\boldsymbol{\lambda})\eta(x;\boldsymbol{\lambda})^n + \text{lower degrees in } \eta(x;\boldsymbol{\lambda}), \tag{2.23}$$

which leads to the *solvability*. The eigenvalue $\mathcal{E}(n;\boldsymbol{\lambda})$ can be easily obtained as the coefficient of the highest degree monomial in $\eta(x;\boldsymbol{\lambda})$. Five different types of the sinusoidal coordinates are known [1, 9]:

$$\eta(x;\boldsymbol{\lambda}) = x, \quad x(x+d), \quad 1-q^x, \quad q^{-x}-1, \quad (q^{-x}-1)(1-dq^x), \tag{2.24}$$

in which d is a real parameter. They all satisfy the universal boundary condition

$$\eta(0;\boldsymbol{\lambda}) = 0. \tag{2.25}$$

The eigenpolynomials are chosen to satisfy the *universal boundary condition*

$$\check{P}_n(0;\boldsymbol{\lambda}) = P_n\bigl(\eta(0;\boldsymbol{\lambda});\boldsymbol{\lambda}\bigr) = P_n(0;\boldsymbol{\lambda}) = 1 \quad (n = 0, 1, 2, \ldots). \tag{2.26}$$

They satisfy the orthogonality condition $(n, m = 0, 1, \ldots)$

$$(\phi_n, \phi_m) = \sum_{x=0}^{\infty} \phi_0(x;\boldsymbol{\lambda})^2 P_n\bigl(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda}\bigr) P_m\bigl(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda}\bigr) = \frac{\delta_{nm}}{d_n(\boldsymbol{\lambda})^2}, \tag{2.27}$$

in which the orthogonality weight $\phi_0(x;\boldsymbol{\lambda})^2$ is expressed in terms of (q-)shifted factorials [1].

The above triangularity (2.23) does not necessarily imply the existence of explicit expressions of the corresponding eigenpolynomials, like the hypergeometric form. A stronger condition governing the parameter dependence of the Hamiltonian or the potential functions, called *shape invariance* (see section IV of I for more details) plays that role. The shape invariance provides the explicit forms of all eigenvalues $\{\mathcal{E}(n;\boldsymbol{\lambda})\}$ (2.31) and the universal Rodrigues type formula of the eigenpolynomials $\{P_n(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda})\}$ (2.40).

The shape invariance condition is

$$\mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^\dagger = \kappa\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta})^\dagger\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta}) + \mathcal{E}(1;\boldsymbol{\lambda}), \tag{2.28}$$

in which $\boldsymbol{\delta}$ denotes the shift of the parameters, κ is a positive constant and $\mathcal{E}(1;\boldsymbol{\lambda})$ is the eigenvalue of the first excited state $\mathcal{E}(1) > 0$ with the explicit parameter dependence. In terms of the potential functions the above condition means the following two relations:

$$B(x+1;\boldsymbol{\lambda})D(x+1;\boldsymbol{\lambda}) = \kappa^2 B(x;\boldsymbol{\lambda}+\boldsymbol{\delta})D(x+1;\boldsymbol{\lambda}+\boldsymbol{\delta}), \tag{2.29}$$

$$B(x;\boldsymbol{\lambda}) + D(x+1;\boldsymbol{\lambda}) = \kappa\big(B(x;\boldsymbol{\lambda}+\boldsymbol{\delta}) + D(x;\boldsymbol{\lambda}+\boldsymbol{\delta})\big) + \mathcal{E}(1;\boldsymbol{\lambda}). \tag{2.30}$$

By using the discrete counterpart of Crum's theorem [11, 25, 26], the spectrum is simply generated by $\mathcal{E}(1,\boldsymbol{\lambda})$:

$$\mathcal{E}(n;\boldsymbol{\lambda}) = \sum_{s=0}^{n-1}\kappa^s\mathcal{E}(1;\boldsymbol{\lambda}+s\boldsymbol{\delta}), \tag{2.31}$$

and the corresponding eigenvectors are generated from the known form of the ground state eigenvector $\phi_0(x;\boldsymbol{\lambda})$ (2.12) together with the multiple action of the successive $\mathcal{A}(\boldsymbol{\lambda})^\dagger$ operator:

$$\phi_n(x;\boldsymbol{\lambda}) \propto \mathcal{A}(\boldsymbol{\lambda})^\dagger\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta})^\dagger\cdots\mathcal{A}\big(\boldsymbol{\lambda}+(n-1)\boldsymbol{\delta}\big)^\dagger\phi_0(x;\boldsymbol{\lambda}+n\boldsymbol{\delta}). \tag{2.32}$$

Let us introduce an auxiliary function $\varphi(x)$ defined on the entire complex x-plane (I.4.12), (I.4.23),

$$\varphi(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{\eta(x+1;\boldsymbol{\lambda}) - \eta(x;\boldsymbol{\lambda})}{\eta(1;\boldsymbol{\lambda})}. \tag{2.33}$$

On the non-negative integer lattice it satisfies

$$\varphi(x;\boldsymbol{\lambda}) = \sqrt{\frac{B(0;\boldsymbol{\lambda})}{B(x;\boldsymbol{\lambda})}}\frac{\phi_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta})}{\phi_0(x;\boldsymbol{\lambda})}. \tag{2.34}$$

Then we have

$$\phi_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta})^{-1}\circ\mathcal{A}(\boldsymbol{\lambda})\circ\phi_0(x;\boldsymbol{\lambda}) = \sqrt{B(0;\boldsymbol{\lambda})}\,\varphi(x;\boldsymbol{\lambda})^{-1}(1-e^{\partial}), \tag{2.35}$$

$$\begin{aligned}\phi_0(x;\boldsymbol{\lambda})^{-1}\circ\mathcal{A}(\boldsymbol{\lambda})^\dagger\circ\phi_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta}) &= \frac{1}{\sqrt{B(0;\boldsymbol{\lambda})}}\big(B(x;\boldsymbol{\lambda}) - D(x;\boldsymbol{\lambda})e^{-\partial}\big)\varphi(x;\boldsymbol{\lambda}) \\ &= \sqrt{B(0;\boldsymbol{\lambda})}\,\phi_0(x;\boldsymbol{\lambda})^{-2}\circ(1-e^{-\partial})\varphi(x;\boldsymbol{\lambda})^{-1}\circ\phi_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta})^2.\end{aligned} \tag{2.36}$$

In terms of the auxiliary function $\varphi(x;\boldsymbol{\lambda})$, let us introduce the forward and backward shift operators $\mathcal{F}(\boldsymbol{\lambda})$ (I.4.18) and $\mathcal{B}(\boldsymbol{\lambda})$ (I.4.19),

$$\mathcal{F}(\boldsymbol{\lambda}) = B(0;\boldsymbol{\lambda})\varphi(x;\boldsymbol{\lambda})^{-1}(1-e^{\partial}), \quad \mathcal{B}(\boldsymbol{\lambda}) = \frac{1}{B(0;\boldsymbol{\lambda})}\big(B(x;\boldsymbol{\lambda}) - D(x;\boldsymbol{\lambda})e^{-\partial}\big)\varphi(x;\boldsymbol{\lambda}). \tag{2.37}$$

The following forward and backward shift relations

$$\mathcal{F}(\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda}) = \mathcal{E}(n;\boldsymbol{\lambda})\check{P}_{n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}), \quad \mathcal{B}(\boldsymbol{\lambda})\check{P}_{n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}) = \check{P}_n(x;\boldsymbol{\lambda}), \tag{2.38}$$

are the consequences of the shape invariance for the polynomials satisfying the universal boundary condition (2.26). Starting from $P_0(\eta;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} 1$, $\check{P}_n(x;\boldsymbol{\lambda})$ can be written as

$$\check{P}_n(x;\boldsymbol{\lambda}) = \mathcal{B}(\boldsymbol{\lambda})\mathcal{B}(\boldsymbol{\lambda}+\boldsymbol{\delta})\cdots\mathcal{B}\big(\boldsymbol{\lambda}+(n-1)\boldsymbol{\delta}\big)\cdot 1. \tag{2.39}$$

The universal Rodrigues formula for all the hypergeometric orthogonal polynomials (of a discrete variable) satisfying the universal boundary conditions $P_n(0;\boldsymbol{\lambda}) = 1$ (2.26), $\phi_0(0;\boldsymbol{\lambda}) = 1$ (I.2.19) and $\phi_0(x)^2 = 0$ ($x \in \mathbb{Z}_{<0}$) (2.14) reads:

$$P_n\big(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda}\big) = \phi_0(x;\boldsymbol{\lambda})^{-2}\,\mathcal{D}(\boldsymbol{\lambda})\mathcal{D}(\boldsymbol{\lambda}+\boldsymbol{\delta})\cdots\mathcal{D}\big(\boldsymbol{\lambda}+(n-1)\boldsymbol{\delta}\big)\cdot\phi_0(x;\boldsymbol{\lambda}+n\boldsymbol{\delta})^2, \tag{2.40}$$

$$\mathcal{D}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (1-e^{-\partial})\varphi(x;\boldsymbol{\lambda})^{-1}. \tag{2.41}$$

For the orthogonal polynomials having Jackson integral measures, both of the conditions $P_n(0;\boldsymbol{\lambda}) = 1$ (2.26) and $\phi_0(0;\boldsymbol{\lambda}) = 1$ are not satisfied. Thus the universal Rodrigues formula gets certain modification as shown in section 5.2. It should be emphasised that the conventional Rodrigues formulas [6, 7] were derived one by one for each polynomial.

The dual polynomial arises naturally as the solution of the original eigenvalue problem (2.10) or (2.22) obtained in a different way. The similarity transformed eigenvalue problem $\widetilde{\mathcal{H}}\,v(x) = \mathcal{E}\,v(x)$ (2.22) can be rewritten into an explicit matrix form with the change of the notation $v(x) \to {}^t(Q_0, Q_1, \ldots, Q_x, \ldots)$

$$\sum_y \widetilde{\mathcal{H}}_{x,y}Q_y = \mathcal{E}Q_x \quad (x = 0,1,\ldots). \tag{2.42}$$

Because of the tri-diagonality of $\widetilde{\mathcal{H}}$, it is in fact a three term recurrence relation for $\{Q_x\}$ as polynomials in $\mathcal{E}$:

$$\mathcal{E}Q_x(\mathcal{E}) = B(x)\big(Q_x(\mathcal{E}) - Q_{x+1}(\mathcal{E})\big) + D(x)\big(Q_x(\mathcal{E}) - Q_{x-1}(\mathcal{E})\big) \quad (x = 0,1,\ldots). \tag{2.43}$$

Starting with the boundary (initial) condition

$$Q_0 = 1, \tag{2.44}$$

we determine $Q_x(\mathcal{E})$ as a degree x polynomial in $\mathcal{E}$. It is easy to see

$$Q_x(0) = 1 \quad (x = 0,1,\ldots). \tag{2.45}$$

Note that the boundary condition $D(0)=0$ guarantees that Q_{-1} decouples from the above three term recurrence relation (2.43). When $\mathcal{E}$ is replaced by the actual value of the n-th eigenvalue $\mathcal{E}(n)$ (2.23) in $Q_x(\mathcal{E})$, we obtain the explicit form of the eigenvector

$$\sum_y \widetilde{\mathcal{H}}_{x,y} Q_y\bigl(\mathcal{E}(n)\bigr) = \mathcal{E}(n) Q_x\bigl(\mathcal{E}(n)\bigr) \quad (x=0,1,\ldots). \tag{2.46}$$

The two expressions (polynomials) for the eigenvectors of the problem (2.10) belonging to the eigenvalue $\mathcal{E}(n)$, $P_n(\eta(x))$ and $Q_x(\mathcal{E}(n))$ are in fact equal on the integer lattice points

$$P_n\bigl(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda}\bigr) = Q_x\bigl(\mathcal{E}(n;\boldsymbol{\lambda});\boldsymbol{\lambda}\bigr) \quad (n=0,1,\ldots\,;\ x=0,1,\ldots), \tag{2.47}$$

due to the simplicity of the spectrum of Jacobi matrices and the boundary conditions (2.26) and (2.45). When $n \in \mathbb{Z}_{\geq 0}$ is fixed, the hypergeometric expression of the polynomial $P_n(\eta(x))$, say (I.5.28) for the Hahn polynomial, or (I.5.73) for the q-Racah polynomial, etc.,

$$\text{Hahn}: \ P_n\bigl(\eta(x);\boldsymbol{\lambda}\bigr) = {}_3F_2\Bigl(\genfrac{}{}{0pt}{}{-n,\ n+a+b-1,\ -x}{a,\ -N}\Bigm| 1\Bigr), \qquad \boldsymbol{\lambda}=(a,b,N), \tag{I.5.28}$$

$$q\text{-Racah}: \ P_n\bigl(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda}\bigr) = {}_4\phi_3\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\ \tilde{d}q^n,\ q^{-x},\ dq^x}{a,\ b,\ c}\Bigm| q\,;q\Bigr), \quad q^{\boldsymbol{\lambda}}=(a,b,c,d), \tag{I.5.73}$$

is a degree n polynomial in $\eta(x;\boldsymbol{\lambda})$ (x for the Hahn and $(q^{-x}-1)(1-dq^x)$ for the q-Racah), whereas when $x \in \mathbb{Z}_{\geq 0}$ is fixed, it is a degree x polynomial in $\mathcal{E}(n;\boldsymbol{\lambda})$ ($n(n+a+b-1)$ for the Hahn and $(q^{-n}-1)(1-\tilde{d}q^n)$, $\tilde{d} \stackrel{\text{def}}{=} abcd^{-1}q^{-1}$ for the q-Racah). When one wants to express the dual polynomial of $P_n(\eta(x))$, *i.e.* $Q_x(\mathcal{E}(n))$, as a degree n polynomial on the x lattice, it is

$$P_n^{\text{d}}\bigl(\eta^{\text{d}}(x;\boldsymbol{\lambda});\boldsymbol{\lambda}\bigr) \stackrel{\text{def}}{=} Q_n\bigl(\mathcal{E}(x;\boldsymbol{\lambda});\boldsymbol{\lambda}\bigr). \tag{2.48}$$

Its explicit form is obtained by the interchange $x \leftrightarrow n$ in the hypergeometric expression, for example:

$$\text{dual Hahn}: \ P_n^{\text{d}}\bigl(\eta^{\text{d}}(x;\boldsymbol{\lambda});\boldsymbol{\lambda}\bigr) = {}_3F_2\Bigl(\genfrac{}{}{0pt}{}{-x,\ x+a+b-1,\ -n}{a,\ -N}\Bigm| 1\Bigr), \quad \eta^{\text{d}}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathcal{E}(x;\boldsymbol{\lambda}),$$

$$\text{dual } q\text{-Racah}: \ P_n^{\text{d}}\bigl(\eta^{\text{d}}(x;\boldsymbol{\lambda});\boldsymbol{\lambda}\bigr) = {}_4\phi_3\Bigl(\genfrac{}{}{0pt}{}{q^{-x},\ \tilde{d}q^x,\ q^{-n},\ dq^n}{a,\ b,\ c}\Bigm| q\,;q\Bigr), \qquad \mathcal{E}^{\text{d}}(n;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \eta(n;\boldsymbol{\lambda}).$$

Here is the summary of the dual correspondence:

$$x \leftrightarrow n, \quad \eta(x) \leftrightarrow \mathcal{E}(n), \quad \eta(0)=0 \leftrightarrow \mathcal{E}(0)=0, \tag{2.49}$$

$$B(x)\leftrightarrow -A_n,\quad D(x)\leftrightarrow -C_n,\quad D(0)=0=C_0,\quad \frac{\phi_0(x)}{\phi_0(0)}\leftrightarrow\frac{d_n}{d_0},\tag{2.50}$$

in which A_n and C_n are the coefficients of the three term recurrence relation of the orthogonal polynomials $P_n(\eta(x))$, (3.8).

The orthogonality relation of the dual polynomials is related to the *normalised* orthogonality relation of the original polynomials:

$$(\hat{\phi}_n,\hat{\phi}_m)=\sum_{x=0}^{\infty}\phi_0(x;\boldsymbol{\lambda})^2 d_n d_m P_n\big(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda}\big)P_m\big(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda}\big)=\delta_{nm}\quad (n,m=0,1,\ldots),\tag{2.51}$$

$$\hat{\phi}_n(x;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\phi_0(x;\boldsymbol{\lambda})d_n(\boldsymbol{\lambda})P_n\big(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda}\big).\tag{2.52}$$

By multiplying $\hat{\phi}_n(y)$ to (2.51) and summing over n produces, under certain conditions, the completeness relation

$$\begin{aligned}\sum_{n=0}^{\infty}\hat{\phi}_n(x)\hat{\phi}_n(y)&=\sum_{n=0}^{\infty}\phi_0(x)\phi_0(y)d_n^2P_n\big(\eta(x)\big)P_n\big(\eta(y)\big)=\delta_{xy}\quad(x,y=0,1,\ldots)\\&=\sum_{n=0}^{\infty}\phi_0(x)\phi_0(y)d_n^2Q_x\big(\mathcal{E}(n)\big)Q_y\big(\mathcal{E}(n)\big),\end{aligned}\tag{2.53}$$

or the dual orthogonality relation

$$\begin{aligned}&\sum_{n=0}^{\infty}d_n(\boldsymbol{\lambda})^2P_n\big(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda}\big)P_n\big(\eta(y;\boldsymbol{\lambda});\boldsymbol{\lambda}\big)\\&=\sum_{n=0}^{\infty}d_n(\boldsymbol{\lambda})^2Q_x\big(\mathcal{E}(n;\boldsymbol{\lambda});\boldsymbol{\lambda}\big)Q_y\big(\mathcal{E}(n;\boldsymbol{\lambda});\boldsymbol{\lambda}\big)=\frac{\delta_{xy}}{\phi_0(x;\boldsymbol{\lambda})^2}.\end{aligned}\tag{2.54}$$

As is well known, the above completeness relation (2.53) simply means that any square summable vector $f(x)$, $\|f\|<\infty$ in the ℓ^2 Hilbert space can be expanded by $\{\hat{\phi}_n\}$:

$$f(x)=\sum_{n=0}^{\infty}(\hat{\phi}_n,f)\hat{\phi}_n(x),$$

and in particular $f(x)=\delta_{xy}$ leads to (2.53). For the orthogonal polynomials defined on a finite lattice, for example, the (q-)Racah polynomials, the completeness relation is an automatic consequence of the original orthogonality relation (2.51). However, for some orthogonal polynomials defined over non-negative integers, *e.g.* the q-Meixner and the q-Charlier polynomials, the completeness relation (2.53) or the dual orthogonality relation (2.54) fails [20]. In the upcoming section we will show that the failure is due to the breakdown of the self-adjointness of the naive dual polynomial system, and present the prescription to derive the correct orthogonality relation, which will be termed as the two component Hamiltonian formalism.

Before closing this section, let us emphasise again that the simple dual correspondence relation (2.47) is the consequence of the universal boundary conditions (2.25), (2.26), (2.45). As will be shown in subsequent sections, the universal boundary conditions cannot be imposed for those polynomials having Jackson integral measures. For them, *e.g.* the big q-Jacobi family, and the q-Meixner (q-Charlier), the dual correspondence relation contains an extra constant factor reflecting the boundary conditions. The duality does not exist for polynomials defined on the full integer lattice, *i.e.* the discrete q-Hermite II and the q-Laguerre to be discussed in section 5.

3 Dual q-Meixner and Two Component Hamiltonian

3.1 Naive dual q-Meixner polynomials

Here we recapitulate the basic data of the q-Meixner polynomials (to be abbreviated as qM hereafter) presented in I:

$$q^{\boldsymbol{\lambda}} = (b, c), \quad \boldsymbol{\delta} = (1, -1), \quad \kappa = q, \quad 0 < b < q^{-1} \text{ or } 0 < -b < c^{-1}q^{-1}, \quad c > 0, \tag{3.1}$$

$$B(x; \boldsymbol{\lambda}) = cq^x(1 - bq^{x+1}), \quad D(x; \boldsymbol{\lambda}) = (1 - q^x)(1 + bcq^x), \tag{3.2}$$

$$\mathcal{E}(n) = 1 - q^n, \quad \eta(x) = q^{-x} - 1, \tag{3.3}$$

$$\check{P}_n(x; \boldsymbol{\lambda}) = P_n\big(\eta(x); \boldsymbol{\lambda}\big) = {}_2\phi_1\Big(\begin{matrix} q^{-n}, q^{-x} \\ bq \end{matrix} \Big| \, q\,; -c^{-1}q^{n+1}\Big) = M_n(q^{-x}; b, c; q), \tag{3.4}$$

$$\phi_0(x; \boldsymbol{\lambda})^2 = c^x q^{\frac{1}{2}x(x-1)} \frac{(bq; q)_x}{(q, -bcq; q)_x} \quad \big(\Leftarrow \phi_0(0; \boldsymbol{\lambda}) = 1\big), \tag{3.5}$$

$$d_n(\boldsymbol{\lambda})^2 = \frac{q^n(bq; q)_n}{(q, -c^{-1}q; q)_n} \times d_0(\boldsymbol{\lambda})^2, \quad d_0(\boldsymbol{\lambda})^2 = \frac{(-bcq; q)_\infty}{(-c; q)_\infty}, \tag{3.6}$$

$$A_n(\boldsymbol{\lambda}) = -cq^{-2n-1}(1 - bq^{n+1}), \quad C_n(\boldsymbol{\lambda}) = -q^{-2n}(1 - q^n)(q^n + c). \tag{3.7}$$

The last entry (3.7) is the coefficients of the three term recurrence relation of the qM ($n = 0, 1, \ldots$),

$$\eta(x)\check{P}_n(x; \boldsymbol{\lambda}) = A_n(\boldsymbol{\lambda})\check{P}_{n+1}(x; \boldsymbol{\lambda}) - \big(A_n(\boldsymbol{\lambda}) + C_n(\boldsymbol{\lambda})\big)\check{P}_n(x; \boldsymbol{\lambda}) + C_n(\boldsymbol{\lambda})\check{P}_n(x; \boldsymbol{\lambda}). \tag{3.8}$$

Note that the potentials are positive for the extended range of b as in (3.1).

Based on these data, we reported that the "dual q-Meixner polynomial" is given by

$$P_n^{\mathrm{d}}\big(\eta^{\mathrm{d}}(x);\boldsymbol{\lambda}\big)={}_2\phi_1\Big(\begin{matrix}q^{-n},\,q^{-x}\\ bq\end{matrix}\Bigm|q\,;-c^{-1}q^{x+1}\Big),\tag{3.9}$$

which satisfies the difference equation ($n=0,1,\ldots$)

$$\big(B^{\mathrm{d}}(x;\boldsymbol{\lambda})(1-e^{\partial})+D^{\mathrm{d}}(x;\boldsymbol{\lambda})(1-e^{-\partial})\big)P_n^{\mathrm{d}}\big(\eta^{\mathrm{d}}(x);\boldsymbol{\lambda}\big)=\mathcal{E}^{\mathrm{d}}(n)P_n^{\mathrm{d}}\big(\eta^{\mathrm{d}}(x);\boldsymbol{\lambda}\big),\tag{3.10}$$

$$B^{\mathrm{d}}(x;\boldsymbol{\lambda})=cq^{-2x-1}(1-bq^{x+1}),\quad D^{\mathrm{d}}(x;\boldsymbol{\lambda})=q^{-2x}(1-q^x)(q^x+c),\tag{3.11}$$

$$\mathcal{E}^{\mathrm{d}}(n)=q^{-n}-1,\quad \eta^{\mathrm{d}}(x)=1-q^x,\tag{3.12}$$

$$\phi_0^{\mathrm{d}}(x;\boldsymbol{\lambda})^2=\phi_0^{\mathrm{d}}(0;\boldsymbol{\lambda})^2\frac{q^x(bq;q)_x}{(q,-c^{-1}q;q)_x},\tag{3.13}$$

$$A_n^{\mathrm{d}}(\boldsymbol{\lambda})=-cq^n(1-bq^{n+1}),\quad C_n^{\mathrm{d}}(\boldsymbol{\lambda})=-(1-q^n)(1+bcq^n).\tag{3.14}$$

The difference equation (3.10) is just rewriting of the above three term recurrence relation (3.8) in terms of the dual correspondence (2.49)–(2.50). But the claim of this "dual q-Meixner polynomial" is not totally correct, since the completeness relation or the dual orthogonality relation (2.54)

$$\sum_{n=0}^{\infty}d_n(\boldsymbol{\lambda})^2P_n\big(\eta(x);\boldsymbol{\lambda}\big)P_n\big(\eta(y);\boldsymbol{\lambda}\big)=\frac{\delta_{xy}}{\phi_0(x;\boldsymbol{\lambda})^2},\tag{3.15}$$

does not hold. The non-orthogonality can be easily seen for certain parameter ranges. For $x=0$, $y=1$, we obtain $P_n(0;\boldsymbol{\lambda})=1$, $P_n(\eta(1);\boldsymbol{\lambda})=1+c^{-1}(q^n-1)/(1-bq)$. In the parameter range $A\stackrel{\text{def}}{=}1-c^{-1}(1-bq)^{-1}>0$, the left hand side of (3.15) does not vanish, l.h.s.$=\sum_{n=0}^{\infty}d_n(\boldsymbol{\lambda})^2\big(A(1-q^n)+q^n\big)>0$.

This is due to the *breakdown of the self-adjointness* of the naive dual Hamiltonian

$$\mathcal{H}^{\mathrm{d}}(\boldsymbol{\lambda})=\mathcal{A}^{\mathrm{d}}(\boldsymbol{\lambda})^{\dagger}\mathcal{A}^{\mathrm{d}}(\boldsymbol{\lambda}),\quad \mathcal{A}^{\mathrm{d}}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}\sqrt{B^{\mathrm{d}}(x;\boldsymbol{\lambda})}-e^{\partial}\sqrt{D^{\mathrm{d}}(x;\boldsymbol{\lambda})},\tag{3.16}$$

$$\mathcal{H}^{\mathrm{d}}(\boldsymbol{\lambda})\phi_n^{\mathrm{d}}(x;\boldsymbol{\lambda})=\mathcal{E}^{\mathrm{d}}(n)\phi_n^{\mathrm{d}}(x;\boldsymbol{\lambda})\ \ (n=0,1,\ldots),\tag{3.17}$$

$$\phi_n^{\mathrm{d}}(x;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\phi_0^{\mathrm{d}}(x;\boldsymbol{\lambda})P_n^{\mathrm{d}}\big(\eta^{\mathrm{d}}(x);\boldsymbol{\lambda}\big).\tag{3.18}$$

The generic element of the corresponding Hilbert space has the form $f(x)=\phi_0^{\mathrm{d}}(x;\boldsymbol{\lambda})\mathcal{P}(\eta^{\mathrm{d}}(x))$, $\|f\|<\infty$, with a smooth function $\mathcal{P}$. At large N, we have

$$\begin{aligned}&\mathcal{P}\big(\eta^{\mathrm{d}}(N)\big)\simeq\mathcal{P}(1)-\mathcal{P}'(1)q^N+O(q^{2N}),\\ &\quad\Rightarrow\mathcal{P}\big(\eta^{\mathrm{d}}(N+1)\big)\mathcal{Q}\big(\eta^{\mathrm{d}}(N)\big)-\mathcal{P}\big(\eta^{\mathrm{d}}(N)\big)\mathcal{Q}\big(\eta^{\mathrm{d}}(N+1)\big)\simeq(1-q)\mathrm{W}[\mathcal{P},\mathcal{Q}](1)q^N+O(q^{2N}),\\ &\phi_0^{\mathrm{d}}(N)^2\simeq q^N\big(1+O(q^N)\big)\times\text{const},\quad B^{\mathrm{d}}(N)\simeq cq^{-2N-1}\big(1+O(q^N)\big),\end{aligned}$$

in which $\mathrm{W}[f,g](x)=f(x)g'(x)-f'(x)g(x)$ is the Wronskian. These show that the right hand side of the criterion formula (2.20) is finite in the $N\to\infty$ limit. The self-adjointness is broken mainly by the very strong increase ($\sim q^{-2x}$) of the potential functions $B^{\mathrm{d}}(x)$, $D^{\mathrm{d}}(x)$ at $x\to\infty$ (3.11) and the almost constant behaviour of the polynomial $P_n^{\mathrm{d}}(\eta^{\mathrm{d}}(x);\boldsymbol{\lambda})$ at $x\to\infty$.

3.2 Two component Hamiltonian system

A clue for the recovery of the self-adjointness of the naive dual q-Meixner (dqM) Hamiltonian system (3.11), (3.16)–(3.18) is the fact that it is accompanied by *another Hamiltonian* sharing the same eigenvalues and the corresponding eigenvectors which are obtained by rescaling the sinusoidal coordinate. A close look at the potential functions (3.11) of the naive Hamiltonian shows that the rescaling of the variable

$$q^x\to -cq^x, \tag{3.19}$$

provides another valid Hamiltonian. For later discussion, let us denote the original naive Hamiltonian and its potential functions etc., by superscript $(+)$ and those of the rescaled one by $(-)$:

$$\mathcal{H}^{(\pm)}=\mathcal{A}^{(\pm)\dagger}\mathcal{A}^{(\pm)},\qquad \mathcal{A}^{(\pm)}\stackrel{\text{def}}{=}\sqrt{B^{(\pm)}(x)}-e^{\partial}\sqrt{D^{(\pm)}(x)}, \tag{3.20}$$

$$B^{(+)}(x)=cq^{-2x-1}(1-bq^{x+1}),\qquad D^{(+)}(x)=q^{-2x}(1-q^x)(q^x+c), \tag{3.21}$$

$$\Downarrow\qquad\qquad q^x\to -cq^x\qquad\qquad\Downarrow$$

$$B^{(-)}(x)=c^{-1}q^{-2x-1}(1+bcq^{x+1}),\quad D^{(-)}(x)=c^{-1}q^{-2x}(1-q^x)(1+cq^x), \tag{3.22}$$

$$\check{P}_n^{(+)}(x)={}_2\phi_1\Bigl(\begin{matrix}q^{-n},\,q^{-x}\\ bq\end{matrix}\Bigm|q\,;-c^{-1}q^{x+1}\Bigr),\ \ \check{P}_n^{(-)}(x)={}_2\phi_1\Bigl(\begin{matrix}q^{-n},\,-c^{-1}q^{-x}\\ bq\end{matrix}\Bigm|q\,;q^{x+1}\Bigr), \tag{3.23}$$

$$\mathcal{H}^{(\pm)}\phi_n^{(\pm)}(x)=\mathcal{E}^{\mathrm{d}}(n)\phi_n^{(\pm)}(x),\quad \phi_n^{(\pm)}(x)=\phi_0^{(\pm)}(x)\check{P}_n^{(\pm)}(x)\ \ (n=0,1,\ldots), \tag{3.24}$$

$$\mathcal{A}^{(\pm)}\phi_0^{(\pm)}(x)=0,\quad \phi_0^{(\pm)}(x)=\phi_0^{(\pm)}(0)\prod_{y=0}^{x-1}\sqrt{\frac{B^{(\pm)}(y)}{D^{(\pm)}(y+1)}}, \tag{3.25}$$

$$\widetilde{\mathcal{H}}^{(\pm)}=\phi_0^{(\pm)}(x)^{-1}\circ\mathcal{H}^{(\pm)}\circ\phi_0^{(\pm)}(x)=B^{(\pm)}(x)(1-e^{\partial})+D^{(\pm)}(x)(1-e^{-\partial}). \tag{3.26}$$

Here we have suppressed the parameter $\boldsymbol{\lambda}$ dependence for simplicity. Except for the ratio of $\phi_0^{(\pm)}(0)$ (3.25), every quantity in the $(-)$ system is determined from that of the $(+)$ system by rescaling (3.19). The positivity of $B^{(-)}(x)$, $D^{(-)}(x)$ and the boundary condition of $D^{(-)}(x)$ are obviously satisfied. The triangularity of $\widetilde{\mathcal{H}}^{(-)}$ (3.26) is trivial, except for the vanishing of q^{-x} term in $\widetilde{\mathcal{H}}^{(-)}q^x$. This is inherited from $\widetilde{\mathcal{H}}^{(+)}$, as the cancellation is achieved by the coefficients of q^{-2x} terms in $B^{(+)}(x)$, $D^{(+)}(x)$. The situation is unchanged by the rescaling (3.19). The eigenvalues of $\widetilde{\mathcal{H}}^{(\pm)}$ (3.26) are the same, since they are determined by the $(q^x)^0$ terms in the potential functions, which are unchanged by the rescaling (3.19). It is easy to verify that both Hamiltonians $\mathcal{H}^{(\pm)}$ are shape invariant.

Now we have two closely related Hamiltonian systems $\mathcal{H}^{(\pm)}$, each of which is not self-adjoint separately. We will demonstrate that the self-adjointness can be recovered in the combined system, to be called two component Hamiltonian system, by adjusting the ratio of $\phi_0^{(\pm)}(0)$ (3.25). Since the rescaling factor is always negative, *e.g.* $q^x \to -cq^x$, this approach leads naturally to the orthogonality measure of the polynomials of *Jackson integral* type. The Jackson integral is defined by [6]

$$\int_0^\alpha d_qy\, f(y) \stackrel{\text{def}}{=} (1-q)\alpha\sum_{k=0}^\infty f(\alpha q^k)q^k, \quad \int_\alpha^\beta d_qy\, f(y) \stackrel{\text{def}}{=} \int_0^\beta d_qy\, f(y) - \int_0^\alpha d_qy\, f(y).$$

Let us introduce appropriate notation for generic two component Hamiltonian systems. The vector $\boldsymbol{f}$ and its inner product are

$$\boldsymbol{f}(x) = \begin{pmatrix} f^{(+)}(x) \\ f^{(-)}(x) \end{pmatrix} \quad (x \in \mathbb{Z}_{\geq 0}), \tag{3.27}$$

$$\begin{aligned} (\!(\boldsymbol{f},\boldsymbol{g})\!) &= \big(f^{(+)}, g^{(+)}\big) + \big(f^{(-)}, g^{(-)}\big) \\ &= \lim_{N\to\infty} (\!(\boldsymbol{f},\boldsymbol{g})\!)_N = \lim_{N\to\infty}\Big(\big(f^{(+)}, g^{(+)}\big)_N + \big(f^{(-)}, g^{(-)}\big)_N\Big). \end{aligned} \tag{3.28}$$

The Hamiltonian is a direct sum

$$\underline{\mathcal{H}} = \begin{pmatrix} \mathcal{H}^{(+)} & 0 \\ 0 & \mathcal{H}^{(-)} \end{pmatrix}. \tag{3.29}$$

The Schrödinger equation is

$$\underline{\mathcal{H}}\boldsymbol{\phi}_n(x) = \mathcal{E}(n)\boldsymbol{\phi}_n(x) \quad (n=0,1,\ldots), \quad 0 = \mathcal{E}(0) < \mathcal{E}(1) < \cdots,$$

where the eigenvector $\boldsymbol{\phi}_n(x) = \left(\begin{smallmatrix} \phi_n^{(+)}(x) \\ \phi_n^{(-)}(x) \end{smallmatrix}\right)$ has a finite norm. We define $\underline{B}(x)$, $\underline{D}(x)$, $\underline{\mathcal{A}}$, etc. as the direct sum as follows:

$$\underline{B}(x) = \begin{pmatrix} B^{(+)}(x) & 0 \\ 0 & B^{(-)}(x) \end{pmatrix}, \quad \underline{D}(x) = \begin{pmatrix} D^{(+)}(x) & 0 \\ 0 & D^{(-)}(x) \end{pmatrix},$$

$$\underline{\mathcal{A}} = \begin{pmatrix} \mathcal{A}^{(+)} & 0 \\ 0 & \mathcal{A}^{(-)} \end{pmatrix}, \quad \underline{\mathcal{A}}^\dagger = \begin{pmatrix} \mathcal{A}^{(+)\dagger} & 0 \\ 0 & \mathcal{A}^{(-)\dagger} \end{pmatrix},$$

$$
\begin{aligned}
&\underline{\widetilde{\mathcal{H}}} = \begin{pmatrix} \widetilde{\mathcal{H}}^{(+)} & 0 \\ 0 & \widetilde{\mathcal{H}}^{(-)} \end{pmatrix}, \quad \underline{\phi_0}(x) = \begin{pmatrix} \phi_0^{(+)}(x) & 0 \\ 0 & \phi_0^{(-)}(x) \end{pmatrix}, \quad \underline{e^{\pm\partial}} = \begin{pmatrix} e^{\pm\partial} & 0 \\ 0 & e^{\pm\partial} \end{pmatrix}, \\
&\underline{\mathcal{H}} = \underline{\mathcal{A}}^\dagger \underline{\mathcal{A}} = -\sqrt{\underline{B}(x)}\, \underline{e^{\partial}} \sqrt{\underline{D}(x)} - \sqrt{\underline{D}(x)}\, \underline{e^{-\partial}} \sqrt{\underline{B}(x)} + \underline{B}(x) + \underline{D}(x), \\
&\underline{\mathcal{A}} = \sqrt{\underline{B}(x)} - \underline{e^{\partial}} \sqrt{\underline{D}(x)}, \quad \underline{\mathcal{A}}^\dagger = \sqrt{\underline{B}(x)} - \sqrt{\underline{D}(x)}\, \underline{e^{-\partial}}, \\
&\underline{\widetilde{\mathcal{H}}} = \underline{\phi_0}(x)^{-1} \circ \underline{\mathcal{H}} \circ \underline{\phi_0}(x) = \underline{B}(x)(1 - \underline{e^{\partial}}) + \underline{D}(x)(1 - \underline{e^{-\partial}}).
\end{aligned} \tag{3.30}
$$

Since $B^{(\pm)}(x)$ and $D^{(\pm)}(x)$ are functions of two sinusoidal coordinates $\eta^{(\pm)}(x)$ of the form

$$
\eta^{(\pm)}(x) = \alpha^{(\pm)} q^x, \quad \alpha^{(\pm)} \in \mathbb{R}, \quad \alpha^{(+)}\alpha^{(-)} < 0, \tag{3.31}
$$

these four input potentials can be expressed by two functions $B^{\mathrm{J}}(\eta)$ and $D^{\mathrm{J}}(\eta)$ as

$$
B^{(\pm)}(x) = B^{\mathrm{J}}\big(\eta^{(\pm)}(x)\big), \quad D^{(\pm)}(x) = D^{\mathrm{J}}\big(\eta^{(\pm)}(x)\big), \tag{3.32}
$$

in which J stands for 'Jackson'. It should be stressed that the sinusoidal coordinates $\eta^{(\pm)}(x)$ in the two component Hamiltonian formalism are not required the universal boundary condition $\eta(0) = 0$ (2.25), which was imposed in [9].

For two vectors $\boldsymbol{f}$ with $f^{(\pm)}(x) = \phi_0^{(\pm)}(x)\check{\mathcal{P}}^{(\pm)}(x)$ and $\boldsymbol{g}$ with $g^{(\pm)}(x) = \phi_0^{(\pm)}(x)\check{\mathcal{Q}}^{(\pm)}(x)$, the criterion of the self-adjointness (2.20) now reads

$$
\begin{aligned}
0 &= \lim_{N\to\infty} \big((\!(\boldsymbol{f}, \underline{\mathcal{H}}\,\boldsymbol{g})\!)_N - (\!(\underline{\mathcal{H}}\,\boldsymbol{f}, \boldsymbol{g})\!)_N \big) \\
&= \lim_{N\to\infty} \Big(\phi_0^{(+)}(N)^2 B^{(+)}(N) \big(\check{\mathcal{P}}^{(+)}(N+1)\check{\mathcal{Q}}^{(+)}(N) - \check{\mathcal{P}}^{(+)}(N)\check{\mathcal{Q}}^{(+)}(N+1) \big) \\
&\qquad + \phi_0^{(-)}(N)^2 B^{(-)}(N) \big(\check{\mathcal{P}}^{(-)}(N+1)\check{\mathcal{Q}}^{(-)}(N) - \check{\mathcal{P}}^{(-)}(N)\check{\mathcal{Q}}^{(-)}(N+1) \big) \Big).
\end{aligned} \tag{3.33}
$$

For smooth functions $\mathcal{P}$ and $\mathcal{Q}$, we have the following large N behaviours:

$$
\begin{aligned}
&\mathcal{P}(\alpha q^N) \simeq \mathcal{P}(0) + \mathcal{P}'(0)\alpha q^N + O(q^{2N}), \\
&\mathcal{P}\big(\eta^{(\pm)}(N+1)\big)\mathcal{Q}\big(\eta^{(\pm)}(N)\big) - \mathcal{P}\big(\eta^{(\pm)}(N)\big)\mathcal{Q}\big(\eta^{(\pm)}(N+1)\big) \\
&\simeq (1-q)\mathrm{W}[\mathcal{P}, \mathcal{Q}](0)\alpha^{(\pm)} q^N + O(q^{2N}).
\end{aligned} \tag{3.34}
$$

Now we are in a position to demonstrate the self-adjointness and to derive the orthogonality relation for the dqM. Two functions $B^{\mathrm{J}}(\eta)$ and $D^{\mathrm{J}}(\eta)$ are

$$
B^{\mathrm{J}}(\eta) \stackrel{\text{def}}{=} \eta^{-2} c q^{-1}(1 - bq\eta), \quad D^{\mathrm{J}}(\eta) \stackrel{\text{def}}{=} \eta^{-2}(1-\eta)(\eta + c). \tag{3.35}
$$

The explicit forms of the ground state eigenfunctions $\phi_0^{(\pm)}(x)$ are

$$
\phi_0^{(+)}(x)^2 = \phi_0^{(+)}(0)^2 \frac{q^x (bq;q)_x}{(q, -c^{-1}q;q)_x}, \quad \phi_0^{(-)}(x)^2 = \phi_0^{(-)}(0)^2 \frac{q^x (-bcq;q)_x}{(q, -cq;q)_x}. \tag{3.36}
$$

Let us introduce new sinusoidal coordinates $\eta^{(\pm)}(x)$, as the original and rescaled q^x:

$$\eta^{(+)}(x) \stackrel{\text{def}}{=} q^x > 0, \quad \eta^{(-)}(x) \stackrel{\text{def}}{=} -cq^x < 0.$$

In terms of these sinusoidal coordinates, the above ground state eigenfunctions (3.36) have a unified expression:

$$\begin{aligned}
\phi_0^{(\pm)}(x)^2 &= \pm A^{(\pm)}\eta^{(\pm)}(x)\frac{(q\eta^{(\pm)}(x), -c^{-1}q\eta^{(\pm)}(x);q)_\infty}{(bq\eta^{(\pm)}(x);q)_\infty},\\
A^{(+)} &\stackrel{\text{def}}{=} \frac{\phi_0^{(+)}(0)^2(bq;q)_\infty}{(q,-c^{-1}q;q)_\infty}, \quad A^{(-)} \stackrel{\text{def}}{=} \frac{\phi_0^{(-)}(0)^2(-bcq;q)_\infty}{c(q,-cq;q)_\infty},\\
\phi_0^{(\pm)}(N)^2 &\simeq \pm A^{(\pm)}\eta^{(\pm)}(N)\big(1+O(q^N)\big) \quad (N\to\infty).
\end{aligned}$$

Then the r.h.s. of the criterion (3.33) reads for $\check{\mathcal{P}}^{(\pm)}(x) = \mathcal{P}(\eta^{(\pm)}(x))$ and $\check{\mathcal{Q}}^{(\pm)}(x) = \mathcal{Q}(\eta^{(\pm)}(x))$:

$$\begin{aligned}
&\lim_{N\to\infty}\Big(A^{(+)}q^N cq^{-2N-1}(1-q)\mathrm{W}[\mathcal{P},\mathcal{Q}](0)q^N + A^{(-)}cq^N c^{-1}q^{-2N-1}(1-q)\mathrm{W}[\mathcal{P},\mathcal{Q}](0)(-c)q^N\Big)\\
&= \big(A^{(+)}-A^{(-)}\big)cq^{-1}(1-q)\mathrm{W}[\mathcal{P},\mathcal{Q}](0),
\end{aligned}$$

implying that the self-adjointness of the two component Hamiltonian system is achieved by the choice

$$A^{(+)} = A^{(-)} = 1, \quad \phi_0^{(+)}(0)^2 \stackrel{\text{def}}{=} \frac{(q,-c^{-1}q;q)_\infty}{(bq;q)_\infty}, \quad \phi_0^{(-)}(0)^2 \stackrel{\text{def}}{=} \frac{c(q,-cq;q)_\infty}{(-bcq;q)_\infty}.$$

The orthogonality relation can also be expressed in terms of the Jackson integral ($n,m = 0,1,\ldots$):

$$\begin{aligned}
(\!(\boldsymbol{\phi}_n,\boldsymbol{\phi}_m)\!) &= \sum_{\epsilon=\pm}\sum_{x=0}^{\infty}\phi_0^{(\epsilon)}(x;\boldsymbol{\lambda})^2 P_n\big(\eta^{(\epsilon)}(x;\boldsymbol{\lambda});\boldsymbol{\lambda}\big)P_m\big(\eta^{(\epsilon)}(x;\boldsymbol{\lambda});\boldsymbol{\lambda}\big) = \frac{\delta_{nm}}{d_n(\boldsymbol{\lambda})^2} && (3.37)\\
&= \sum_{k=0}^{\infty}\frac{(q^{k+1},-c^{-1}q^{k+1};q)_\infty}{(bq^{k+1};q)_\infty}P_n(q^k;\boldsymbol{\lambda})P_m(q^k;\boldsymbol{\lambda})q^k\\
&\quad + c\sum_{k=0}^{\infty}\frac{(q^{k+1},-cq^{k+1};q)_\infty}{(-bcq^{k+1};q)_\infty}P_n(-cq^k;\boldsymbol{\lambda})P_m(-cq^k;\boldsymbol{\lambda})q^k\\
&= \frac{1}{1-q}\int_{-c}^{1}d_qy\,\frac{(qy,-c^{-1}qy;q)_\infty}{(bqy;q)_\infty}P_n(y;\boldsymbol{\lambda})P_m(y;\boldsymbol{\lambda}), && (3.38)
\end{aligned}$$

$$d_n(\boldsymbol{\lambda})^2 = c^n q^{\frac{1}{2}n(n-1)}\frac{(bq;q)_n}{(q,-bcq;q)_n}\times d_0(\boldsymbol{\lambda})^2, \quad d_0(\boldsymbol{\lambda})^2 = \frac{(bq,-bcq;q)_\infty}{(q,-c,-c^{-1}q;q)_\infty}. \qquad (3.39)$$

Note that $d_n(\boldsymbol{\lambda})^2/d_0(\boldsymbol{\lambda})^2$ in (3.39) and $\phi_0(x;\boldsymbol{\lambda})^2$ in (3.5) have the same form. Here the polynomial $P_n(y;\boldsymbol{\lambda})$ is obtained from the original expression (3.9) by the replacement $q^x \to y$:

$$P_n(y;\boldsymbol{\lambda}) = {}_2\phi_1\Bigl(\begin{matrix} q^{-n},\ y^{-1} \\ bq \end{matrix} \Bigm| q\,;-c^{-1}qy\Bigr).$$

The three term recurrence relation of the above dual q-Meixner polynomials is

$$(1-y)P_n(y) = A_n^{\mathrm{d}}P_{n+1}(y) - (A_n^{\mathrm{d}}+C_n^{\mathrm{d}})P_n(y) + C_n^{\mathrm{d}}P_{n-1}(y), \tag{3.40}$$

in which A_n^{d} and C_n^{d} are given in (3.14). This is related to the big q-Laguerre polynomial by the rescaling of the variable y and redefinition of the parameters.

3.3 Dual q-Charlier polynomials

The naive dual q-Charlier polynomials presented in I do not satisfy the orthogonality relation, due to the breakdown of the self-adjointness condition. As for the q-Meixner system, the two component Hamiltonian formulation offers the remedy.

The Hamiltonian system for the q-Charlier polynomials (qC) is obtained from that of qM by putting $b=0$ and change of the parameter $c \to a$. The basic data presented in I are:

$$q^{\boldsymbol{\lambda}} = a, \quad \boldsymbol{\delta} = -1, \quad \kappa = q, \quad a > 0,$$

$$B(x;\boldsymbol{\lambda}) = aq^x, \quad D(x) = 1-q^x, \quad \mathcal{E}(n) = 1-q^n, \quad \eta(x) = q^{-x}-1, \tag{3.41}$$

$$\check{P}_n(x;\boldsymbol{\lambda}) = P_n\bigl(\eta(x);\boldsymbol{\lambda}\bigr) = {}_2\phi_1\Bigl(\begin{matrix} q^{-n},\ q^{-x} \\ 0 \end{matrix} \Bigm| q\,;-a^{-1}q^{n+1}\Bigr) = C(q^{-x};a;q), \tag{3.42}$$

$$\phi_0(x;\boldsymbol{\lambda})^2 = \frac{a^x q^{\frac{1}{2}x(x-1)}}{(q;q)_x} \quad \bigl(\Leftarrow \phi_0(0;\boldsymbol{\lambda}) = 1\bigr), \tag{3.43}$$

$$d_n(\boldsymbol{\lambda})^2 = \frac{q^n}{(q,-a^{-1}q;q)_n} \times d_0(\boldsymbol{\lambda})^2, \quad d_0(\boldsymbol{\lambda})^2 = \frac{1}{(-a;q)_\infty}, \tag{3.44}$$

$$A_n(\boldsymbol{\lambda}) = -aq^{-2n-1}, \quad C_n(\boldsymbol{\lambda}) = -q^{-2n}(1-q^n)(q^n+a). \tag{3.45}$$

The $(+)$ part of the dual q-Charlier (dqC) system is

$$\check{P}_n^{(+)}(x;\boldsymbol{\lambda}) = P_n\bigl(\eta^{(+)}(x);\boldsymbol{\lambda}\bigr) = {}_2\phi_1\Bigl(\begin{matrix} q^{-n},\ q^{-x} \\ 0 \end{matrix} \Bigm| q\,;-a^{-1}q^{x+1}\Bigr), \tag{3.46}$$

$$B^{(+)}(x;\boldsymbol{\lambda}) = aq^{-2x-1}, \quad D^{(+)}(x;\boldsymbol{\lambda}) = q^{-2x}(1-q^x)(q^x+a), \tag{3.47}$$

$$\mathcal{E}(n) = q^{-n}-1, \quad \eta^{(+)}(x) = q^x, \tag{3.48}$$

$$\phi_0^{(+)}(x;\boldsymbol{\lambda})^2 = (q,-a^{-1}q;q)_\infty \frac{q^x}{(q,-a^{-1}q;q)_x} = q^x(q^{x+1},-a^{-1}q^{x+1};q)_\infty, \tag{3.49}$$

$$A_n(\boldsymbol{\lambda}) = -aq^n, \quad C_n = -(1-q^n). \tag{3.50}$$

The self-adjointness of the $(+)$ part Hamiltonian is broken as in the dqM case. The quantities of the $(-)$ part are obtained by the replacement $q^x \to -aq^x$:

$$\check{P}_n^{(-)}(x;\boldsymbol{\lambda}) = P_n\bigl(\eta^{(-)}(x);\boldsymbol{\lambda}\bigr) = {}_2\phi_1\Bigl(\begin{matrix}q^{-n},\ -a^{-1}q^{-x}\\ 0\end{matrix}\Bigm|q\,;q^{x+1}\Bigr), \quad \eta^{(-)}(x) = -aq^x, \tag{3.51}$$

$$B^{(-)}(x;\boldsymbol{\lambda}) = a^{-1}q^{-2x-1}, \quad D^{(-)}(x;\boldsymbol{\lambda}) = a^{-1}q^{-2x}(1-q^x)(1+aq^x), \tag{3.52}$$

$$\phi_0^{(-)}(x;\boldsymbol{\lambda})^2 = a(q,-aq;q)_\infty\frac{q^x}{(q,-aq;q)_x} = aq^x(q^{x+1},-aq^{x+1};q)_\infty. \tag{3.53}$$

With the ground state eigenfunctions $\phi_0^{(\pm)}(x;\boldsymbol{\lambda})^2$ (3.49), (3.53), the self-adjointness is achieved and the orthogonality relation is

$$(\!(\boldsymbol{\phi}_n,\boldsymbol{\phi}_m)\!) = \sum_{\epsilon=\pm}\sum_{x=0}^{\infty}\phi_0^{(\epsilon)}(x;\boldsymbol{\lambda})^2 P_n\bigl(\eta^{(\epsilon)}(x;\boldsymbol{\lambda});\boldsymbol{\lambda}\bigr)P_m\bigl(\eta^{(\epsilon)}(x;\boldsymbol{\lambda});\boldsymbol{\lambda}\bigr) = \frac{\delta_{nm}}{d_n(\boldsymbol{\lambda})^2} \tag{3.54}$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty}(q^{k+1},-a^{-1}q^{k+1};q)_\infty P_n(q^k;\boldsymbol{\lambda})P_m(q^k;\boldsymbol{\lambda})q^k\\
&\quad + a\sum_{k=0}^{\infty}(q^{k+1},-aq^{k+1};q)_\infty P_n(-aq^k;\boldsymbol{\lambda})P_m(-aq^k;\boldsymbol{\lambda})q^k\\
&= \frac{1}{1-q}\int_{-a}^{1}d_qy\,(qy,-a^{-1}qy;q)_\infty P_n(y;\boldsymbol{\lambda})P_m(y;\boldsymbol{\lambda}),
\end{aligned} \tag{3.55}$$

$$d_n(\boldsymbol{\lambda})^2 = \frac{a^nq^{\frac12 n(n-1)}}{(q;q)_n}\times d_0(\boldsymbol{\lambda})^2, \quad d_0(\boldsymbol{\lambda})^2 = \frac{1}{(q,-a,-a^{-1}q;q)_\infty}, \tag{3.56}$$

$$P_n(y;\boldsymbol{\lambda}) = {}_2\phi_1\Bigl(\begin{matrix}q^{-n},\ y^{-1}\\ 0\end{matrix}\Bigm|q\,;-a^{-1}qy\Bigr), \tag{3.57}$$

in which $\{P_n(y)\}$ satisfy the three term recurrence relation (3.40) with A_n^{d} and C_n^{d} replaced by $A_n(\boldsymbol{\lambda})$, C_n in (3.50). This is related to the Al-Salam-Carlitz I polynomial by the redefinition of the parameter.

4 Orthogonal Polynomials and Jackson Integral

In this section we discuss polynomials having orthogonality measures of Jackson integral type and their duals. As exemplified by the dqM in the previous section, the Jackson integral measures are closely related to the two component Hamiltonian formalism. The discrete q-Hermite polynomial II has some exceptional features, which will be discussed separately in § 5.

4.1 General structure

Let us begin with the general structures of the two component Hamiltonian systems. The most generic potential functions have the following form (The superscript J stands for 'Jackson,' see (3.32))

$$B^{\mathrm{J}}(\eta)=\eta^{-2}p_1(\eta),\quad D^{\mathrm{J}}(\eta)=\eta^{-2}p_2(\eta), \tag{4.1}$$

in which $p_1(\eta)$ is a polynomial in $\eta\propto q^x$ (3.31) of at most degree 2, whereas $p_2(\eta)$ is a polynomial of exact degree 2 in η. Thus the system has at most six real parameters. Among them, one corresponding to the overall normalisation of the eigenvalues $\{\mathcal{E}(n)\}$ can be fixed to 1. This constrains the degree two terms of $p_1(\eta)$ and $p_2(\eta)$. Since the rescaling of η does not change the form (4.1), one parameter can be reduced. The boundary condition of $D^{(\pm)}(0)=0$ determines $\alpha^{(\pm)}$ in $\eta^{(\pm)}(x)$ (3.31). The triangularity of $\widetilde{\mathcal{H}}^{(\pm)}$ provides the third constraint, requiring

$$\widetilde{\mathcal{H}}^{(\pm)}\eta=c_1\eta+c_0+0\times\eta^{-1}. \tag{4.2}$$

This gives the constraint on the constant terms of $p_1(\eta)$ and $p_2(\eta)$, as $0=p_1(0)(1-q)+p_2(0)(1-q^{-1})$. With this condition, the triangularity of $\widetilde{\mathcal{H}}^{(\pm)}$ with respect to the basis of $\{1,\eta,\eta^2\ldots,\eta^n\}$ is obvious. The strong increase $\sim q^{-2x}$ of the potentials at $x\to\infty$ and the corresponding eigenpolynomials in q^x require the two component Hamiltonian formalism for self-adjointness. The positivity of $B^{(\pm)}(x)$ and $D^{(\pm)}(x)$ for $x\in\mathbb{Z}_{\geq 0}$ restricts the ranges of the parameters. Thus the most generic two component Hamiltonian systems have three independent parameters. In contrast, the systems with the least number of parameters correspond to the case $p_1(\eta)=const.$ and the dqC discussed in § 3.3 belong to this class. When these conditions are satisfied, the solvability of the Hamiltonian is guaranteed independent of the parametrisation. However, very special parametrisation is required for the shape invariance (2.28)–(2.30) which ensures the existence of explicit expressions of the polynomials of the Rodrigues type (2.40). Choosing specific parametrisation of the two polynomials $p_1(\eta)$ and $p_2(\eta)$ in (4.1) is tantamount to discussion of the corresponding specific polynomial system, which will be presented subsequently. Before closing this subsection, let us mention that any higher degree generalisation of the above input (4.1), for example,

$$B^{\mathrm{J}}(\eta)=\eta^{-3}\bar{p}_1(\eta),\quad D^{\mathrm{J}}(\eta)=\eta^{-3}\bar{p}_2(\eta),$$

with cubic polynomials $\bar{p}_1$ and $\bar{p}_2$, cannot work. The triangularity of $\widetilde{\mathcal{H}}^{(\pm)}$ is broken, since the two conditions $\widetilde{\mathcal{H}}^{(\pm)}\eta=c_1\eta+c_0+0\times\eta^{-1}+0\times\eta^{-2}$ and $\widetilde{\mathcal{H}}^{(\pm)}\eta^2=c_2'\eta^2+c_1'\eta+c_0'+0\times\eta^{-1}$

are incompatible.

4.2 Big q-Jacobi

The big q-Jacobi (to be abbreviated as bqJ hereafter) system has three real parameters on top of q and it is the most generic member having the Jackson integral measure. Its data are:

$$q^{\boldsymbol{\lambda}}=(a,b,c),\quad \boldsymbol{\delta}=(1,1,1),\quad \kappa=q^{-1},\quad 0<a<q^{-1},\quad 0<b<q^{-1},\quad c<0, \tag{4.3}$$

$$\mathcal{E}(n;\boldsymbol{\lambda})\stackrel{\text{def}}{=}(q^{-n}-1)(1-abq^{n+1}),\quad \eta^{(+)}(x;\boldsymbol{\lambda})\stackrel{\text{def}}{=}aq^{x+1},\quad \eta^{(-)}(x;\boldsymbol{\lambda})\stackrel{\text{def}}{=}cq^{x+1}, \tag{4.4}$$

$$B^{\text{J}}(\eta;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\eta^{-2}aq(1-\eta)(b\eta-c),\quad D^{\text{J}}(\eta;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\eta^{-2}(aq-\eta)(\eta-cq). \tag{4.5}$$

It is easy to verify the eigenvalues $\mathcal{E}(n;\boldsymbol{\lambda})$ (4.4) and the triangularity (4.2). To be more explicit, the potential functions (3.32) are

$$\begin{aligned}B^{(+)}(x;\boldsymbol{\lambda})&=-a^{-1}cq^{-2x-1}(1-aq^{x+1})(1-abc^{-1}q^{x+1}),\\ D^{(+)}(x;\boldsymbol{\lambda})&=q^{-2x}(1-q^x)(q^x-a^{-1}c),\end{aligned} \tag{4.6}$$

$$\begin{aligned}B^{(-)}(x;\boldsymbol{\lambda})&=-ac^{-1}q^{-2x-1}(1-cq^{x+1})(1-bq^{x+1}),\\ D^{(-)}(x;\boldsymbol{\lambda})&=q^{-2x}(1-q^x)(q^x-ac^{-1}).\end{aligned} \tag{4.7}$$

For the parameter range (4.3), the positivity and the boundary conditions are satisfied

$$B^{(\pm)}(x;\boldsymbol{\lambda})>0\ \ (x\geq 0),\quad D^{(\pm)}(x;\boldsymbol{\lambda})>0\ \ (x\geq 1),\quad D^{(\pm)}(0;\boldsymbol{\lambda})=0.$$

It is straightforward to verify that the shape invariance conditions (2.29)–(2.30) are satisfied and that the same eigenvalues as above (4.4) are obtained by (2.31). The sinusoidal coordinates $\eta^{(\pm)}(x;\boldsymbol{\lambda})$ (4.4) satisfy the following relations

$$\eta^{(\pm)}(x+1;\boldsymbol{\lambda})=q\eta^{(\pm)}(x;\boldsymbol{\lambda})=\eta^{(\pm)}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}), \tag{4.8}$$

$$\eta^{(\pm)}(x+1;\boldsymbol{\lambda})+\eta^{(\pm)}(x-1;\boldsymbol{\lambda})=(q+q^{-1})\eta^{(\pm)}(x;\boldsymbol{\lambda}), \tag{4.9}$$

$$\eta^{(\pm)}(x+1;\boldsymbol{\lambda})\eta^{(\pm)}(x-1;\boldsymbol{\lambda})=\eta^{(\pm)}(x;\boldsymbol{\lambda})^2. \tag{4.10}$$

The big q-Jacobi polynomial $P_n(\eta;\boldsymbol{\lambda})$ ($n\in\mathbb{Z}_{\geq 0}$), which is a degree n polynomial in η satisfying the second order difference equation,

$$B^{\text{J}}(\eta;\boldsymbol{\lambda})\big(P_n(\eta;\boldsymbol{\lambda})-P_n(q\eta;\boldsymbol{\lambda})\big)+D^{\text{J}}(\eta;\boldsymbol{\lambda})\big(P_n(\eta;\boldsymbol{\lambda})-P_n(q^{-1}\eta;\boldsymbol{\lambda})\big)=\mathcal{E}(n;\boldsymbol{\lambda})P_n(\eta;\boldsymbol{\lambda}), \tag{4.11}$$

has a truncated hypergeometric expression [6]

$$P_n(\eta;\boldsymbol{\lambda})\stackrel{\text{def}}{=}P_n(\eta;a,b,c;q)\stackrel{\text{def}}{=}{}_3\phi_2\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\,abq^{n+1},\,\eta}{aq,\,cq}\Bigm|q\,;q\Bigr),\quad P_n(1;\boldsymbol{\lambda})=1. \tag{4.12}$$

The explicit expressions of the two component eigenpolynomials are

$$\check{P}_n^{(\pm)}(x;\boldsymbol{\lambda})\stackrel{\text{def}}{=}P_n\bigl(\eta^{(\pm)}(x;\boldsymbol{\lambda});\boldsymbol{\lambda}\bigr), \tag{4.13}$$

$$\check{P}_n^{(+)}(x;\boldsymbol{\lambda})={}_3\phi_2\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\,abq^{n+1},\,aq^{x+1}}{aq,\,cq}\Bigm|q\,;q\Bigr), \tag{4.14}$$

$$\check{P}_n^{(-)}(x;\boldsymbol{\lambda})={}_3\phi_2\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\,abq^{n+1},\,cq^{x+1}}{aq,\,cq}\Bigm|q\,;q\Bigr). \tag{4.15}$$

The coefficients of the three term recurrence relation for the big q-Jacobi polynomial $P_n(\eta;\boldsymbol{\lambda})$

$$(1-\eta)P_n(\eta;\boldsymbol{\lambda})=A_n(\boldsymbol{\lambda})P_{n+1}(\eta;\boldsymbol{\lambda})-\bigl(A_n(\boldsymbol{\lambda})+C_n(\boldsymbol{\lambda})\bigr)P_n(\eta;\boldsymbol{\lambda})+C_n(\boldsymbol{\lambda})P_{n-1}(\eta;\boldsymbol{\lambda}), \tag{4.16}$$

are

$$A_n(\boldsymbol{\lambda})\stackrel{\text{def}}{=}-\frac{(1-aq^{n+1})(1-abq^{n+1})(1-cq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})}, \tag{4.17}$$

$$C_n(\boldsymbol{\lambda})\stackrel{\text{def}}{=}acq^{n+1}\frac{(1-q^n)(1-abc^{-1}q^n)(1-bq^n)}{(1-abq^{2n})(1-abq^{2n+1})}. \tag{4.18}$$

Let us note that the recurrence relation with the initial condition $P_0(\eta;\boldsymbol{\lambda})=1$ implies $P_n(1;\boldsymbol{\lambda})=1$ $(n=1,2,\ldots)$ (4.12).

From (2.12), the ground state vectors $\phi_0^{(\pm)}(x;\boldsymbol{\lambda})>0$ are

$$\phi_0^{(+)}(x;\boldsymbol{\lambda})^2=\phi_0^{(+)}(0;\boldsymbol{\lambda})^2\,\frac{q^x(aq,abc^{-1}q;q)_x}{(q,ac^{-1}q;q)_x},$$

$$\phi_0^{(-)}(x;\boldsymbol{\lambda})^2=\phi_0^{(-)}(0;\boldsymbol{\lambda})^2\,\frac{q^x(cq,bq;q)_x}{(q,a^{-1}cq;q)_x},$$

in which the ratio of $\phi_0^{(\pm)}(0;\boldsymbol{\lambda})^2$ is as yet unspecified but the overall normalisation is immaterial. They can be rewritten with the newly introduced constant factor $A^{(\pm)}$ as

$$\phi_0^{(\pm)}(x;\boldsymbol{\lambda})^2=\pm A^{(\pm)}\eta^{(\pm)}(x;\boldsymbol{\lambda})\frac{\bigl(a^{-1}\eta^{(\pm)}(x;\boldsymbol{\lambda}),c^{-1}\eta^{(\pm)}(x;\boldsymbol{\lambda});q\bigr)_\infty}{\bigl(\eta^{(\pm)}(x;\boldsymbol{\lambda}),bc^{-1}\eta^{(\pm)}(x;\boldsymbol{\lambda});q\bigr)_\infty}, \tag{4.19}$$

$$A^{(+)}\stackrel{\text{def}}{=}\frac{\phi_0^{(+)}(0;\boldsymbol{\lambda})^2(aq,abc^{-1}q;q)_\infty}{aq(q,ac^{-1}q;q)_\infty},\quad A^{(-)}\stackrel{\text{def}}{=}\frac{\phi_0^{(-)}(0;\boldsymbol{\lambda})^2(cq,bq;q)_\infty}{-cq(q,a^{-1}cq;q)_\infty}.$$

The other eigenvectors of the Hamiltonians $\mathcal{H}^{(\pm)}(\boldsymbol{\lambda})$ are

$$\phi_n^{(\pm)}(x;\boldsymbol{\lambda})=\phi_0^{(\pm)}(x;\boldsymbol{\lambda})\check{P}_n^{(\pm)}(x;\boldsymbol{\lambda}), \tag{4.20}$$

$$\mathcal{H}^{(\pm)}(\boldsymbol{\lambda})\phi_n^{(\pm)}(x;\boldsymbol{\lambda})=\mathcal{E}(n;\boldsymbol{\lambda})\phi_n^{(\pm)}(x;\boldsymbol{\lambda})\quad(n=0,1,\ldots).\tag{4.21}$$

Now we demonstrate the self-adjointness. For two smooth functions $\mathcal{P}$ and $\mathcal{Q}$, we take two vectors $\boldsymbol{f}$ and $\boldsymbol{g}$ as follows:

$$f^{(\pm)}(x)=\phi_0^{(\pm)}(x)\mathcal{P}\bigl(\eta^{(\pm)}(x;\boldsymbol{\lambda})\bigr),\quad g^{(\pm)}(x)=\phi_0^{(\pm)}(x)\mathcal{Q}\bigl(\eta^{(\pm)}(x;\boldsymbol{\lambda})\bigr).$$

The asymptotic forms of $\phi_0^{(\pm)}(N;\boldsymbol{\lambda})^2$ and $B^{(\pm)}(N;\boldsymbol{\lambda})$ at large N,

$$\phi_0^{(\pm)}(N;\boldsymbol{\lambda})^2\simeq\pm A^{(\pm)}\eta^{(\pm)}(N;\boldsymbol{\lambda})\bigl(1+O(q^N)\bigr),\quad B^{(\pm)}(N;\boldsymbol{\lambda})\simeq\frac{-acq}{\eta^{(\pm)}(N;\boldsymbol{\lambda})^2}\bigl(1+O(q^N)\bigr),$$

and (3.34) show that the criterion for the self-adjointness (2.20) of the separate Hamiltonians $\mathcal{H}^{(\pm)}$ does not hold,

$$\lim_{N\to\infty}\bigl((f^{(\pm)},\mathcal{H}^{(\pm)}g^{(\pm)})_N-(\mathcal{H}^{(\pm)}f^{(\pm)},g^{(\pm)})_N\bigr)=\pm A^{(\pm)}(-acq)(1-q)\mathrm{W}[\mathcal{P},\mathcal{Q}](0)\neq 0.$$

For the two component Hamiltonian $\underline{\mathcal{H}}$, that is, $\mathcal{H}^{(+)}$ combined with $\mathcal{H}^{(-)}$, the self-adjointness is recovered for the choice

$$A^{(+)}=A^{(-)}=1,\tag{4.22}$$

as the criterion (3.33) is satisfied

$$\lim_{N\to\infty}\bigl((\!(\boldsymbol{f},\underline{\mathcal{H}}\,\boldsymbol{g})\!)_N-(\!(\underline{\mathcal{H}}\,\boldsymbol{f},\boldsymbol{g})\!)_N\bigr)=\bigl(A^{(+)}-A^{(-)}\bigr)(-acq)(1-q)\mathrm{W}[\mathcal{P},\mathcal{Q}](0)=0.$$

With the above choice of $A^{(\pm)}$, the ground state vectors (4.19) read

$$\phi_0^{(\pm)}(x;\boldsymbol{\lambda})^2=\pm\eta^{(\pm)}(x;\boldsymbol{\lambda})\frac{\bigl(a^{-1}\eta^{(\pm)}(x;\boldsymbol{\lambda}),c^{-1}\eta^{(\pm)}(x;\boldsymbol{\lambda});q\bigr)_\infty}{\bigl(\eta^{(\pm)}(x;\boldsymbol{\lambda}),bc^{-1}\eta^{(\pm)}(x;\boldsymbol{\lambda});q\bigr)_\infty},$$

implying that the inner product of $\boldsymbol{f}$ and $\boldsymbol{g}$ (3.28) is expressed by the Jackson integral,

$$(\!(\boldsymbol{f},\boldsymbol{g})\!)=\frac{1}{1-q}\int_{cq}^{aq}d_qy\,\frac{(a^{-1}y,c^{-1}y;q)_\infty}{(y,bc^{-1}y;q)_\infty}\mathcal{P}(y)\mathcal{Q}(y).$$

Thus we arrive at the orthogonality relation of the big q-Jacobi polynomials

$$\begin{aligned}(\!(\boldsymbol{\phi}_n,\boldsymbol{\phi}_m)\!)&=\bigl(\phi_n^{(+)},\phi_m^{(+)}\bigr)+\bigl(\phi_n^{(-)},\phi_m^{(-)}\bigr)=\frac{\delta_{nm}}{d_n(\boldsymbol{\lambda})^2}\quad(n,m=0,1,\ldots)\qquad(4.23)\\&=\sum_{\epsilon=\pm}\sum_{x=0}^{\infty}\phi_0^{(\epsilon)}(x;\boldsymbol{\lambda})^2P_n\bigl(\eta^{(\epsilon)}(x;\boldsymbol{\lambda});\boldsymbol{\lambda}\bigr)P_m\bigl(\eta^{(\epsilon)}(x;\boldsymbol{\lambda});\boldsymbol{\lambda}\bigr)\\&=aq\sum_{k=0}^{\infty}\frac{(q^{k+1},ac^{-1}q^{k+1};q)_\infty}{(aq^{k+1},abc^{-1}q^{k+1};q)_\infty}P_n(aq^{k+1};\boldsymbol{\lambda})P_m(aq^{k+1};\boldsymbol{\lambda})q^k\end{aligned}$$

$$
\begin{aligned}
&-cq\sum_{k=0}^{\infty}\frac{(q^{k+1},a^{-1}cq^{k+1};q)_\infty}{(cq^{k+1},bq^{k+1};q)_\infty}P_n(cq^{k+1};\boldsymbol{\lambda})P_m(cq^{k+1};\boldsymbol{\lambda})q^k\\
&=\frac{1}{1-q}\int_{cq}^{aq}d_qy\,\frac{(a^{-1}y,c^{-1}y;q)_\infty}{(y,bc^{-1}y;q)_\infty}P_n(y;\boldsymbol{\lambda})P_m(y;\boldsymbol{\lambda}),
\end{aligned}\tag{4.24}
$$

where the normalisation constant $d_n(\boldsymbol{\lambda})^2$ is

$$
\begin{aligned}
d_n(\boldsymbol{\lambda})^2&\stackrel{\text{def}}{=}(-acq^2)^{-n}q^{-\frac12 n(n-1)}\frac{1-abq^{2n+1}}{1-abq^{n+1}}\frac{(aq,abq^2,cq;q)_n}{(q,bq,abc^{-1}q;q)_n}\times d_0(\boldsymbol{\lambda})^2,\\
d_0(\boldsymbol{\lambda})^2&\stackrel{\text{def}}{=}\frac{(aq,bq,cq,abc^{-1}q;q)_\infty}{aq(q,abq^2,ac^{-1}q,a^{-1}c;q)_\infty}.
\end{aligned}\tag{4.25}
$$

The normalised eigenvectors are

$$
\hat{\boldsymbol{\phi}}_n(x;\boldsymbol{\lambda})=\begin{pmatrix}\hat\phi_n^{(+)}(x;\boldsymbol{\lambda})\\ \hat\phi_n^{(-)}(x;\boldsymbol{\lambda})\end{pmatrix},\quad \hat\phi_n^{(\pm)}(x;\boldsymbol{\lambda})=d_n(\boldsymbol{\lambda})\phi_0^{(\pm)}(x;\boldsymbol{\lambda})\check{P}_n^{(\pm)}(x;\boldsymbol{\lambda}),
$$

and they satisfy the orthogonality relation

$$
(\!(\hat{\boldsymbol{\phi}}_n,\hat{\boldsymbol{\phi}}_m)\!)=\big(\hat\phi_n^{(+)},\hat\phi_n^{(+)}\big)+\big(\hat\phi_n^{(-)},\hat\phi_n^{(-)}\big)=\delta_{nm}.
$$

The forward/backward shift relations for the polynomial $P_n(\eta;\boldsymbol{\lambda})$ are

$$
\begin{aligned}
&\frac{(1-aq)(1-cq)}{q\eta}\big(P_n(\eta;\boldsymbol{\lambda})-P_n(q\eta;\boldsymbol{\lambda})\big)=\mathcal{E}(n;\boldsymbol{\lambda})P_{n-1}(q\eta;\boldsymbol{\lambda}+\boldsymbol{\delta}),\\
&\big(B^{\text{J}}(\eta;\boldsymbol{\lambda})P_{n-1}(q\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})-q^{-1}D^{\text{J}}(\eta;\boldsymbol{\lambda})P_{n-1}(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})\big)\frac{q\eta}{(1-aq)(1-cq)}=P_n(\eta;\boldsymbol{\lambda}).
\end{aligned}
$$

These relations can be rewritten as the forward/backward shift relations for the polynomials $\check{P}_n^{(\pm)}(x;\boldsymbol{\lambda})$ on the x-lattice, (2.37)–(2.38)

$$
\mathcal{F}^{(\pm)}(\boldsymbol{\lambda})=\varphi^{(\pm)}(x;\boldsymbol{\lambda})^{-1}(1-e^{\partial}),\quad \mathcal{B}^{(\pm)}(\boldsymbol{\lambda})=\big(B^{(\pm)}(x;\boldsymbol{\lambda})-D^{(\pm)}(x;\boldsymbol{\lambda})e^{-\partial}\big)\varphi^{(\pm)}(x;\boldsymbol{\lambda}),\tag{4.26}
$$

$$
\mathcal{F}^{(\pm)}(\boldsymbol{\lambda})\check{P}_n^{(\pm)}(x;\boldsymbol{\lambda})=\mathcal{E}(n;\boldsymbol{\lambda})\check{P}_{n-1}^{(\pm)}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}),\quad \mathcal{B}^{(\pm)}(\boldsymbol{\lambda})\check{P}_{n-1}^{(\pm)}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})=\check{P}_n^{(\pm)}(x;\boldsymbol{\lambda}),\tag{4.27}
$$

where the auxiliary functions $\varphi^{(\pm)}(x;\boldsymbol{\lambda})$ are

$$
\varphi^{(\pm)}(x;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\frac{q\eta^{(\pm)}(x;\boldsymbol{\lambda})}{(1-aq)(1-cq)}.\tag{4.28}
$$

It is interesting to note that the following parameter substitution (involution)

$$
(a,b,c)\to(c,abc^{-1},a),\tag{4.29}
$$

gives rise to the interchange of the $(+)$ and $(-)$ systems

$$
\begin{aligned}
&B^{(+)}(x;\boldsymbol{\lambda}) \leftrightarrow B^{(-)}(x;\boldsymbol{\lambda}), \qquad \eta^{(+)}(x;\boldsymbol{\lambda}) \leftrightarrow \eta^{(-)}(x;\boldsymbol{\lambda}),\\
&D^{(+)}(x;\boldsymbol{\lambda}) \leftrightarrow D^{(-)}(x;\boldsymbol{\lambda}), \qquad \check{P}_n^{(+)}(x;\boldsymbol{\lambda}) \leftrightarrow \check{P}_n^{(-)}(x;\boldsymbol{\lambda}),\\
&B^{\mathrm{J}}(\eta;\boldsymbol{\lambda}),\ D^{\mathrm{J}}(\eta;\boldsymbol{\lambda}),\ \mathcal{E}(n;\boldsymbol{\lambda}),\ P_n(\eta;\boldsymbol{\lambda}):\ \text{invariant},
\end{aligned}
\tag{4.30}
$$

but the parameter range (4.3) is not preserved.

In [9] we have presented a unified prescription to obtain (quasi-)exactly solvable difference Schrödinger equations in which the potential functions are constructed in terms of the sinusoidal coordinates. Here we show how the bqJ system fits in that scheme, although the solvability is obvious from the start in the two component Hamiltonian formalism. By taking $v_{k,l}$ as

$$
\begin{aligned}
&v_{0,0} = -(1-q)(1-q^2)ac,\\
&v_{1,0} = (1-q)\bigl(a(b+c)-q(a+c)\bigr), \quad v_{0,1} = (1-q)\bigl(a+c-qa(b+c)\bigr),\\
&v_{2,0} = (1-q)(1-ab)+v_{0,2}, \qquad\quad v_{1,1} = (1-q^{-1})(1-abq^2)-(q+q^{-1})v_{0,2},
\end{aligned}
\tag{4.31}
$$

($v_{0,2}$ is arbitrary), the potential functions are expressed as follows:

$$
B^{(\pm)}(x;\boldsymbol{\lambda}) = \frac{\sum_{\substack{k,l\geq 0\\ k+l\leq 2}} v_{k,l}\eta^{(\pm)}(x;\boldsymbol{\lambda})^k\eta^{(\pm)}(x+1;\boldsymbol{\lambda})^l}{\bigl(\eta^{(\pm)}(x+1;\boldsymbol{\lambda})-\eta^{(\pm)}(x;\boldsymbol{\lambda})\bigr)\bigl(\eta^{(\pm)}(x+1;\boldsymbol{\lambda})-\eta^{(\pm)}(x-1;\boldsymbol{\lambda})\bigr)}, \tag{4.32}
$$

$$
D^{(\pm)}(x;\boldsymbol{\lambda}) = \frac{\sum_{\substack{k,l\geq 0\\ k+l\leq 2}} v_{k,l}\eta^{(\pm)}(x;\boldsymbol{\lambda})^k\eta^{(\pm)}(x-1;\boldsymbol{\lambda})^l}{\bigl(\eta^{(\pm)}(x-1;\boldsymbol{\lambda})-\eta^{(\pm)}(x;\boldsymbol{\lambda})\bigr)\bigl(\eta^{(\pm)}(x-1;\boldsymbol{\lambda})-\eta^{(\pm)}(x+1;\boldsymbol{\lambda})\bigr)}. \tag{4.33}
$$

All the above parameters $v_{k,l}$ (4.31) are invariant under the parameter substitution (4.29). In the logic of [9], the upper triangularity of the similarity transformed Hamiltonians $\widetilde{\mathcal{H}}^{(\pm)}$ with respect to the bases $\{\eta^{(\pm)}(x;\boldsymbol{\lambda})^k\}_{k=0,1,\ldots}$ is shown through the expansion:

$$
\frac{\eta^{(\pm)}(x+1;\boldsymbol{\lambda})^{n+1}-\eta^{(\pm)}(x-1;\boldsymbol{\lambda})^{n+1}}{\eta^{(\pm)}(x+1;\boldsymbol{\lambda})-\eta^{(\pm)}(x-1;\boldsymbol{\lambda})} = \sum_{k=0}^{n} g_n^{(k)\,(\pm)}\eta^{(\pm)}(x;\boldsymbol{\lambda})^{n-k},
$$

which is the consequence of the properties (4.9)–(4.10). In fact, the above formula is rather trivial in the present case.

4.2.1 dual big q-Jacobi polynomials

As explained in § 2, dual polynomials are usually defined from the original polynomials by interchanging the roles of x and n (2.49)–(2.50). Fixing $x \in \mathbb{Z}_{\geq 0}$ in $\check{P}_n^{(\pm)}(x;\boldsymbol{\lambda})$ (4.14)–(4.15), however, does not provide degree x polynomials in $\mathcal{E}(n;\boldsymbol{\lambda})$ (4.4). The reason is that the original eigenpolynomials $\check{P}_n^{(\pm)}(x;\boldsymbol{\lambda})$ (4.13) fail to satisfy the universal normalisation condition $\check{P}_n^{(\pm)}(0;\boldsymbol{\lambda})\neq 1$ (2.26).

The dual big q-Jacobi polynomials (db$q$J) are defined from the original polynomials $\check{P}_n^{(\pm)}(x;\boldsymbol{\lambda})$ (4.13) by removing the n dependent factor $\alpha_n^{(\pm)}(\boldsymbol{\lambda})$, which can be identified by the transformation formula of the truncated ${}_3\phi_2$ function listed as eq.(III.12) in the Appendix III of [5]:

$${}_3\phi_2\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\,b,\,c}{d,\,e}\Bigm|q\,;q\Bigr)=c^n\frac{(c^{-1}e;q)_n}{(e;q)_n}{}_3\phi_2\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\,c,\,b^{-1}d}{d,\,ce^{-1}q^{1-n}}\Bigm|q\,;\frac{bq}{e}\Bigr).$$

Let us introduce $\check{Q}_x^{(\pm)}(n;\boldsymbol{\lambda})$ by

$$\check{P}_n^{(\pm)}(x;\boldsymbol{\lambda})=\alpha_n^{(\pm)}(\boldsymbol{\lambda})\check{Q}_x^{(\pm)}(n;\boldsymbol{\lambda}),\quad \check{Q}_x^{(\pm)}(0;\boldsymbol{\lambda})=1,\quad (\pm1)^n\alpha_n^{(\pm)}(\boldsymbol{\lambda})>0,\tag{4.34}$$

$$\check{Q}_x^{(+)}(n;\boldsymbol{\lambda})\stackrel{\text{def}}{=}Q_x^{(+)}\bigl(\mathcal{E}(n;\boldsymbol{\lambda});\boldsymbol{\lambda}\bigr)={}_3\phi_2\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\,abq^{n+1},\,q^{-x}}{aq,\,abc^{-1}q}\Bigm|q\,;ac^{-1}q^{x+1}\Bigr),\tag{4.35}$$

$$\check{Q}_x^{(-)}(n;\boldsymbol{\lambda})\stackrel{\text{def}}{=}Q_x^{(-)}\bigl(\mathcal{E}(n;\boldsymbol{\lambda});\boldsymbol{\lambda}\bigr)={}_3\phi_2\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\,abq^{n+1},\,q^{-x}}{bq,\,cq}\Bigm|q\,;a^{-1}cq^{x+1}\Bigr),\tag{4.36}$$

$$\alpha_n^{(+)}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}(-c)^nq^{\frac12n(n+1)}\frac{(abc^{-1}q;q)_n}{(cq;q)_n},\quad \alpha_n^{(-)}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}(-a)^nq^{\frac12n(n+1)}\frac{(bq;q)_n}{(aq;q)_n}.\tag{4.37}$$

For fixed $x\in\mathbb{Z}_{\geq0}$, $Q_x^{(\pm)}(\mathcal{E}(n;\boldsymbol{\lambda});\boldsymbol{\lambda})$ are degree x polynomials in $\mathcal{E}(n;\boldsymbol{\lambda})$ (4.4). The difference equation for the bqJ (4.11) can be rewritten as the three term recurrence relation for $\check{Q}_x^{(\pm)}(n;\boldsymbol{\lambda})$

$$\begin{aligned}\mathcal{E}(n;\boldsymbol{\lambda})\check{Q}_x^{(\pm)}(n;\boldsymbol{\lambda})&=-B^{(\pm)}(x;\boldsymbol{\lambda})\check{Q}_{x+1}^{(\pm)}(n;\boldsymbol{\lambda})-D^{(\pm)}(x;\boldsymbol{\lambda})\check{Q}_{x-1}^{(\pm)}(n;\boldsymbol{\lambda})\\&\quad+\bigl(B^{(\pm)}(x;\boldsymbol{\lambda})+D^{(\pm)}(x;\boldsymbol{\lambda})\bigr)\check{Q}_x^{(\pm)}(n;\boldsymbol{\lambda}).\end{aligned}\tag{4.38}$$

In order to write down the difference equations for the dual polynomials $\check{Q}_x^{(\pm)}(n;\boldsymbol{\lambda})$, let us introduce $A_n^{(\pm)}(\boldsymbol{\lambda})$, $C_n^{(\pm)}(\boldsymbol{\lambda})$ by rescaling the coefficients $A_n(\boldsymbol{\lambda})$ and $C_n(\boldsymbol{\lambda})$ (4.17)–(4.18) of the three term recurrence relation of the big q-Jacobi polynomial (4.16) by $\alpha_n^{(\pm)}(\boldsymbol{\lambda})$:

$$A_n^{(\pm)}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}\pm\frac{\alpha_{n+1}^{(\pm)}(\boldsymbol{\lambda})}{\alpha_n^{(\pm)}(\boldsymbol{\lambda})}A_n(\boldsymbol{\lambda}),\quad C_n^{(\pm)}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}\pm\frac{\alpha_{n-1}^{(\pm)}(\boldsymbol{\lambda})}{\alpha_n^{(\pm)}(\boldsymbol{\lambda})}C_n(\boldsymbol{\lambda}).\tag{4.39}$$

To be more explicit, we have

$$\begin{aligned}A_n^{(+)}(\boldsymbol{\lambda})&=cq^{n+1}\frac{(1-aq^{n+1})(1-abq^{n+1})(1-abc^{-1}q^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})},\\A_n^{(-)}(\boldsymbol{\lambda})&=-aq^{n+1}\frac{(1-bq^{n+1})(1-abq^{n+1})(1-cq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})},\end{aligned}$$

$$C_n^{(+)}(\boldsymbol{\lambda})=-aq\frac{(1-q^n)(1-bq^n)(1-cq^n)}{(1-abq^{2n})(1-abq^{2n+1})},$$
$$C_n^{(-)}(\boldsymbol{\lambda})=cq\frac{(1-q^n)(1-aq^n)(1-abc^{-1}q^n)}{(1-abq^{2n})(1-abq^{2n+1})}.$$

The sign change factor $\pm$ in (4.39) is to compensate the sign change of $\alpha_n^{(-)}(\boldsymbol{\lambda})$ (4.37). The potential functions of the dual Hamiltonians are *bounded* and positive semi-definite

$$-A_n^{(\pm)}(\boldsymbol{\lambda})>0\ \ (n\geq 0),\quad -C_n^{(\pm)}(\boldsymbol{\lambda})>0\ \ (n\geq 1),\ \ C_0^{(\pm)}(\boldsymbol{\lambda})=0,$$

and they satisfy the relations

$$A_n(\boldsymbol{\lambda})C_{n+1}(\boldsymbol{\lambda})=A_n^{(\pm)}(\boldsymbol{\lambda})C_{n+1}^{(\pm)}(\boldsymbol{\lambda}),\tag{4.40}$$
$$A_n(\boldsymbol{\lambda})+C_n(\boldsymbol{\lambda})=\pm\big(A_n^{(\pm)}(\boldsymbol{\lambda})+C_n^{(\pm)}(\boldsymbol{\lambda})\big)-\big(1-\eta^{(\pm)}(0;\boldsymbol{\lambda})\big).\tag{4.41}$$

Under the parameter substitution (4.29), the $(+)$ and $(-)$ quantities are interchanged

$$\check{Q}_x^{(+)}(n;\boldsymbol{\lambda})\leftrightarrow\check{Q}_x^{(-)}(n;\boldsymbol{\lambda}),\quad A_n^{(+)}(\boldsymbol{\lambda})\leftrightarrow -A_n^{(-)}(\boldsymbol{\lambda}),\quad C_n^{(+)}(\boldsymbol{\lambda})\leftrightarrow -C_n^{(-)}(\boldsymbol{\lambda}).$$

Thanks to (4.41) and $\eta^{(\pm)}(x;\boldsymbol{\lambda})=\eta^{(\pm)}(0;\boldsymbol{\lambda})q^x$, the three term recurrence relation of the bqJ (4.16) is rewritten as the difference equations for the dual bqJ (4.35)–(4.36):

$$\begin{aligned}&-A_n^{(\pm)}(\boldsymbol{\lambda})\big(\check{Q}_x^{(\pm)}(n;\boldsymbol{\lambda})-\check{Q}_x^{(\pm)}(n+1;\boldsymbol{\lambda})\big)-C_n^{(\pm)}(\boldsymbol{\lambda})\big(\check{Q}_x^{(\pm)}(n;\boldsymbol{\lambda})-\check{Q}_x^{(\pm)}(n-1;\boldsymbol{\lambda})\big)\\&=\pm\eta^{(\pm)}(0;\boldsymbol{\lambda})\eta(x)\check{Q}_x^{(\pm)}(n;\boldsymbol{\lambda}),\quad \eta(x)\stackrel{\text{def}}{=}1-q^x,\end{aligned}\tag{4.42}$$

with positive semi-definite and *bounded* eigenvalues

$$\mathcal{E}^{\text{d}\,(\pm)}(x;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\pm\eta^{(\pm)}(0;\boldsymbol{\lambda})\eta(x)=\left\{\begin{array}{rl}aq(1-q^x)&:(+)\\-cq(1-q^x)&:(-)\end{array}\right..\tag{4.43}$$

In terms of the similarity transformed dual Hamiltonians

$$\widetilde{\mathcal{H}}^{\text{d}\,(\pm)}(\boldsymbol{\lambda})=-A_n^{(\pm)}(\boldsymbol{\lambda})(1-e^{\partial_n})-C_n^{(\pm)}(\boldsymbol{\lambda})(1-e^{-\partial_n}),\tag{4.44}$$

the above equations (4.42) reads succinctly

$$\widetilde{\mathcal{H}}^{\text{d}\,(\pm)}(\boldsymbol{\lambda})\check{Q}_x^{(\pm)}(n;\boldsymbol{\lambda})=\pm\eta^{(\pm)}(0;\boldsymbol{\lambda})\eta(x)\check{Q}_x^{(\pm)}(n;\boldsymbol{\lambda}).$$

It is straightforward to verify that the similarity transformed dual Hamiltonians $\widetilde{\mathcal{H}}^{\text{d}\,(\pm)}(\boldsymbol{\lambda})$ (4.44) are triangular in the basis $\{1,\mathcal{E}(n;\boldsymbol{\lambda}),\ldots,\mathcal{E}(n;\boldsymbol{\lambda})^k\}$ with the above positive semi-definite eigenvalues. Their eigenpolynomials $\check{Q}_x^{(\pm)}(n;\boldsymbol{\lambda})\sim\mathcal{E}(n;\boldsymbol{\lambda})^x\simeq q^{-nx}$ grow rapidly as the coordinate $n\to\infty$.

Let us define two dual Hamiltonians $\mathcal{H}^{\mathrm{d}\,(\pm)}(\boldsymbol{\lambda})$ as follows:

$$\begin{aligned}
\mathcal{H}^{\mathrm{d}\,(\pm)}(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} \mp\sqrt{A^{(\pm)}_n(\boldsymbol{\lambda})C^{(\pm)}_{n+1}(\boldsymbol{\lambda})}\,e^{\partial_n} \mp \sqrt{A^{(\pm)}_{n-1}(\boldsymbol{\lambda})C^{(\pm)}_n(\boldsymbol{\lambda})}\,e^{-\partial_n} - \bigl(A^{(\pm)}_n(\boldsymbol{\lambda})+C^{(\pm)}_n(\boldsymbol{\lambda})\bigr)\\
&= \mathcal{A}^{\mathrm{d}\,(\pm)}(\boldsymbol{\lambda})^\dagger\mathcal{A}^{\mathrm{d}\,(\pm)}(\boldsymbol{\lambda}), \qquad (4.45)\\
\mathcal{A}^{\mathrm{d}\,(\pm)}(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} \sqrt{-A^{(\pm)}_n(\boldsymbol{\lambda})} \mp e^{\partial_n}\sqrt{-C^{(\pm)}_n(\boldsymbol{\lambda})}.
\end{aligned}$$

Note that $\mathcal{H}^{\mathrm{d}\,(-)}(\boldsymbol{\lambda})$ and $\mathcal{A}^{\mathrm{d}\,(-)}(\boldsymbol{\lambda})$ are related to the standard forms (2.6)–(2.7) by the similarity transformation in terms of the diagonal matrix $(-1)^n=\mathrm{diag}(1,-1,1,-1,\ldots)$,

$$\begin{aligned}
(-1)^n\circ\mathcal{H}^{\mathrm{d}\,(-)}\circ(-1)^n &= -\sqrt{A^{(-)}_nC^{(-)}_{n+1}}\,e^{\partial_n} - \sqrt{A^{(-)}_{n-1}C^{(-)}_n}\,e^{-\partial_n} - \bigl(A^{(-)}_n+C^{(-)}_n\bigr),\\
(-1)^n\circ\mathcal{A}^{\mathrm{d}\,(-)}\circ(-1)^n &= \sqrt{-A^{(-)}_n} - e^{\partial_n}\sqrt{-C^{(\pm)}_n},
\end{aligned}$$

where we have suppressed $\boldsymbol{\lambda}$. The ground state eigenvectors $\phi^{\mathrm{d}\,(\pm)}_0(n;\boldsymbol{\lambda})$ characterised by $\mathcal{A}^{\mathrm{d}\,(\pm)}(\boldsymbol{\lambda})\phi^{\mathrm{d}\,(\pm)}_0(n;\boldsymbol{\lambda})=0$ and $\phi^{\mathrm{d}\,(\pm)}_0(0;\boldsymbol{\lambda})=1$ are

$$\phi^{\mathrm{d}\,(\pm)}_0(n;\boldsymbol{\lambda}) = (\pm1)^n\prod_{m=0}^{n-1}\sqrt{\frac{A^{(\pm)}_m(\boldsymbol{\lambda})}{C^{(\pm)}_{m+1}(\boldsymbol{\lambda})}} = \alpha^{(\pm)}_n(\boldsymbol{\lambda})\frac{d_n(\boldsymbol{\lambda})}{d_0(\boldsymbol{\lambda})}, \quad (\pm1)^n\phi^{\mathrm{d}\,(\pm)}_0(n;\boldsymbol{\lambda})>0. \qquad (4.46)$$

The dual polynomials $\check{Q}^{(\pm)}_x(n;\boldsymbol{\lambda})$ give eigenvectors

$$\phi^{\mathrm{d}\,(\pm)}_x(n;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \phi^{\mathrm{d}\,(\pm)}_0(n;\boldsymbol{\lambda})\check{Q}^{(\pm)}_x(n;\boldsymbol{\lambda}), \qquad (4.47)$$

$$\mathcal{H}^{\mathrm{d}\,(\pm)}(\boldsymbol{\lambda})\phi^{\mathrm{d}\,(\pm)}_x(n;\boldsymbol{\lambda}) = \mathcal{E}^{\mathrm{d}\,(\pm)}(x;\boldsymbol{\lambda})\phi^{\mathrm{d}\,(\pm)}_x(n;\boldsymbol{\lambda}). \qquad (4.48)$$

By using (4.40)–(4.41), we can show that the two dual Hamiltonians $\mathcal{H}^{\mathrm{d}\,(\pm)}(\boldsymbol{\lambda})$ are related as follows:

$$\mathcal{H}^{\mathrm{d}\,(+)}(\boldsymbol{\lambda}) - \eta^{(+)}(0;\boldsymbol{\lambda}) = -\mathcal{H}^{\mathrm{d}\,(-)}(\boldsymbol{\lambda}) - \eta^{(-)}(0;\boldsymbol{\lambda}). \qquad (4.49)$$

Therefore $\mathcal{H}^{\mathrm{d}\,(\pm)}(\boldsymbol{\lambda})$ have another set of eigenvectors $\phi^{\mathrm{d}\,(\mp)}_x(n;\boldsymbol{\lambda})$,

$$\mathcal{H}^{\mathrm{d}\,(\pm)}(\boldsymbol{\lambda})\phi^{\mathrm{d}\,(\mp)}_x(n;\boldsymbol{\lambda}) = \mathcal{E}'^{\,\mathrm{d}\,(\pm)}(x;\boldsymbol{\lambda})\phi^{\mathrm{d}\,(\mp)}_x(n;\boldsymbol{\lambda}), \quad \mathcal{E}'^{\,\mathrm{d}\,(\pm)}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \left\{\begin{array}{ll} q(a-cq^x) & :(+)\\ q(-c+aq^x) & :(-)\end{array}\right. . \qquad (4.50)$$

The ‘dual Hamiltonian’ $\mathcal{H}^{\mathrm{d}}(\boldsymbol{\lambda})$ (4.51) may be constructed from the original coefficients $A_n(\boldsymbol{\lambda})$ and $C_n(\boldsymbol{\lambda})$ (4.17)–(4.18) of the three term recurrence relation of the bqJ:

$$\begin{aligned}
\mathcal{H}^{\mathrm{d}}(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} -\sqrt{A_n(\boldsymbol{\lambda})C_{n+1}(\boldsymbol{\lambda})}\,e^{\partial_n} - \sqrt{A_{n-1}(\boldsymbol{\lambda})C_n(\boldsymbol{\lambda})}\,e^{-\partial_n} - \bigl(A_n(\boldsymbol{\lambda})+C_n(\boldsymbol{\lambda})\bigr) \qquad (4.51)\\
&= \pm\mathcal{H}^{\mathrm{d}\,(\pm)}(\boldsymbol{\lambda}) + 1 - \eta^{(\pm)}(0;\boldsymbol{\lambda}).
\end{aligned}$$

However, the zero mode of this ‘dual Hamiltonian’ $\mathcal{H}^{\mathrm{d}}(\boldsymbol{\lambda})$ is not square summable

$$\phi_0^{\mathrm{d}}(n;\boldsymbol{\lambda})^2 \propto \prod_{m=0}^{n-1}\frac{A_m(\boldsymbol{\lambda})}{C_{m+1}(\boldsymbol{\lambda})}=\frac{d_n(\boldsymbol{\lambda})^2}{d_0(\boldsymbol{\lambda})^2}\simeq(-ac)^{-n}q^{-\frac{1}{2}n(n+3)}\bigl(1+O(q^n)\bigr)\times\text{const}\quad(n\to\infty),$$

and it fails to deliver the dual big q-Jacobi polynomials. This is why we have to introduce two dual Hamiltonians $\mathcal{H}^{\mathrm{d}\,(\pm)}(\boldsymbol{\lambda})$ (4.45).

We arrive at the dual orthogonality relations $(x,y=0,1,\ldots)$

$$(\phi_x^{\mathrm{d}\,(\pm)},\phi_y^{\mathrm{d}\,(\pm)})=\sum_{n=0}^{\infty}\phi_0^{\mathrm{d}\,(\pm)}(n;\boldsymbol{\lambda})^2\check{Q}_x^{(\pm)}(n;\boldsymbol{\lambda})\check{Q}_y^{(\pm)}(n;\boldsymbol{\lambda})=\frac{\delta_{xy}}{\phi_0^{(\pm)}(x;\boldsymbol{\lambda})^2 d_0(\boldsymbol{\lambda})^2},\tag{4.52}$$

$$(\phi_x^{\mathrm{d}\,(+)},\phi_y^{\mathrm{d}\,(-)})=\sum_{n=0}^{\infty}\phi_0^{\mathrm{d}\,(+)}(n;\boldsymbol{\lambda})\phi_0^{\mathrm{d}\,(-)}(n;\boldsymbol{\lambda})\check{Q}_x^{(+)}(n;\boldsymbol{\lambda})\check{Q}_y^{(-)}(n;\boldsymbol{\lambda})=0,\tag{4.53}$$

$$(\hat{\phi}_x^{\mathrm{d}\,(\epsilon)},\hat{\phi}_y^{\mathrm{d}\,(\epsilon')})=\delta_{\epsilon\,\epsilon'}\delta_{xy},\quad \hat{\phi}_x^{\mathrm{d}\,(\pm)}(n;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\phi_0^{(\pm)}(x;\boldsymbol{\lambda})d_0(\boldsymbol{\lambda})\phi_x^{\mathrm{d}\,(\pm)}(n;\boldsymbol{\lambda}).\tag{4.54}$$

These are the consequences of the self-adjointness of the Hamiltonian $\mathcal{H}^{\mathrm{d}\,(+)}(\boldsymbol{\lambda})$ (4.45), for which $\{\phi_x^{\mathrm{d}\,(+)}(n;\boldsymbol{\lambda})\}$ and $\{\phi_x^{\mathrm{d}\,(-)}(n;\boldsymbol{\lambda})\}$ are eigenvectors. The spectrum of the latter $\{\mathcal{E}'^{\,\mathrm{d}\,(+)}(x;\boldsymbol{\lambda})=q(a-cq^x)\}$ (4.50), which is monotonously decreasing with x, lies above that of the former, $\{\mathcal{E}^{\mathrm{d}\,(+)}(x)=aq(1-q^x)\}$ (4.48), which is monotonously increasing with x. They share the same accumulation point aq. The ground state vector $\phi_0^{\mathrm{d}\,(+)}(n;\boldsymbol{\lambda})$ is positive and the excited state vector $\phi_x^{\mathrm{d}\,(+)}(n;\boldsymbol{\lambda})$ has x ‘zeros’ due to x zeros of the polynomial $Q_x^{(+)}(\mathcal{E};\boldsymbol{\lambda})$. Here the number of ‘zeros’ means the number of sign changing in $n\in[0,\infty)$. All the eigenvectors $\{\phi_x^{\mathrm{d}\,(-)}(n;\boldsymbol{\lambda})\}$ have infinitely many ‘zeros’ due to the infinitely many eigenlevels of $\{\phi_x^{\mathrm{d}\,(+)}(n;\boldsymbol{\lambda})\}$ lying below. These infinite ‘zeros’ are partly taken care of by the alternating sign factor $(-1)^n$ or $\alpha_n^{(-)}(\boldsymbol{\lambda})$ in $\phi_0^{\mathrm{d}\,(-)}(n;\boldsymbol{\lambda})$ (4.46). Since zeros of $Q_x^{(-)}(\mathcal{E};\boldsymbol{\lambda})$ cancels the sign change of $(-1)^n$, it is easy to “understand” that the number of ‘zeros’ in $\phi_x^{\mathrm{d}\,(-)}(n;\boldsymbol{\lambda})$ would increase as the degree x decreases. This is consistent with the oscillation theorem. The orthogonality relation (4.53) simply means $\sum_{n=0}^{\infty}\phi_0^{\mathrm{d}\,(+)}(n;\boldsymbol{\lambda})\phi_0^{\mathrm{d}\,(-)}(n;\boldsymbol{\lambda})\mathcal{E}(n;\boldsymbol{\lambda})^k=0$, namely

$$\sum_{n=0}^{\infty}(-1)^n\frac{q^{\frac{1}{2}n(n-1)}}{(q;q)_n}\frac{1-abq^{2n+1}}{1-abq^{n+1}}(abq^2;q)_n\,\mathcal{E}(n;\boldsymbol{\lambda})^k=0\quad(k\in\mathbb{Z}_{\geq0}),\tag{4.55}$$

as mentioned in [23].

The matrix elements of $\mathcal{H}^{\mathrm{d}\,(-)}(\boldsymbol{\lambda})$ (4.45) are all non-negative. The eigenvectors and eigenvalues are $\{\phi_x^{\mathrm{d}\,(-)}(n;\boldsymbol{\lambda})\}$ with $\{\mathcal{E}^{\mathrm{d}\,(-)}(x)=-cq(1-q^x)\}$ (4.48) and $\{\phi_x^{\mathrm{d}\,(+)}(n;\boldsymbol{\lambda})\}$ with $\{\mathcal{E}'^{\,\mathrm{d}\,(-)}(x;\boldsymbol{\lambda})=q(-c+aq^x)\}$ (4.50). The maximum eigenvalue is $\mathcal{E}'^{\,\mathrm{d}\,(-)}(0;\boldsymbol{\lambda})=q(a-c)$ and the corresponding eigenvector $\phi_0^{\mathrm{d}\,(+)}(n;\boldsymbol{\lambda})$ is positive everywhere. This is consistent with the Perron-Frobenius theorem. The ground state vector $\phi_0^{\mathrm{d}\,(-)}(n;\boldsymbol{\lambda})$ has infinitely many 'zeros'.

The alternating sign factor has no effect for the orthogonality among $\{\check{Q}_x^{(-)}(n;\boldsymbol{\lambda})\}$ (4.52), which is just an ordinary orthogonality relation among polynomials. It should be stressed that the weight function for the orthogonality of the two different kinds of polynomials, $\{\check{Q}_x^{(+)}(n;\boldsymbol{\lambda})\}$ and $\{\check{Q}_x^{(-)}(n;\boldsymbol{\lambda})\}$ (4.53) is *not positive definite* due to the alternating sign factor in $\phi_0^{\mathrm{d}\,(-)}(n;\boldsymbol{\lambda})$ (4.46). If the weight function were positive definite, the orthogonality of $\check{Q}_0^{(+)}(n;\boldsymbol{\lambda}) = 1 = \check{Q}_0^{(-)}(n;\boldsymbol{\lambda})$ could not be attained. As is well known, the overall scale of the Hamiltonian is immaterial. In order to get the common eigenvalues $\{1 - q^x\}$ for both Hamiltonians $\mathcal{H}^{\mathrm{d}\,(\pm)}(\boldsymbol{\lambda})$ (4.45), one should divide $A_n^{(+)}$, $C_n^{(+)}$ by aq in $(+)$ sector and $A_n^{(-)}$, $C_n^{(-)}$ by $-cq$ in $(-)$ sector.

As discussed in § 2, dual polynomials are defined through the completeness relation (2.53) of the original orthogonal polynomials. In the case of big q-Jacobi polynomials, one multiplies normalised eigenfunction $\hat{\phi}_n^{(\epsilon)}(y) \stackrel{\text{def}}{=} d_n\phi_n^{(\epsilon)}(y)$ to the original normalised orthogonality relation (4.23) $(\hat{\phi}_n^{(+)}, \hat{\phi}_m^{(+)}) + (\hat{\phi}_n^{(-)}, \hat{\phi}_m^{(-)}) = \delta_{nm}$ and sum over n. Since the original orthogonality relation is uniformly convergent, one obtains the completeness relation or the dual orthogonality relation

$$\sum_{n=0}^{\infty}\hat{\phi}_n^{(+)}(x)\hat{\phi}_n^{(+)}(y) = \delta_{xy}, \quad \sum_{n=0}^{\infty}\hat{\phi}_n^{(-)}(x)\hat{\phi}_n^{(-)}(y) = \delta_{xy}, \quad \sum_{n=0}^{\infty}\hat{\phi}_n^{(+)}(x)\hat{\phi}_n^{(-)}(y) = 0. \tag{4.56}$$

Rewriting the bqJ $\check{P}_n^{(\pm)}(x;\boldsymbol{\lambda})$ in the above orthogonality relations (4.56) in terms of the dual polynomials $\check{Q}_x^{(\pm)}(n;\boldsymbol{\lambda})$ (4.34) provides the orthogonality relations (4.52)–(4.53).

The dual big q-Jacobi polynomials have been introduced and discussed in [21, 23].

4.3 Limits of big q-Jacobi

By restricting the parameters or by taking certain limits of the bqJ system presented in the previous subsection, we obtain three systems described by the big q-Laguerre (bqL), Al-Salam-Carlitz I (ASC I) and discrete q-Hermite I (dqH I) polynomials. The orthogonality of these polynomials requires the two component Hamiltonian formalism for the self-adjointness and their orthogonality measures are of Jackson integral type. The duals of these polynomials have also been discussed in [23] in some detail. We will provide main results skipping most of derivation. The coordinate x takes non-negative integer values, $x \in \mathbb{Z}_{\geq 0}$.

4.3.1 big q-Laguerre

The big q-Laguerre polynomial is obtained from the big q-Jacobi (bqJ) polynomial by setting $(a^{\text{b}q\text{J}}, b^{\text{b}q\text{J}}, c^{\text{b}q\text{J}}) = (a, 0, b)$. There is no difficulty in this limit. The basic data are:

$$
\begin{aligned}
&q^{\boldsymbol{\lambda}} = (a, b), \quad \boldsymbol{\delta} = (1, 1), \quad \kappa = q^{-1}, \quad 0 < a < q^{-1}, \quad b < 0,\\
&\mathcal{E}(n) \stackrel{\text{def}}{=} q^{-n} - 1, \quad \eta^{(+)}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} aq^{x+1}, \quad \eta^{(-)}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} bq^{x+1}, \qquad (4.57)\\
&B^{\text{J}}(\eta; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \eta^{-2}(-abq)(1 - \eta), \quad D^{\text{J}}(\eta; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \eta^{-2}(aq - \eta)(\eta - bq). \qquad (4.58)
\end{aligned}
$$

The potential functions (3.32) are

$$
\begin{aligned}
&B^{(+)}(x; \boldsymbol{\lambda}) = -a^{-1}bq^{-2x-1}(1 - aq^{x+1}), \quad D^{(+)}(x; \boldsymbol{\lambda}) = q^{-2x}(1 - q^x)(q^x - a^{-1}b), \qquad (4.59)\\
&B^{(-)}(x; \boldsymbol{\lambda}) = -ab^{-1}q^{-2x-1}(1 - bq^{x+1}), \quad D^{(-)}(x; \boldsymbol{\lambda}) = q^{-2x}(1 - q^x)(q^x - ab^{-1}). \qquad (4.60)
\end{aligned}
$$

It is easy to verify the triangularity (4.2) and the eigenvalues $\mathcal{E}(n)$ (4.57). The sinusoidal coordinate $\eta^{(\pm)}(x; \boldsymbol{\lambda})$ satisfies (4.8)–(4.10).

The big q-Laguerre polynomial $P_n(\eta; a, b; q)$ $(n \in \mathbb{Z}_{\geq 0})$,

$$
\begin{aligned}
P_n(\eta; \boldsymbol{\lambda}) &\stackrel{\text{def}}{=} P_n(\eta; a, b; q) \stackrel{\text{def}}{=} {}_3\phi_2\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\, 0,\, \eta}{aq,\, bq} \Bigm| q\,;q\Bigr)\\
&= \frac{(-b)^n q^{\frac{1}{2}n(n+1)}}{(bq;q)_n}\, {}_2\phi_1\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\, aq\eta^{-1}}{aq} \Bigm| q\,; b^{-1}\eta\Bigr), \quad P_n(1; \boldsymbol{\lambda}) = 1, \qquad (4.61)
\end{aligned}
$$

is the polynomial solution of degree n in η of the second order difference equation (4.11) with $B^{\text{J}}(\eta)$ and $D^{\text{J}}(\eta)$ in (4.58). It also satisfies the same type of three term recurrence relation as that of the bqJ (4.16) with the coefficients

$$
A_n(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -(1 - aq^{n+1})(1 - bq^{n+1}), \quad C_n(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} abq^{n+1}(1 - q^n). \qquad (4.62)
$$

By setting $\eta \to aq^{x+1}$ in the second expression (4.61), we see that the big q-Laguerre polynomial is proportional to the dual q-Meixner polynomial (3.9) introduced in § 3.1.

The eigenvectors of the two Hamiltonians $\mathcal{H}^{(\pm)}(\boldsymbol{\lambda})$ have the forms (4.20) with (4.13), in which $\phi_0^{(\pm)}(x; \boldsymbol{\lambda})$ are calculated to be

$$
\begin{aligned}
&\phi_0^{(\pm)}(x; \boldsymbol{\lambda})^2 = \pm A^{(\pm)} \eta^{(\pm)}(x; \boldsymbol{\lambda}) \frac{\bigl(a^{-1}\eta^{(\pm)}(x; \boldsymbol{\lambda}), b^{-1}\eta^{(\pm)}(x; \boldsymbol{\lambda}); q\bigr)_\infty}{\bigl(\eta^{(\pm)}(x; \boldsymbol{\lambda}); q\bigr)_\infty}, \qquad (4.63)\\
&A^{(+)} = \frac{\phi_0^{(+)}(0; \boldsymbol{\lambda})^2 (aq;q)_\infty}{aq(q, ab^{-1}q; q)_\infty}, \quad A^{(-)} = \frac{\phi_0^{(-)}(0; \boldsymbol{\lambda})^2 (bq;q)_\infty}{-bq(q, a^{-1}bq; q)_\infty}.
\end{aligned}
$$

As shown in § 4.2, the choice $A^{(+)} = A^{(-)} = 1$ (4.22) leads to the self-adjoint two component Hamiltonian system $\mathcal{H}^{(\pm)}(\boldsymbol{\lambda})$ with the potential functions $B^{(\pm)}(x;\boldsymbol{\lambda})$ and $D^{(\pm)}(x;\boldsymbol{\lambda})$ (4.59)–(4.60). The ground state vectors $\phi_0^{(\pm)}(x;\boldsymbol{\lambda}) > 0$ (4.63) read explicitly

$$\phi_0^{(+)}(x;\boldsymbol{\lambda})^2 = aq^{x+1}\frac{(q^{x+1}, ab^{-1}q^{x+1};q)_\infty}{(aq^{x+1};q)_\infty}, \quad \phi_0^{(-)}(x;\boldsymbol{\lambda})^2 = -bq^{x+1}\frac{(q^{x+1}, a^{-1}bq^{x+1};q)_\infty}{(bq^{x+1};q)_\infty}. \quad (4.64)$$

We arrive at the orthogonality relation with the Jackson integral measure

$$\begin{aligned}
(\!(\boldsymbol{\phi}_n, \boldsymbol{\phi}_m)\!) &= \big(\phi_n^{(+)}, \phi_m^{(+)}\big) + \big(\phi_n^{(-)}, \phi_m^{(-)}\big) = \frac{\delta_{nm}}{d_n(\boldsymbol{\lambda})^2} && (4.65)\\
&= aq\sum_{k=0}^{\infty}\frac{(q^{k+1}, ab^{-1}q^{k+1};q)_\infty}{(aq^{k+1};q)_\infty}P_n(aq^{k+1};\boldsymbol{\lambda})P_m(aq^{k+1};\boldsymbol{\lambda})q^k\\
&\quad - bq\sum_{k=0}^{\infty}\frac{(q^{k+1}, a^{-1}bq^{k+1};q)_\infty}{(bq^{k+1};q)_\infty}P_n(bq^{k+1};\boldsymbol{\lambda})P_m(bq^{k+1};\boldsymbol{\lambda})q^k\\
&= \frac{1}{1-q}\int_{bq}^{aq}d_qy\,\frac{(a^{-1}y, b^{-1}y;q)_\infty}{(y;q)_\infty}P_n(y;\boldsymbol{\lambda})P_m(y;\boldsymbol{\lambda}), && (4.66)
\end{aligned}$$

where the normalisation constant $d_n(\boldsymbol{\lambda})^2$ is

$$d_n(\boldsymbol{\lambda})^2 = (-abq^2)^{-n}q^{-\frac{1}{2}n(n-1)}\frac{(aq, bq;q)_n}{(q;q)_n}\times d_0(\boldsymbol{\lambda})^2, \quad d_0(\boldsymbol{\lambda})^2 = \frac{(aq, bq;q)_\infty}{aq(q, ab^{-1}q, a^{-1}b;q)_\infty}. \quad (4.67)$$

The system is shape invariant, since the potential functions $B^{(\pm)}(x;\boldsymbol{\lambda})$ and $D^{(\pm)}(x;\boldsymbol{\lambda})$ satisfy (2.29)–(2.30). The forward/backward shift relations for the bqL have the form (4.26)–(4.27) with the auxiliary functions $\varphi^{(\pm)}(x;\boldsymbol{\lambda})$,

$$\varphi^{(\pm)}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{q\eta^{(\pm)}(x;\boldsymbol{\lambda})}{(1-aq)(1-bq)}. \quad (4.68)$$

Under the parameter substitution (involution) $(a, b) \to (b, a)$, the $(+)$ and $(-)$ systems are interchanged as (4.30).

4.3.2 dual big q-Laguerre

The two types of the dual big q-Laguerre (db$q$L) polynomials are

$$\check{Q}_x^{(+)}(n;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} Q_x^{(+)}\big(\mathcal{E}(n);\boldsymbol{\lambda}\big) \stackrel{\text{def}}{=} {}_2\phi_1\Big(\begin{matrix} q^{-n},\ q^{-x}\\ aq\end{matrix}\,\Big|\,q\,;ab^{-1}q^{x+1}\Big), \quad (4.69)$$

$$\check{Q}_x^{(-)}(n;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} Q_x^{(-)}\big(\mathcal{E}(n);\boldsymbol{\lambda}\big) \stackrel{\text{def}}{=} {}_2\phi_1\Big(\begin{matrix} q^{-n},\ q^{-x}\\ bq\end{matrix}\,\Big|\,q\,;a^{-1}bq^{x+1}\Big). \quad (4.70)$$

They are degree x polynomials in $\mathcal{E}(n)$ (4.57) and related to $\check{P}_n^{(\pm)}(x;\boldsymbol{\lambda})$ as (4.34) with

$$\alpha_n^{(+)}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{(-b)^n q^{\frac{1}{2}n(n+1)}}{(bq;q)_n}, \quad \alpha_n^{(-)}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{(-a)^n q^{\frac{1}{2}n(n+1)}}{(aq;q)_n}. \tag{4.71}$$

Following the argument in § 4.2.1, $A_n^{(\pm)}(\boldsymbol{\lambda})$ and $C_n^{(\pm)}(\boldsymbol{\lambda})$ (4.39) and $\mathcal{H}^{\mathrm{d}\,(\pm)}(\boldsymbol{\lambda})$ (4.45) are introduced:

$$\begin{aligned}
&A_n^{(+)}(\boldsymbol{\lambda}) = bq^{n+1}(1-aq^{n+1}), \quad &&C_n^{(+)}(\boldsymbol{\lambda}) = -aq(1-q^n)(1-bq^n),\\
&A_n^{(-)}(\boldsymbol{\lambda}) = -aq^{n+1}(1-bq^{n+1}), &&C_n^{(-)}(\boldsymbol{\lambda}) = bq(1-q^n)(1-aq^n).
\end{aligned}$$

They satisfy (4.40)–(4.41), which implies (4.49). The eigenvectors of $\mathcal{H}^{\mathrm{d}\,(\pm)}(\boldsymbol{\lambda})$ are

$$\begin{aligned}
&\phi_x^{\mathrm{d}\,(\pm)}(n;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \phi_0^{\mathrm{d}\,(\pm)}(n;\boldsymbol{\lambda})\check{Q}_x^{(\pm)}(n;\boldsymbol{\lambda}), &&(4.72)\\
&\phi_0^{\mathrm{d}\,(\pm)}(n;\boldsymbol{\lambda}) = (\pm 1)^n \prod_{m=0}^{n-1}\sqrt{\frac{A_m^{(\pm)}(\boldsymbol{\lambda})}{C_{m+1}^{(\pm)}(\boldsymbol{\lambda})}} = \alpha_n^{(\pm)}(\boldsymbol{\lambda})\frac{d_n(\boldsymbol{\lambda})}{d_0(\boldsymbol{\lambda})}, \quad (\pm 1)^n\phi_0^{\mathrm{d}\,(\pm)}(n;\boldsymbol{\lambda}) > 0,\\
&\mathcal{H}^{\mathrm{d}\,(\pm)}(\boldsymbol{\lambda})\phi_x^{\mathrm{d}\,(\pm)}(n;\boldsymbol{\lambda}) = \mathcal{E}^{\mathrm{d}\,(\pm)}(x;\boldsymbol{\lambda})\phi_x^{\mathrm{d}\,(\pm)}(n;\boldsymbol{\lambda}),\\
&\mathcal{H}^{\mathrm{d}\,(\pm)}(\boldsymbol{\lambda})\phi_x^{\mathrm{d}\,(\mp)}(n;\boldsymbol{\lambda}) = \mathcal{E}'^{\,\mathrm{d}\,(\pm)}(x;\boldsymbol{\lambda})\phi_x^{\mathrm{d}\,(\mp)}(n;\boldsymbol{\lambda}),\\
&\mathcal{E}^{\mathrm{d}\,(\pm)}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} aq(1-q^x) & :(+)\\ -bq(1-q^x) & :(-)\end{cases}, \quad \mathcal{E}'^{\,\mathrm{d}\,(\pm)}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} q(a-bq^x) & :(+)\\ q(-b+aq^x) & :(-)\end{cases}. &&(4.73)
\end{aligned}$$

They satisfy the dual orthogonality relations in the same form as (4.52)–(4.53), in which $\phi_0^{(\pm)}(x;\boldsymbol{\lambda})^2$ and $d_0(\boldsymbol{\lambda})^2$ are defined in (4.64) and (4.67). The orthogonality relation (4.53) means simply

$$\sum_{n=0}^{\infty}(-1)^n\frac{q^{\frac{1}{2}n(n-1)}}{(q;q)_n}q^{-kn} = 0 \quad (k\in\mathbb{Z}_{\geq 0}), \quad \phi_0^{\mathrm{d}\,(+)}(n;\boldsymbol{\lambda})\phi_0^{\mathrm{d}\,(-)}(n;\boldsymbol{\lambda}) = (-1)^n\frac{q^{\frac{1}{2}n(n-1)}}{(q;q)_n}, \tag{4.74}$$

which can be shown by [6]

$$\sum_{n=0}^{\infty}\frac{q^{\frac{1}{2}n(n-1)}}{(q;q)_n}z^n = (-z;q)_\infty.$$

As expected from the big q-Laguerre and the dual q-Meixner correspondence, the dual big q-Laguerre polynomials (4.69)–(4.70) are equal to the q-Meixner polynomial and its completeness partner (4.93) under certain parameter identification.

4.3.3 Al-Salam-Carlitz I

The Al-Salam-Carlitz I polynomial is obtained from the big q-Laguerre (bqL) polynomial by setting $(a^{\mathrm{bqL}}, b^{\mathrm{bqL}}) = (t, ta)$, $\eta^{\mathrm{bqL}} = tq\eta$ and taking $t\to 0$ limit. In this limit we need appropriate overall rescalings for various quantities. The basic data are:

$$
\begin{aligned}
&q^{\boldsymbol{\lambda}}=a,\quad \boldsymbol{\delta}=0,\quad \kappa=q^{-1},\quad a<0,\\
&\mathcal{E}(n)=q^{-n}-1,\quad \eta^{(+)}(x;\boldsymbol{\lambda})\stackrel{\text{def}}{=}q^x,\quad \eta^{(-)}(x;\boldsymbol{\lambda})\stackrel{\text{def}}{=}aq^x,\\
&B^{\text{J}}(\eta;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\eta^{-2}(-aq^{-1}),\quad D^{\text{J}}(\eta;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\eta^{-2}(1-\eta)(\eta-a).
\end{aligned}\tag{4.75}
$$

The potential functions (3.32) are

$$
B^{(+)}(x;\boldsymbol{\lambda})=-aq^{-2x-1},\qquad D^{(+)}(x;\boldsymbol{\lambda})=q^{-2x}(1-q^x)(q^x-a),\tag{4.76}
$$

$$
B^{(-)}(x;\boldsymbol{\lambda})=-a^{-1}q^{-2x-1},\quad D^{(-)}(x;\boldsymbol{\lambda})=q^{-2x}(1-q^x)(q^x-a^{-1}).\tag{4.77}
$$

The two sinusoidal coordinates satisfy (4.9)–(4.10). Throughout this paper, our definition of Al-Salam-Carlitz I (ASC I) polynomial is a simple q-hypergeometric function

$$
P_n(\eta;\boldsymbol{\lambda})\stackrel{\text{def}}{=}{}_2\phi_1\Bigl(\begin{matrix}q^{-n},\,\eta^{-1}\\ 0\end{matrix}\Bigm|q\,;a^{-1}q\eta\Bigr)=(-a)^{-n}q^{-\frac12 n(n-1)}U_n^{(a)}(\eta;q),\quad P_n(1;\boldsymbol{\lambda})=1,\tag{4.78}
$$

which is obtained by rescaling the conventional Al-Salam-Carlitz I polynomial $U_n^{(a)}(\eta;q)$ ($n\in\mathbb{Z}_{\geq 0}$) [6]. It is the polynomial solution of degree n in η of the difference equation (4.11) with $B^{\text{J}}(\eta)$ and $D^{\text{J}}(\eta)$ in (4.75). With the rescaling our ASC I satisfy the same form of the three term recurrence relation as that of bqJ (4.16) and its coefficients are

$$
A_n(\boldsymbol{\lambda})\stackrel{\text{def}}{=}aq^n,\quad C_n\stackrel{\text{def}}{=}-(1-q^n).\tag{4.79}
$$

The forward/backward shift relations are

$$
\begin{aligned}
&\frac{-a}{q\eta}\bigl(P_n(\eta;\boldsymbol{\lambda})-P_n(q\eta;\boldsymbol{\lambda})\bigr)=\mathcal{E}(n)P_{n-1}(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta}),\\
&\bigl(B^{\text{J}}(\eta;\boldsymbol{\lambda})P_{n-1}(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})-q^{-1}D^{\text{J}}(\eta;\boldsymbol{\lambda})P_{n-1}(q^{-1}\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})\bigr)\frac{q\eta}{-a}=P_n(\eta;\boldsymbol{\lambda}).
\end{aligned}
$$

We strongly advocate this natural definition of Al-Salam-Carlitz I polynomial (4.78), since many important formulas take the same form as those for the bqJ family.

By setting $\eta\to q^x$ ($-aq^x$) in (4.78), we see that the ASC I polynomial $P_n(\eta;\boldsymbol{\lambda})$ is equal to the dual q-Charlier polynomial (3.46) ((3.51)) introduced in § 3.3 with parameter identification $a\to -a$.

The eigenvectors of the two Hamiltonians $\mathcal{H}^{(\pm)}(\boldsymbol{\lambda})$ are (4.20) with (4.13), in which $\phi_0^{(\pm)}(x;\boldsymbol{\lambda})$ are calculated as

$$
\phi_0^{(\pm)}(x;\boldsymbol{\lambda})^2=\pm A^{(\pm)}\eta^{(\pm)}(x;\boldsymbol{\lambda})(q\eta^{(\pm)}(x;\boldsymbol{\lambda}),a^{-1}q\eta^{(\pm)}(x;\boldsymbol{\lambda});q)_\infty,\tag{4.80}
$$

$$A^{(+)}=\frac{\phi_0^{(+)}(0;\boldsymbol{\lambda})^2}{(q,a^{-1}q;q)_\infty},\quad A^{(-)}=\frac{\phi_0^{(-)}(0;\boldsymbol{\lambda})^2}{-a(q,aq;q)_\infty}.$$

As shown in § 4.2, the choice $A^{(+)}=A^{(-)}=1$ (4.22) leads to the self-adjoint two component Hamiltonian system $\mathcal{H}^{(\pm)}(\boldsymbol{\lambda})$ with the potential functions $B^{(\pm)}(x;\boldsymbol{\lambda})$ and $D^{(\pm)}(x;\boldsymbol{\lambda})$ (4.76)–(4.77). The ground state vectors $\phi_0^{(\pm)}(x;\boldsymbol{\lambda})>0$ (4.80) read explicitly

$$\phi_0^{(+)}(x;\boldsymbol{\lambda})^2=q^x(q^{x+1},a^{-1}q^{x+1};q)_\infty,\quad \phi_0^{(-)}(x;\boldsymbol{\lambda})^2=-aq^x(q^{x+1},aq^{x+1};q)_\infty. \tag{4.81}$$

We arrive at the orthogonality relation with the Jackson integral measure

$$\begin{aligned}(\!(\boldsymbol{\phi}_n,\boldsymbol{\phi}_m)\!)&=\big(\phi_n^{(+)},\phi_m^{(+)}\big)+\big(\phi_n^{(-)},\phi_m^{(-)}\big)=\frac{\delta_{nm}}{d_n(\boldsymbol{\lambda})^2} &(4.82)\\ &=\sum_{k=0}^{\infty}(q^{k+1},a^{-1}q^{k+1};q)_\infty P_n(q^k;\boldsymbol{\lambda})P_m(q^k;\boldsymbol{\lambda})q^k\\ &\quad -a\sum_{k=0}^{\infty}(q^{k+1},aq^{k+1};q)_\infty P_n(aq^k;\boldsymbol{\lambda})P_m(aq^k;\boldsymbol{\lambda})q^k\\ &=\frac{1}{1-q}\int_a^1 d_qy\,(qy,a^{-1}qy;q)_\infty P_n(y;\boldsymbol{\lambda})P_m(y;\boldsymbol{\lambda}), &(4.83)\end{aligned}$$

where the normalisation constant $d_n(\boldsymbol{\lambda})^2$ is

$$d_n(\boldsymbol{\lambda})^2=\frac{(-a)^nq^{\frac12 n(n-1)}}{(q;q)_n}\times d_0(\boldsymbol{\lambda})^2,\quad d_0(\boldsymbol{\lambda})^2=\frac{1}{(q,a,a^{-1}q;q)_\infty}. \tag{4.84}$$

The system is shape invariant, since the potential functions $B^{(\pm)}(x;\boldsymbol{\lambda})$ and $D^{(\pm)}(x;\boldsymbol{\lambda})$ satisfy (2.29)–(2.30). The forward/backward shift relations for the ASC I have the form (4.26)–(4.27) with the auxiliary functions $\varphi^{(\pm)}(x;\boldsymbol{\lambda})$,

$$\varphi^{(\pm)}(x;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\frac{q\eta^{(\pm)}(x;\boldsymbol{\lambda})}{-a}. \tag{4.85}$$

4.3.4 dual Al-Salam-Carlitz I

The two types of the dual Al-Salam-Carlitz I (dASC I) polynomials are

$$\check{Q}_x^{(+)}(n;\boldsymbol{\lambda})\stackrel{\text{def}}{=}Q_x^{(+)}\big(\mathcal{E}(n);\boldsymbol{\lambda}\big)\stackrel{\text{def}}{=}{}_2\phi_1\Big(\begin{matrix}q^{-n},\ q^{-x}\\ 0\end{matrix}\Big|\,q\,;a^{-1}q^{x+1}\Big), \tag{4.86}$$

$$\check{Q}_x^{(-)}(n;\boldsymbol{\lambda})\stackrel{\text{def}}{=}Q_x^{(-)}\big(\mathcal{E}(n);\boldsymbol{\lambda}\big)\stackrel{\text{def}}{=}{}_2\phi_1\Big(\begin{matrix}q^{-n},\ q^{-x}\\ 0\end{matrix}\Big|\,q\,;aq^{x+1}\Big). \tag{4.87}$$

They are degree x polynomials in $\mathcal{E}(n)$ and related to $\check{P}_n^{(\pm)}(x;\boldsymbol{\lambda})$ as (4.34) with

$$\alpha_n^{(+)}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}1,\quad \alpha_n^{(-)}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}a^{-n}. \tag{4.88}$$

Following the argument in § 4.2.1, $A^{(\pm)}_n(\boldsymbol{\lambda})$ and $C^{(\pm)}_n(\boldsymbol{\lambda})$ (4.39) and $\mathcal{H}^{\text{d}\,(\pm)}(\boldsymbol{\lambda})$ (4.45) are introduced:

$$A^{(+)}_n(\boldsymbol{\lambda}) = aq^n, \quad A^{(-)}_n(\boldsymbol{\lambda}) = -q^n, \quad C^{(+)}_n(\boldsymbol{\lambda}) = -(1-q^n), \quad C^{(-)}_n(\boldsymbol{\lambda}) = a(1-q^n).$$

They satisfy (4.40)–(4.41), which implies (4.49). The eigenvectors of $\mathcal{H}^{\text{d}\,(\pm)}(\boldsymbol{\lambda})$ are

$$\phi^{\text{d}\,(\pm)}_x(n;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \phi^{\text{d}\,(\pm)}_0(n;\boldsymbol{\lambda})\check{Q}^{(\pm)}_x(n;\boldsymbol{\lambda}), \tag{4.89}$$

$$\phi^{\text{d}\,(\pm)}_0(n;\boldsymbol{\lambda}) = (\pm 1)^n \prod_{m=0}^{n-1}\sqrt{\frac{A^{(\pm)}_m(\boldsymbol{\lambda})}{C^{(\pm)}_{m+1}(\boldsymbol{\lambda})}} = \alpha^{(\pm)}_n(\boldsymbol{\lambda})\frac{d_n(\boldsymbol{\lambda})}{d_0(\boldsymbol{\lambda})}, \quad (\pm 1)^n\phi^{\text{d}\,(\pm)}_0(n;\boldsymbol{\lambda}) > 0, \tag{4.90}$$

$$\mathcal{H}^{\text{d}\,(\pm)}(\boldsymbol{\lambda})\phi^{\text{d}\,(\pm)}_x(n;\boldsymbol{\lambda}) = \mathcal{E}^{\text{d}\,(\pm)}(x;\boldsymbol{\lambda})\phi^{\text{d}\,(\pm)}_x(n;\boldsymbol{\lambda}),$$

$$\mathcal{H}^{\text{d}\,(\pm)}(\boldsymbol{\lambda})\phi^{\text{d}\,(\mp)}_x(n;\boldsymbol{\lambda}) = \mathcal{E}'^{\,\text{d}\,(\pm)}(x;\boldsymbol{\lambda})\phi^{\text{d}\,(\mp)}_x(n;\boldsymbol{\lambda}),$$

$$\mathcal{E}^{\text{d}\,(\pm)}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} 1-q^x & : (+) \\ -a(1-q^x) & : (-) \end{cases}, \quad \mathcal{E}'^{\,\text{d}\,(\pm)}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} 1-aq^x & : (+) \\ -a+q^x & : (-) \end{cases}. \tag{4.91}$$

They satisfy the dual orthogonality relations in the same form as (4.52)–(4.53), in which $\phi^{(\pm)}_0(x;\boldsymbol{\lambda})^2$ and $d_0(\boldsymbol{\lambda})^2$ are given in (4.81) and (4.84). As in the big q-Laguerre case, the orthogonality relation (4.53) means the same relation as (4.74). In line with the Al-Salam-Carlitz I and the dual q-Charlier correspondence, the dual Al-Salam-Carlitz I polynomials (4.86)–(4.87) are equal to the q-Charlier polynomial and its completeness supplement (4.100)–(4.101).

4.3.5 discrete q-Hermite I

The discrete q-Hermite polynomial $h_n(\eta;q)$ is obtained from the Al-Salam-Carlitz I (ASC I) polynomial by setting $a^{\text{ASC I}} = -1$: $h_n(\eta;q) = U^{(-1)}_n(\eta;q)$. Various formulas are easily obtained from those of ASC I by setting $a^{\text{ASC I}} = -1$.

4.4 Complete set involving q-Meixner polynomials

By constructing the duals of the dual q-Meixner polynomials in a similar way as the dual big q-Jacobi and big q-Laguerre cases developed in § 4.2.1, § 4.3.2, we obtain the complete set of orthogonal vectors involving the q-Meixner polynomials. Since the dual of $\check{P}^{(+)}_n(x)$ (3.23) is the original q-Meixner (3.4), that of $\check{P}^{(-)}_n(x)$

$${}_2\phi_1\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\ -c^{-1}q^{-x}}{bq}\Bigm|q\,;q^{x+1}\Bigr) = \frac{(-bcq;q)_n}{(-c)^n(bq;q)_n}\,{}_2\phi_1\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\ q^{-x}}{-bcq}\Bigm|q\,;-cq^{x+1}\Bigr),$$

provides the supplementary orthogonal vectors. By interchanging $x \leftrightarrow n$ of $\check{P}_n^{(\pm)}(x)$, we obtain two polynomials in $\eta(x) = q^{-x} - 1$,

$$\check{P}_n(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_n\bigl(\eta(x);\boldsymbol{\lambda}\bigr) \stackrel{\text{def}}{=} {}_2\phi_1\Bigl(\begin{matrix} q^{-n}, q^{-x} \\ bq \end{matrix}\Bigm| q\,; -c^{-1}q^{n+1}\Bigr) = M_n(q^{-x}; b, c; q), \tag{4.92}$$

$$\check{P}_n^{(-)}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_n^{(-)}\bigl(\eta(x);\boldsymbol{\lambda}\bigr) \stackrel{\text{def}}{=} {}_2\phi_1\Bigl(\begin{matrix} q^{-n}, q^{-x} \\ -bcq \end{matrix}\Bigm| q\,; -cq^{n+1}\Bigr) = M_n(q^{-x}; -bc, c^{-1}; q), \tag{4.93}$$

i.e. the original q-Meixner (3.4) and its supplement. The parameter change (involution)

$$(b, c) \to (-bc, c^{-1}),$$

exchanges these two and it also provides $B^{(-)}(x)$, $D^{(-)}(x)$, $A_n^{(-)}$ and $C_n^{(-)}$ governing the latter

$$\begin{aligned} &B^{(-)}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} c^{-1}q^x(1 + bcq^{x+1}), \quad D^{(-)}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (1 - q^x)(1 - bq^x), \\ &A_n^{(-)}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -c^{-1}q^{-2n-1}(1 + bcq^{n+1}), \quad C_n^{(-)}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -q^{-2n}(1 - q^n)(q^n + c^{-1}), \end{aligned} \tag{4.94}$$

from $B(x)$, $D(x)$, A_n and C_n for the original q-Meixner (3.2), (3.7). By noting

$$\begin{aligned} B(x;\boldsymbol{\lambda})D(x+1;\boldsymbol{\lambda}) &= c^2 B^{(-)}(x;\boldsymbol{\lambda})D^{(-)}(x+1;\boldsymbol{\lambda}), \\ B(x;\boldsymbol{\lambda}) + D(x;\boldsymbol{\lambda}) - 1 &= -c\bigl(B^{(-)}(x;\boldsymbol{\lambda}) + D^{(-)}(x;\boldsymbol{\lambda}) - 1\bigr), \end{aligned}$$

we find that the two Hamiltonians $\mathcal{H}(\boldsymbol{\lambda})$ and $\mathcal{H}^{(-)}(\boldsymbol{\lambda})$

$$\begin{aligned} \mathcal{H}(\boldsymbol{\lambda}) &= -\sqrt{B(x;\boldsymbol{\lambda})D(x+1;\boldsymbol{\lambda})}\, e^{\partial} - \sqrt{B(x-1;\boldsymbol{\lambda})D(x;\boldsymbol{\lambda})}\, e^{-\partial} + B(x;\boldsymbol{\lambda}) + D(x;\boldsymbol{\lambda}), \\ \mathcal{H}^{(-)}(\boldsymbol{\lambda}) &= \sqrt{B^{(-)}(x;\boldsymbol{\lambda})D^{(-)}(x+1;\boldsymbol{\lambda})}\, e^{\partial} + \sqrt{B^{(-)}(x-1;\boldsymbol{\lambda})D^{(-)}(x;\boldsymbol{\lambda})}\, e^{-\partial} \\ &\quad + B^{(-)}(x;\boldsymbol{\lambda}) + D^{(-)}(x;\boldsymbol{\lambda}), \end{aligned}$$

are linearly related

$$c\mathcal{H}^{(-)} = -\mathcal{H} + 1 + c.$$

The eigenvectors are

$$\phi_n(x;\boldsymbol{\lambda}) = \phi_0(x;\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda}), \quad \phi_0(x;\boldsymbol{\lambda})^2 = c^x q^{\frac{1}{2}x(x-1)} \frac{(bq;q)_x}{(q, -bcq;q)_x}, \quad \phi_0(x;\boldsymbol{\lambda}) > 0, \tag{4.95}$$

$$\begin{aligned} &\phi_n^{(-)}(x;\boldsymbol{\lambda}) = \phi_0^{(-)}(x;\boldsymbol{\lambda})\check{P}_n^{(-)}(x;\boldsymbol{\lambda}), \quad \phi_0^{(-)}(x;\boldsymbol{\lambda}) = (-1)^x \prod_{y=0}^{x-1} \sqrt{\frac{B^{(-)}(y;\boldsymbol{\lambda})}{D^{(-)}(y+1;\boldsymbol{\lambda})}}, \\ &\phi_0^{(-)}(x;\boldsymbol{\lambda})^2 = c^{-x} q^{\frac{1}{2}x(x-1)} \frac{(-bcq;q)_x}{(q, bq;q)_x}, \quad (-1)^x \phi_0^{(-)}(x;\boldsymbol{\lambda}) > 0, \end{aligned} \tag{4.96}$$

$$\mathcal{H}(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda})=\mathcal{E}(n)\phi_n(x;\boldsymbol{\lambda}),\ \mathcal{H}^{(-)}(\boldsymbol{\lambda})\phi_n^{(-)}(x;\boldsymbol{\lambda})=\mathcal{E}(n)\phi_n^{(-)}(x;\boldsymbol{\lambda}),\ \mathcal{E}(n)=1-q^n.$$

The original Hamiltonian $\mathcal{H}(\boldsymbol{\lambda})$ governing the q-Meixner polynomials has another infinite set of eigenvectors $\{\phi_n^{(-)}(x;\boldsymbol{\lambda})\}$ (4.96),

$$\mathcal{H}(\boldsymbol{\lambda})\phi_n^{(-)}(x;\boldsymbol{\lambda})=\mathcal{E}'(n;\boldsymbol{\lambda})\phi_n^{(-)}(x;\boldsymbol{\lambda}),\quad \mathcal{E}'(n;\boldsymbol{\lambda})\stackrel{\text{def}}{=}1+cq^n,\tag{4.97}$$

which lies above the original eigenvectors. Similarly we have

$$\mathcal{H}^{(-)}(\boldsymbol{\lambda})\phi_n^{(+)}(x;\boldsymbol{\lambda})=\mathcal{E}'^{(-)}(n;\boldsymbol{\lambda})\phi_n^{(+)}(x;\boldsymbol{\lambda}),\quad \mathcal{E}'^{(-)}(n;\boldsymbol{\lambda})\stackrel{\text{def}}{=}1+c^{-1}q^n.$$

The orthogonality relations are $(n,m=0,1,\ldots)$

$$(\phi_n,\phi_m)=\sum_{x=0}^{\infty}\phi_n(x;\boldsymbol{\lambda})\phi_m(x;\boldsymbol{\lambda})=\frac{\delta_{nm}}{d_n(\boldsymbol{\lambda})^2},\tag{4.98}$$

$$\big(\phi_n^{(-)},\phi_m^{(-)}\big)=\sum_{x=0}^{\infty}\phi_n^{(-)}(x;\boldsymbol{\lambda})\phi_m^{(-)}(x;\boldsymbol{\lambda})=\frac{\delta_{nm}}{d_n^{(-)}(\boldsymbol{\lambda})^2},\quad d_n^{(-)}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}d_n(\boldsymbol{\lambda})\big|_{(b,c)\to(-bc,c^{-1})},$$

$$\big(\phi_n,\phi_m^{(-)}\big)=\sum_{x=0}^{\infty}\phi_n(x;\boldsymbol{\lambda})\phi_m^{(-)}(x;\boldsymbol{\lambda})=0.\tag{4.99}$$

The last orthogonality relation simply means

$$\sum_{x=0}^{\infty}(-1)^x\frac{q^{\frac12x(x-1)}}{(q;q)_x}q^{-kx}=0\quad(k\in\mathbb{Z}_{\ge0}),\quad \phi_0(x;\boldsymbol{\lambda})\phi_0^{(-)}(x;\boldsymbol{\lambda})=(-1)^x\frac{q^{\frac12x(x-1)}}{(q;q)_x},$$

which is the same as (4.74).

4.4.1 complete set involving q-Charlier polynomials

The q-Charlier polynomials are obtained from those of q-Meixner by setting $(b^{q\text{M}},c^{q\text{M}})=(0,a)$. The complete set of orthogonal vectors involving the q-Charlier polynomials are

$$\phi_n(x;\boldsymbol{\lambda})=\phi_0(x;\boldsymbol{\lambda})\,{}_2\phi_1\Big(\genfrac{}{}{0pt}{}{q^{-n},\,q^{-x}}{0}\Big|\,q\,;-a^{-1}q^{n+1}\Big),\ \ \phi_0(x;\boldsymbol{\lambda})=\sqrt{\frac{a^xq^{\frac12x(x-1)}}{(q;q)_x}}\,,\tag{4.100}$$

$$\phi_n^{(-)}(x;\boldsymbol{\lambda})=\phi_0^{(-)}(x;\boldsymbol{\lambda})\,{}_2\phi_1\Big(\genfrac{}{}{0pt}{}{q^{-n},\,q^{-x}}{0}\Big|\,q\,;-aq^{n+1}\Big),\ \ \phi_0^{(-)}(x;\boldsymbol{\lambda})=(-1)^x\sqrt{\frac{a^{-x}q^{\frac12x(x-1)}}{(q;q)_x}}\,,\tag{4.101}$$

$$\mathcal{H}(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda})=\mathcal{E}(n)\phi_n(x;\boldsymbol{\lambda}),\quad \mathcal{E}(n)=1-q^n,\tag{4.102}$$

$$\mathcal{H}(\boldsymbol{\lambda})\phi_n^{(-)}(x;\boldsymbol{\lambda})=\mathcal{E}'(n;\boldsymbol{\lambda})\phi_n^{(-)}(x;\boldsymbol{\lambda}),\quad \mathcal{E}'(n;\boldsymbol{\lambda})\stackrel{\text{def}}{=}1+aq^n.\tag{4.103}$$

The orthogonality relations have the same form as (4.98)–(4.99) with $d_n(\boldsymbol{\lambda})$ given in (3.44) and $d_n^{(-)}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}d_n(\boldsymbol{\lambda})|_{a\to a^{-1}}$. The third one $\big(\phi_n,\phi_m^{(-)}\big)=0$ is the same as (4.74).

5 Discrete q-Hermite II

In this section we discuss the discrete q-Hermite polynomial II, which is defined on the integer lattice of the full line $x \in \mathbb{Z}$. In order to place it under proper perspectives with respect to the other orthogonal polynomials having Jackson integral measures, let us go back to the general structure of the difference equations and the corresponding potential functions (4.1) giving rise to orthogonal polynomials with Jackson integral type measures. The essential point for the Jackson integral measure is that $p_2(\eta)$ ($\eta \propto q^x$) has two distinct roots other than zero. If it has one zero root, $\eta = 0 \leftrightarrow x = +\infty$, then $p_1(\eta)$ must have one zero root, too. Otherwise the zero mode vector $\phi_0(x)$ would not be square summable. This is the case for the little q-Jacobi (Laguerre) polynomials. One could say that their orthogonality measures are of Jackson integral type of the form $\int_0^1 d_q x$. If $p_2(\eta)$ is a linear function of η, the second Hamiltonian cannot be introduced. The difference operator (Hamiltonian) is not self-adjoint due to the rapid growth at $x = \infty$. If p_2 is a constant, however, new possibility of achieving self-adjointness emerges when p_1 is even. In this case the boundary point of the variable x disappears and it can take full integer values, $-\infty < x < \infty$ and the scale of the variable q^x can become arbitrary. Due to the evenness of the potential functions, a polynomial solution $P(cq^x)$ ($c \in \mathbb{R}_{>0}$) is always accompanied by another $P(-cq^x)$. Thus two component Hamiltonian method can be employed to take care of the rapid growth of the potential functions at $x \to +\infty$ and the exponential increase of the eigenpolynomials $P(\pm cq^x)$ at $x \to -\infty$. The fact that x can take negative values makes it impossible to introduce dual polynomials by the interchange $n \leftrightarrow x$ (2.49)–(2.50), since n is always a non-negative integer.

Let us now consider the general difference equation (2.1)–(2.7) on the full integer lattice. Now the potential functions $B(x)$ and $D(x)$ are everywhere positive

$$B(x) > 0, \quad D(x) > 0 \quad (x \in \mathbb{Z}). \tag{5.1}$$

The zero mode equation $\mathcal{A}\phi_0(x) = 0$ (2.11), being a two term recurrence relation, can be solved easily by starting from $x = 0$ into the positive and negative directions:

$$\phi_0(x) = \phi_0(0) \times \begin{cases} \displaystyle\prod_{y=0}^{x-1} \sqrt{\frac{B(y)}{D(y+1)}} & (x \in \mathbb{Z}_{\geq 0}) \\ \displaystyle\prod_{y=0}^{-x-1} \sqrt{\frac{D(-y)}{B(-y-1)}} & (x \in \mathbb{Z}_{<0}) \end{cases}. \tag{5.2}$$

The similarity transformed Hamiltonian $\widetilde{\mathcal{H}}$ has the same expression as in the half line case (2.17)

$$\widetilde{\mathcal{H}} \stackrel{\text{def}}{=} \phi_0(x)^{-1} \circ \mathcal{H} \circ \phi_0(x) = B(x)(1-e^{\partial}) + D(x)(1-e^{-\partial}).$$

The potential functions for the discrete q-Hermite II are

$$B^{\rm J}(\eta) \stackrel{\text{def}}{=} \eta^{-2}(1+\eta^2), \quad D^{\rm J}(\eta) \stackrel{\text{def}}{=} \eta^{-2}q, \quad \mathcal{E}(n) = 1-q^n, \quad \eta(x) \propto q^x. \tag{5.3}$$

It is obvious that the above $\widetilde{\mathcal{H}}$ is triangular with respect to the basis $\{1,\eta,\eta^2,\ldots,\eta^n\}$ and its eigenvalues are $\mathcal{E}(n) = 1-q^n$. This is a quite interesting situation that the similarity transformed Hamiltonian $\widetilde{\mathcal{H}}$ is unbounded but the spectrum is bounded; markedly different from those having the Jackson integral measures, the big q-Jacobi family. The corresponding polynomial solution of degree n in η is the discrete q-Hermite polynomial II [6],

$$\begin{aligned} P_n(\eta) &\stackrel{\text{def}}{=} q^{\frac12 n(n-1)}\tilde{h}_n(\eta;q) \stackrel{\text{def}}{=} i^{-n}{}_2\phi_0\Bigl(\begin{matrix}q^{-n},\, i\eta\\ -\end{matrix}\Bigm| q\,;-q^n\Bigr)\\ &= q^{\frac12 n(n-1)}\eta^n{}_2\phi_1\Bigl(\begin{matrix}q^{-n},\, q^{1-n}\\ 0\end{matrix}\Bigm| q^2\,;-q^2\eta^{-2}\Bigr), \qquad P_n(-i) = (-i)^n, \end{aligned} \tag{5.4}$$

which has definite parity, $P_n(-\eta) = (-1)^n P_n(\eta)$ due to the evenness of the potential functions. It is obtained from the Al-Salam-Carlitz II polynomial $V_n^{(a)}(\eta;q)$ by setting $a=-1$ and $\eta \to i\eta$, $\tilde{h}_n(\eta;q) = i^{-n}V_n^{(-1)}(i\eta;q)$. The difference equation, forward/backward shift relations and three term recurrence relation are

$$B^{\rm J}(\eta)\bigl(P_n(\eta)-P_n(q\eta)\bigr) + D^{\rm J}(\eta)\bigl(P_n(\eta)-P_n(q^{-1}\eta)\bigr) = \mathcal{E}(n)P_n(\eta), \tag{5.5}$$

$$\eta^{-1}\bigl(P_n(\eta)-P_n(q\eta)\bigr) = \mathcal{E}(n)P_{n-1}(q\eta), \tag{5.6}$$

$$\bigl(B^{\rm J}(\eta)P_{n-1}(q\eta) - q^{-1}D^{\rm J}(\eta)P_{n-1}(\eta)\bigr)\eta = P_n(\eta), \tag{5.7}$$

$$\eta P_n(\eta) = A_n P_{n+1}(\eta) + C_n P_{n-1}(\eta), \quad A_n \stackrel{\text{def}}{=} q^{-n}, \quad C_n \stackrel{\text{def}}{=} q^{-n}-1. \tag{5.8}$$

In the big q-Jacobi family, the two zeros of $D(x)$ determine the two scales of q^x appearing in the Jackson integral. In the present case, the scale is arbitrary $\pm cq^x$, $c>0$, since $D^{\rm J}(\eta)$ has no zero and the potentials are even. From now on we consider c as the system parameter:

$$q^{\boldsymbol{\lambda}} = c, \quad \boldsymbol{\delta} = 1, \quad \kappa = q, \quad c > 0. \tag{5.9}$$

As shown for the big q-Jacobi family, in the single component formulation, the self-adjointness is broken by the unboundedness of the potentials at $x \to +\infty$ and the behaviour of the polynomial solutions $P_n(cq^x)$ at $x \to +\infty$. These will lead to the two component Hamiltonian formulation with the sinusoidal coordinates $\eta^{(\pm)}(x;\boldsymbol{\lambda})$:

$$\eta^{(+)}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} cq^x, \quad \eta^{(-)}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -cq^x, \tag{5.10}$$

and the corresponding potential functions (3.32)

$$B^{(\pm)}(x;\boldsymbol{\lambda}) = 1 + c^{-2}q^{-2x} \stackrel{\text{def}}{=} B(x;\boldsymbol{\lambda}) > 0, \quad D^{(\pm)}(x;\boldsymbol{\lambda}) = c^{-2}q^{1-2x} \stackrel{\text{def}}{=} D(x;\boldsymbol{\lambda}) > 0, \tag{5.11}$$

which are equal for $(\pm)$. The eigenpolynomials are different for $(\pm)$:

$$\check{P}_n^{(\pm)}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_n\bigl(\eta^{(\pm)}(x;\boldsymbol{\lambda})\bigr) = i^{-n}{}_2\phi_0\Bigl(\begin{matrix} q^{-n},\ \pm icq^x \\ - \end{matrix}\Bigm| q\,;-q^n\Bigr). \tag{5.12}$$

As in the bqJ family, the appropriate ratio of the two zero mode vectors $\phi_0^{(\pm)}(0;\boldsymbol{\lambda})$ achieve the self-adjointness of the two component Hamiltonians, as demonstrated shortly. Eq. (5.2) gives

$$\frac{\phi_0(x;\boldsymbol{\lambda})^2}{\phi_0(0;\boldsymbol{\lambda})^2} = \begin{cases} q^x(-c^2;q^2)_x & = \dfrac{q^x(-c^2;q^2)_\infty}{(-c^2q^{2x};q^2)_\infty} \quad (x \in \mathbb{Z}_{\geq 0}) \\ \dfrac{c^{2x}q^{x^2}}{(-c^{-2}q^2;q^2)_{-x}} & = \dfrac{c^{2x}q^{x^2}(-c^{-2}q^{2-2x};q^2)_\infty}{(-c^{-2}q^2;q^2)_\infty} \quad (x \in \mathbb{Z}_{<0}) \end{cases}, \tag{5.13}$$

$$= \frac{q^x(-c^2;q^2)_\infty}{(-c^2q^{2x};q^2)_\infty} \quad (x \in \mathbb{Z}). \tag{5.14}$$

In the last equality we have used an identity

$$1 = \frac{(-c^2, -c^{-2}q^2;q^2)_\infty}{c^{2x}q^{x(x-1)}(-c^2q^{2x}, -c^{-2}q^{2-2x};q^2)_\infty} \quad (x \in \mathbb{Z}).$$

Now let us concentrate on the issue of self-adjointness. Since it could be broken either at $x = +\infty$ or $x = -\infty$ or both, the inner product of two single component vectors $f = (f(x))$ and $g = (g(x))$ is defined by

$$(f,g) \stackrel{\text{def}}{=} \lim_{N,N'\to\infty}(f,g)_{N,N'}, \quad (f,g)_{N,N'} \stackrel{\text{def}}{=} \sum_{x=-N'}^{N} f(x)g(x). \tag{5.15}$$

As shown in §2.1 (2.18)–(2.19), for two vectors $f(x) = \phi_0(x)\check{\mathcal{P}}(x)$ and $g(x) = \phi_0(x)\check{\mathcal{Q}}(x)$, we have

$$\begin{aligned}(f,\mathcal{H}g)_{N,N'} = &\sum_{x=-N'}^{N} \phi_0(x)^2\bigl(B(x)+D(x)\bigr)\check{\mathcal{P}}(x)\check{\mathcal{Q}}(x) - \sum_{x=-N'}^{N} \phi_0(x)^2B(x)\check{\mathcal{P}}(x)\check{\mathcal{Q}}(x+1) \\ &- \sum_{x=-N'-1}^{N-1} \phi_0(x)^2B(x)\check{\mathcal{P}}(x+1)\check{\mathcal{Q}}(x),\end{aligned}$$

and obtain

$$
\begin{aligned}
&(f,\mathcal{H}g)_{N,N'}-(\mathcal{H}f,g)_{N,N'}\\
&=\phi_0(N)^2B(N)\big(\check{\mathcal{P}}(N+1)\check{\mathcal{Q}}(N)-\check{\mathcal{P}}(N)\check{\mathcal{Q}}(N+1)\big)\\
&\quad+\phi_0(-N'-1)^2B(-N'-1)\big(\check{\mathcal{P}}(-N'-1)\check{\mathcal{Q}}(-N')-\check{\mathcal{P}}(-N')\check{\mathcal{Q}}(-N'-1)\big).
\end{aligned}
\tag{5.16}
$$

At $x=-N'$, a degree n polynomial grows like $\sim q^{-nN'}$, whereas the potential damps as $B(-N')\sim q^{2N'}$ and the zero mode vector $\phi_0(-N')^2$ provides much stronger damping as given in (5.13), $\phi_0(-N';\boldsymbol{\lambda})^2\sim c^{-2N'}q^{N'^2}$. Thus the second term in (5.16) vanishes as $N'\to\infty$ and the behaviour at $x=-\infty$ causes no problem for self-adjointness. The behaviour at $x=+\infty$ is the same as that for the bqJ family case and we have to rely on the two component formulation. In this case the two Hamiltonians $\mathcal{H}^{(\pm)}$ are exactly the same. The eigenvectors of $\mathcal{H}^{(\pm)}$ have the same structures as those of the big q Jacobi (4.20).

The two component system has been introduced in (3.27)–(3.30). The inner product of two vectors $\boldsymbol{f}(x)=\binom{f^{(+)}(x)}{f^{(-)}(x)}$ and $\boldsymbol{g}(x)=\binom{g^{(+)}(x)}{g^{(-)}(x)}$ is defined by

$$
(\!(\boldsymbol{f},\boldsymbol{g})\!)\stackrel{\text{def}}{=}\lim_{N\to\infty}(\!(\boldsymbol{f},\boldsymbol{g})\!)_N,\quad (\!(\boldsymbol{f},\boldsymbol{g})\!)_N\stackrel{\text{def}}{=}\sum_{x=-\infty}^{N}\big(f^{(+)}(x)g^{(+)}(x)+f^{(-)}(x)g^{(-)}(x)\big).
\tag{5.17}
$$

For two vectors $\boldsymbol{f}$ with $f^{(\pm)}(x)=\phi_0^{(\pm)}(x)\check{\mathcal{P}}^{(\pm)}(x)$ and $\boldsymbol{g}$ with $g^{(\pm)}(x)=\phi_0^{(\pm)}(x)\check{\mathcal{Q}}^{(\pm)}(x)$, eq. (5.16) gives

$$
\begin{aligned}
&(\!(\boldsymbol{f},\underline{\mathcal{H}}\,\boldsymbol{g})\!)_N-(\!(\underline{\mathcal{H}}\,\boldsymbol{f},\boldsymbol{g})\!)_N\\
&=\phi_0^{(+)}(N)^2B(N)\big(\check{\mathcal{P}}^{(+)}(N+1)\check{\mathcal{Q}}^{(+)}(N)-\check{\mathcal{P}}^{(+)}(N)\check{\mathcal{Q}}^{(+)}(N+1)\big)\\
&\quad+\phi_0^{(-)}(N)^2B(N)\big(\check{\mathcal{P}}^{(-)}(N+1)\check{\mathcal{Q}}^{(-)}(N)-\check{\mathcal{P}}^{(-)}(N)\check{\mathcal{Q}}^{(-)}(N+1)\big).
\end{aligned}
\tag{5.18}
$$

The situation is simpler than that in the big q-Jacobi family case, since $\phi_0^{(\pm)}(N)^2$ are only different by the unspecified overall factor (5.14)

$$
\phi_0^{(\pm)}(N;\boldsymbol{\lambda})^2=\phi_0^{(\pm)}(0;\boldsymbol{\lambda})^2\frac{q^N(-c^2;q^2)_\infty}{(-c^2q^{2N};q^2)_\infty}.
$$

It is very easy to see that for the equal choice

$$
\phi_0^{(\pm)}(0;\boldsymbol{\lambda})=\frac{1}{\sqrt{(-c^2;q^2)_\infty}}\ \Rightarrow\phi_0^{(\pm)}(x;\boldsymbol{\lambda})^2=\phi_0(x;\boldsymbol{\lambda})^2=\frac{q^x}{(-c^2q^{2x};q^2)_\infty},
\tag{5.19}
$$

the Hamiltonian is self-adjoint and the inner product of the two component eigenvectors

$$
\boldsymbol{\phi}_n(x;\boldsymbol{\lambda})=\begin{pmatrix}\phi_n^{(+)}(x;\boldsymbol{\lambda})\\ \phi_n^{(-)}(x;\boldsymbol{\lambda})\end{pmatrix}=\phi_0(x;\boldsymbol{\lambda})\begin{pmatrix}P_n(cq^x)\\ P_n(-cq^x)\end{pmatrix},
\tag{5.20}
$$

is given by

$$(\!(\boldsymbol{\phi}_n,\boldsymbol{\phi}_m)\!) = \big(\phi_n^{(+)},\phi_m^{(+)}\big)+\big(\phi_n^{(-)},\phi_m^{(-)}\big)=\frac{2\delta_{nm}}{d_n(\boldsymbol{\lambda})^2}\quad(n,m=0,1,\ldots) \tag{5.21}$$
$$=\sum_{x=-\infty}^{\infty}\big(\phi_n^{(+)}(x;\boldsymbol{\lambda})\phi_m^{(+)}(x;\boldsymbol{\lambda})+\phi_n^{(-)}(x;\boldsymbol{\lambda})\phi_m^{(-)}(x;\boldsymbol{\lambda})\big)$$
$$=\sum_{x=-\infty}^{\infty}\frac{q^x}{(-c^2q^{2x};q^2)_\infty}\big(P_n(cq^x)P_m(cq^x)+P_n(-cq^x)P_m(-cq^x)\big).$$

Here the normalisation constant $d_n(\boldsymbol{\lambda})>0$ is

$$d_n(\boldsymbol{\lambda})^2\stackrel{\text{def}}{=}\frac{q^n}{(q;q)_n}\times d_0(\boldsymbol{\lambda})^2,\quad d_0(\boldsymbol{\lambda})^2\stackrel{\text{def}}{=}\frac{(q,-c^2,-c^{-2}q^2;q^2)_\infty}{(q^2,-c^2q,-c^{-2}q;q^2)_\infty}. \tag{5.22}$$

Due to the definite parity $P_n(-\eta)=(-1)^nP_n(\eta)$, this orthogonal relation is rewritten as

$$\sum_{x=-\infty}^{\infty}\frac{1+(-1)^{n+m}}{2}\phi_n^{(+)}(x;\boldsymbol{\lambda})\phi_m^{(+)}(x;\boldsymbol{\lambda})=\frac{\delta_{nm}}{d_n(\boldsymbol{\lambda})^2}. \tag{5.23}$$

The system is shape invariant, since the potential functions $B^{(\pm)}(x;\boldsymbol{\lambda})$ and $D^{(\pm)}(x;\boldsymbol{\lambda})$ satisfy (2.29)–(2.30). The forward/backward shift relations for the dqH II have the form (4.26)–(4.27) with the auxiliary functions $\varphi^{(\pm)}(x;\boldsymbol{\lambda})$,

$$\varphi^{(\pm)}(x;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\eta^{(\pm)}(x;\boldsymbol{\lambda}). \tag{5.24}$$

The triangularity of the potentials can be explained along the line of [9]. By taking $v_{k,l}$ as

$$\begin{aligned}
&v_{0,0}=q^{-1}(1-q)(1-q^2), && v_{1,0}=v_{0,1}=0,\\
&v_{2,0}=q^{-1}-1+v_{0,2}, && v_{1,1}=q-1-(q+q^{-1})v_{0,2},
\end{aligned}$$

($v_{0,2}$ is arbitrary), the potential functions are expressed as (4.32)–(4.33).

5.1 q-Laguerre

As is well known, the Laguerre polynomials of $\alpha=\mp\frac12$ are related to the Hermite polynomials:

$$H_{2n}(x)=(-1)^n\,2^{2n}n!\,L_n^{(-\frac12)}(x^2),\quad H_{2n+1}(x)=(-1)^n\,2^{2n}n!\,2xL_n^{(\frac12)}(x^2).$$

The q-Laguerre (qL) polynomial $P_n(\eta;\boldsymbol{\lambda})$ ($n\in\mathbb{Z}_{\geq0}$) [6],

$$P_n(\eta;\boldsymbol{\lambda})\stackrel{\text{def}}{=}P_n(\eta;a;q),\quad P_n(-1;\boldsymbol{\lambda})=1, \tag{5.25}$$
$$\stackrel{\text{def}}{=}{}_2\phi_1\Big(\begin{matrix}q^{-n},\ -\eta\\ 0\end{matrix}\Big|\,q\,;aq^{n+1}\Big)=(q;q)_nL_n^{(\alpha)}(\eta;q),\quad a=q^\alpha,$$

$$
\begin{aligned}
&= (-a)^n q^{n^2} \eta^n {}_2\phi_1\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\, a^{-1}q^{-n}}{0}\Bigm|\, q\,; -q\eta^{-1}\Bigr),\\
&= (aq;q)_n\, {}_1\phi_1\Bigl(\genfrac{}{}{0pt}{}{q^{-n}}{aq}\Bigm|\, q\,; -aq^{n+1}\eta\Bigr),
\end{aligned}
$$

is the counterpart of the discrete q-Hermite II polynomial defined on the full integer lattice. For $\alpha = \mp\frac{1}{2}$, they are related in a similar way as the Hermite to the Laguerre:

$$
P^{\mathrm{d}q\mathrm{H\,II}}_{2n}(\eta^{\frac{1}{2}}; q^{\frac{1}{2}}) = (-1)^n P_n(\eta; q^{-\frac{1}{2}}; q), \quad P^{\mathrm{d}q\mathrm{H\,II}}_{2n+1}(\eta^{\frac{1}{2}}; q^{\frac{1}{2}}) = (-1)^n \eta^{\frac{1}{2}} P_n(\eta; q^{\frac{1}{2}}; q). \tag{5.26}
$$

The qL is related to the Laguerre polynomial $L^{(\alpha)}_n(\eta)$ by the following limit,

$$
\lim_{q\to 1}(1-q)^{-n} P_n\bigl((1-q)\eta; q^{\alpha}; q\bigr) = n!\, L^{(\alpha)}_n(\eta). \tag{5.27}
$$

The basic data of qL are

$$
\begin{aligned}
&q^{\boldsymbol{\lambda}} = (a, c), \quad \boldsymbol{\delta} = (1,1), \quad \kappa = q, \quad 0 < a < q^{-1}, \quad c > 0, && (5.28)\\
&\mathcal{E}(n) \stackrel{\text{def}}{=} 1 - q^n, && (5.29)\\
&B^{\mathrm{J}}(\eta) \stackrel{\text{def}}{=} \eta^{-1}(\eta+1), \quad D^{\mathrm{J}}(\eta; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} a^{-1}\eta^{-1}, && (5.30)\\
&A_n(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -a^{-1}q^{-2n-1}, \quad C_n(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -a^{-1}q^{-2n}(1-q^n)(1-aq^n). && (5.31)
\end{aligned}
$$

The qL polynomial $P_n(\eta; \boldsymbol{\lambda})$ is the degree n polynomial solution in η of the second order difference equation

$$
B^{\mathrm{J}}(\eta)\bigl(P_n(\eta; \boldsymbol{\lambda}) - P_n(q\eta; \boldsymbol{\lambda})\bigr) + D^{\mathrm{J}}(\eta; \boldsymbol{\lambda})\bigl(P_n(\eta; \boldsymbol{\lambda}) - P_n(q^{-1}\eta; \boldsymbol{\lambda})\bigr) = \mathcal{E}(n) P_n(\eta; \boldsymbol{\lambda}). \tag{5.32}
$$

Its recurrence relation and the forward/backward shift relations are

$$
\begin{aligned}
&(1+\eta)P_n(\eta; \boldsymbol{\lambda}) = A_n(\boldsymbol{\lambda})P_{n+1}(\eta; \boldsymbol{\lambda}) - \bigl(A_n(\boldsymbol{\lambda}) + C_n(\boldsymbol{\lambda})\bigr)P_n(\eta; \boldsymbol{\lambda}) + C_n(\boldsymbol{\lambda})P_{n-1}(\eta; \boldsymbol{\lambda}), && (5.33)\\
&(-aq\eta)^{-1}\bigl(P_n(\eta; \boldsymbol{\lambda}) - P_n(q\eta; \boldsymbol{\lambda})\bigr) = \mathcal{E}(n)P_{n-1}(q\eta; \boldsymbol{\lambda} + \boldsymbol{\delta}), && (5.34)\\
&\bigl(B^{\mathrm{J}}(\eta)P_{n-1}(q\eta; \boldsymbol{\lambda} + \boldsymbol{\delta}) - q^{-1}D^{\mathrm{J}}(\eta; \boldsymbol{\lambda})P_{n-1}(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta})\bigr)(-aq\eta) = P_n(\eta; \boldsymbol{\lambda}). && (5.35)
\end{aligned}
$$

In terms of the sinusoidal coordinate

$$
\eta(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} cq^x, \quad \check{P}_n(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_n\bigl(\eta(x; \boldsymbol{\lambda}); \boldsymbol{\lambda}\bigr), \tag{5.36}
$$

and the potential functions

$$
B(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} B^{\mathrm{J}}\bigl(\eta(x; \boldsymbol{\lambda})\bigr) = 1 + c^{-1}q^{-x}, \quad D(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} D^{\mathrm{J}}\bigl(\eta(x; \boldsymbol{\lambda}); \boldsymbol{\lambda}\bigr) = a^{-1}c^{-1}q^{-x}, \tag{5.37}
$$

the corresponding Hamiltonian $\mathcal{H}(\boldsymbol{\lambda})$ is defined on the full integer lattice, $x \in \mathbb{Z}$. Its ground state vector $\phi_0(x;\boldsymbol{\lambda})$ is determined in the same way as for the dqH II (5.2),

$$\phi_0(x;\boldsymbol{\lambda})^2 = \frac{(aq)^x}{(-cq^x;q)_\infty}, \tag{5.38}$$

in which we have adopted as the boundary condition $\phi_0(0;\boldsymbol{\lambda})^2 = (-c;q)_\infty^{-1}$. Like the d$q$H II case, this measure function decreases very rapidly at $x \to -\infty$ but the unboundedness of the potentials at large $N > 0$ is milder $\sim q^{-N}$ than that of the dqH II. Thus the single component Hamiltonian $\mathcal{H}(\boldsymbol{\lambda})$ is self-adjoint and its eigenvectors are

$$\mathcal{H}(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda}) = \mathcal{E}(n)\phi_n(x;\boldsymbol{\lambda}), \quad \phi_n(x;\boldsymbol{\lambda}) = \phi_0(x;\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda}) \ \ (n = 0, 1, \ldots). \tag{5.39}$$

The orthogonality relation is the standard one

$$(\phi_n, \phi_m) = \sum_{x=-\infty}^{\infty} \phi_n(x;\boldsymbol{\lambda})\phi_m(x;\boldsymbol{\lambda}) = \frac{\delta_{nm}}{d_n(\boldsymbol{\lambda})^2} \ \ (n, m \in \mathbb{Z}_{\geq 0}), \tag{5.40}$$

in which $d_n(\boldsymbol{\lambda}) > 0$ is given by

$$d_n(\boldsymbol{\lambda})^2 = \frac{q^n}{(q, aq;q)_n} \times d_0(\boldsymbol{\lambda})^2, \quad d_0(\boldsymbol{\lambda})^2 = \frac{(aq, -c, -c^{-1}q;q)_\infty}{(q, -acq, -a^{-1}c^{-1};q)_\infty}. \tag{5.41}$$

The system is shape invariant. The forward/backward shift relations for the qL have the form (4.26)–(4.27) (without superscript $(\pm)$) with the auxiliary functions $\varphi(x;\boldsymbol{\lambda})$,

$$\varphi(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -aq\eta(x;\boldsymbol{\lambda}). \tag{5.42}$$

5.2 Universal Rodrigues formula for polynomials with Jackson integrals

The universal Rodrigues formula for the polynomials with Jackson integral type measures has the same structure (2.40) as that for the rest of the classical orthogonal polynomials of a discrete variable (without superscript $(\pm)$ for qL):

$$\begin{aligned} P_n^{(\pm)}\big(\eta^{(\pm)}(x;\boldsymbol{\lambda});\boldsymbol{\lambda}\big) &= \phi_0^{(\pm)}(x;\boldsymbol{\lambda})^{-2}\,\bar{\mathcal{D}}^{(\pm)}(\boldsymbol{\lambda})\bar{\mathcal{D}}^{(\pm)}(\boldsymbol{\lambda}+\boldsymbol{\delta})\cdots\bar{\mathcal{D}}^{(\pm)}\big(\boldsymbol{\lambda}+(n-1)\boldsymbol{\delta}\big) \\ &\quad\times \phi_0^{(\pm)}(x;\boldsymbol{\lambda}+n\boldsymbol{\delta})^2, \end{aligned} \tag{5.43}$$

$$\bar{\mathcal{D}}^{(\pm)}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (1-e^{-\partial})\bar{\varphi}^{(\pm)}(x;\boldsymbol{\lambda})^{-1}, \quad \bar{\varphi}^{(\pm)}(x;\boldsymbol{\lambda}) \propto \varphi^{(\pm)}(x;\boldsymbol{\lambda}). \tag{5.44}$$

However, reflecting the different normalisation (boundary) conditions for the ground state wavefunctions $\phi_0^{(\pm)}(x;\boldsymbol{\lambda})$ and for the polynomials; $P_n(1;\boldsymbol{\lambda}) = 1$ for bqJ (4.12), bqL (4.61), ASC I (4.78), $P_n(-i;\boldsymbol{\lambda})=(-i)^n$ for dqH II (5.4), $P_n(-1;\boldsymbol{\lambda})=1$ for qL (5.25), the function $\varphi(x;\boldsymbol{\lambda})$ in (2.41) gets an extra constant factor. The modified functions $\bar{\varphi}^{(\pm)}(x;\boldsymbol{\lambda})$ are

$$\bar{\varphi}^{(\pm)}(x;\boldsymbol{\lambda})=\eta^{(\pm)}(x;\boldsymbol{\lambda})\times\begin{cases}(1-aq)(1-cq)(-acq)^{-1} & :\mathrm{b}q\mathrm{J}\\(1-aq)(1-bq)(-abq)^{-1} & :\mathrm{b}q\mathrm{L}\\1 & :\mathrm{ASC\,I},\mathrm{d}q\mathrm{H\,II}\end{cases},$$
$$\bar{\varphi}(x;\boldsymbol{\lambda})=\eta(x;\boldsymbol{\lambda})(-acq)^{-1}\ \ :q\mathrm{L}. \tag{5.45}$$

6 Other Topics

6.1 Birth and Death processes related with Jackson integral measures

As an application of orthogonal polynomials of a discrete variable, here we comment on birth and death processes [4]. A Birth and Death (BD) process is a typical stationary Markov process with a one-dimensional discrete state space and the transitions occur only between nearest neighbours. The transition probability per unit time from x to $x+1$ is $B(x)$ (birth rate) and x to $x-1$ is $D(x)$ (death rate). It was shown in [24] that classical orthogonal polynomials obtained from hermitian matrices in I, *e.g.* the (q-)Racah and (dual) (q-)Hahn polynomials, provide *exactly solvable birth and death processes*. Here we show that the orthogonal polynomials discussed in this paper, *i.e.* those having Jackson integral measures, also supply examples of exactly solvable birth and death processes with two component discrete state spaces. The dual polynomials of the big q-Jacobi family also give rise to interesting examples of exactly solvable BD processes.

Let us summarise the essence of BD process within the framework of 'orthogonal polynomials from hermitian matrices' as explained in § 2. The birth and death equation reads

$$\frac{\partial}{\partial t}\mathcal{P}(x;t)=(L_{\mathrm{BD}}\mathcal{P})(x;t),\quad \mathcal{P}(x;t)\geq 0,\quad \sum_x\mathcal{P}(x;t)=1, \tag{6.1}$$

in which $\mathcal{P}(x;t)$ is the probability distribution over a certain discrete set of the parameter x. The *birth and death operator* L_{BD} is obtained by *inverse similarity transformation* of the Hamiltonian $\mathcal{H}$:

$$L_{\mathrm{BD}}\stackrel{\text{def}}{=}-\phi_0\circ\mathcal{H}\circ\phi_0^{-1}=(e^{-\partial}-1)B(x)+(e^{\partial}-1)D(x), \tag{6.2}$$
$$L_{\mathrm{BD}}\phi_0(x)\phi_n(x)=-\mathcal{E}(n)\phi_0(x)\phi_n(x)\ \ (n=0,1,\ldots).$$

Given an arbitrary initial probability distribution $\mathcal{P}(x;0) \geq 0$ (with $\sum_x \mathcal{P}(x;0) = 1$), the probability distribution at a later time t is

$$\mathcal{P}(x;t) = \hat{\phi}_0(x) \sum_{n=0}^{\infty} c_n \, e^{-\mathcal{E}(n)t} \hat{\phi}_n(x) \quad (t > 0), \tag{6.3}$$

in which the constants $\{c_n\}$ are the expansion coefficients in terms of the *complete set of normalised eigenfunctions* $\{\hat{\phi}_n(x)\}$ of the Hamiltonian $\mathcal{H}$:

$$\mathcal{P}(x;0) = \hat{\phi}_0(x) \sum_{n=0}^{\infty} c_n \hat{\phi}_n(x), \quad c_n = \big(\hat{\phi}_n(x), \hat{\phi}_0(x)^{-1} \mathcal{P}(x;0)\big) \quad (\Rightarrow c_0 = 1). \tag{6.4}$$

For a concentrated initial distribution at y (*e.g.* $\mathcal{P}(x;0) = \delta_{xy}$), the transition probability from y to x is given by

$$\mathcal{P}(x,y;t) = \hat{\phi}_0(x) \Big(\sum_{n=0}^{\infty} e^{-\mathcal{E}(n)t} \hat{\phi}_n(x) \hat{\phi}_n(y) \Big) \hat{\phi}_0(y)^{-1} \quad (t > 0), \tag{6.5}$$

which satisfies the so-called Chapman-Kolmogorov equation [4]

$$\mathcal{P}(x,y;t) = \sum_{z=0}^{\infty} \mathcal{P}(x,z;t-t') \mathcal{P}(z,y;t') \quad (0 < t' < t), \tag{6.6}$$

as the consequence of the normalised eigenfunctions $\sum_{z=0}^{\infty} \hat{\phi}_n(z) \hat{\phi}_m(z) = \delta_{nm}$. The notation for the transition probability here is slightly changed from [24].

6.1.1 two component BD

Now let us formulate the two component BD processes matching with the big q-Jacobi family § 4.2. Corresponding to the two component Hamiltonian (3.27)–(3.30), the state space has also two components,

$$\boldsymbol{\mathcal{P}}(x;t) = \begin{pmatrix} \mathcal{P}^{(+)}(x;t) \\ \mathcal{P}^{(-)}(x;t) \end{pmatrix}, \quad \mathcal{P}^{(\pm)}(x;t) \geq 0, \quad \sum_{x=0}^{\infty} \big(\mathcal{P}^{(+)}(x;t) + \mathcal{P}^{(-)}(x;t) \big) = 1. \tag{6.7}$$

The two component birth and death equation reads

$$\frac{\partial}{\partial t} \boldsymbol{\mathcal{P}}(x;t) = \underline{L}_{\text{BD}} \boldsymbol{\mathcal{P}}(x;t), \tag{6.8}$$

$$\underline{L}_{\text{BD}} = -\underline{\phi_0}(x) \circ \underline{\mathcal{H}} \circ \underline{\phi_0}(x)^{-1} = (\underline{e^{-\partial}} - 1) \underline{B}(x) + (\underline{e^{\partial}} - 1) \underline{D}(x), \tag{6.9}$$

$$\underline{L}_{\text{BD}} \begin{pmatrix} \hat{\phi}_0^{(+)}(x) \hat{\phi}_n^{(+)}(x) \\ \hat{\phi}_0^{(-)}(x) \hat{\phi}_n^{(-)}(x) \end{pmatrix} = -\mathcal{E}(n) \begin{pmatrix} \hat{\phi}_0^{(+)}(x) \hat{\phi}_n^{(+)}(x) \\ \hat{\phi}_0^{(-)}(x) \hat{\phi}_n^{(-)}(x) \end{pmatrix} \quad (n = 0, 1, \ldots). \tag{6.10}$$

For each type $(+)$ and $(-)$, the equation reads

$$\frac{\partial}{\partial t}\mathcal{P}^{(\pm)}(x;t) = \big((e^{-\partial}-1)B^{(\pm)}(x) + (e^{\partial}-1)D^{(\pm)}(x)\big)\mathcal{P}^{(\pm)}(x;t), \tag{6.11}$$

in which $B^{(\pm)}(x)$ and $D^{(\pm)}(x)$ are given in (4.6)–(4.7) for the bqJ, (4.59)–(4.60) for the bqL, and (4.76)–(4.77) for the ASC I. As seen from the equation, the direct interactions are between the nearest neighbours of the same type only. The $(+)$ and $(-)$ sites are correlated through the asymptotic points in such a way that the self-adjointness of the corresponding Hamiltonian is realised. For an arbitrary initial probability distribution $\mathcal{P}^{(\pm)}(x;0) \geq 0$ (with $\sum_\epsilon \sum_x \mathcal{P}^{(\epsilon)}(x;0) = 1$), the probability distribution at a later time t is

$$\boldsymbol{\mathcal{P}}(x;t) = \sum_{n=0}^{\infty} c_n\, e^{-\mathcal{E}(n)t} \begin{pmatrix} \hat{\phi}_0^{(+)}(x)\hat{\phi}_n^{(+)}(x) \\ \hat{\phi}_0^{(-)}(x)\hat{\phi}_n^{(-)}(x) \end{pmatrix} \quad (t>0), \tag{6.12}$$

in which the constants $\{c_n\}$ are the expansion coefficients in terms of the complete set of normalised eigenvectors $\{\hat{\boldsymbol{\phi}}_n(x)\}$ of the Hamiltonian $\underline{\mathcal{H}}$:

$$\big(\!\big(\hat{\boldsymbol{\phi}}_n, \hat{\boldsymbol{\phi}}_m\big)\!\big) = \sum_{x=0}^{\infty}\big(\hat{\phi}_n^{(+)}(x)\hat{\phi}_m^{(+)}(x) + \hat{\phi}_n^{(-)}(x)\hat{\phi}_m^{(-)}(x)\big) = \delta_{nm}, \tag{6.13}$$

$$\begin{aligned} c_n &= \big(\!\big(\hat{\boldsymbol{\phi}}_n(x), \underline{\hat{\phi}}_0(x)^{-1}\boldsymbol{\mathcal{P}}(x;0)\big)\!\big) \quad (\Rightarrow c_0 = 1) \\ &= \sum_{x=0}^{\infty}\big(\hat{\phi}_n^{(+)}(x)\hat{\phi}_0^{(+)}(x)^{-1}\mathcal{P}^{(+)}(x;0) + \hat{\phi}_n^{(-)}(x)\hat{\phi}_0^{(-)}(x)^{-1}\mathcal{P}^{(-)}(x;0)\big). \end{aligned} \tag{6.14}$$

Asymptotically the above probability distribution (6.12) approaches to the stationary distribution

$$\lim_{t\to\infty}\boldsymbol{\mathcal{P}}(x;t) = \begin{pmatrix} \hat{\phi}_0^{(+)}(x)^2 \\ \hat{\phi}_0^{(-)}(x)^2 \end{pmatrix}.$$

There are four types of transition probabilities, starting from $(+)$ or $(-)$ component at site y at $t=0$ and arriving at $(+)$ or $(-)$ component at site x at a later time t. They form a 2×2 matrix generalisation of the one component formula (6.5),

$$\boldsymbol{\mathcal{P}}(x,y;t) = \sum_{n=0}^{\infty} e^{-\mathcal{E}(n)t} \begin{pmatrix} \hat{\phi}_0^{(+)}(x)\hat{\phi}_n^{(+)}(x) \\ \hat{\phi}_0^{(-)}(x)\hat{\phi}_n^{(-)}(x) \end{pmatrix} \begin{pmatrix} \hat{\phi}_n^{(+)}(y)\hat{\phi}_0^{(+)}(y)^{-1} & \hat{\phi}_n^{(-)}(y)\hat{\phi}_0^{(-)}(y)^{-1} \end{pmatrix}, \tag{6.15}$$

which also satisfies the matrix form Chapman-Kolmogorov equation

$$\boldsymbol{\mathcal{P}}(x,y;t) = \sum_{z=0}^{\infty}\boldsymbol{\mathcal{P}}(x,z;t-t')\boldsymbol{\mathcal{P}}(z,y;t') \quad (0<t'<t), \tag{6.16}$$

as a consequence of the above orthogonality relation (6.13).

6.1.2 discrete q-Hermite II

For the BD process corresponding to the discrete q-Hermite II § 5, the above formulas require slight modifications. The site x is now on the full integer lattice and $\hat{\phi}_0^{(+)}(x) = \hat{\phi}_0^{(-)}(x) = \hat{\phi}_0(x)$ and the potentials $B^{(\pm)}(x) = B(x)$, $D^{(\pm)}(x) = D(x)$ are given in (5.11). The transition probability is simply given by

$$\boldsymbol{\mathcal{P}}(x,y;t) = \hat{\phi}_0(x) \sum_{n=0}^{\infty} e^{-\mathcal{E}(n)t} \begin{pmatrix} \hat{\phi}_n^{(+)}(x) \\ \hat{\phi}_n^{(-)}(x) \end{pmatrix} \begin{pmatrix} \hat{\phi}_n^{(+)}(y) & \hat{\phi}_n^{(-)}(y) \end{pmatrix} \hat{\phi}_0(y)^{-1}. \tag{6.17}$$

6.1.3 q-Laguerre

Except for the demographic interpretation, which is not applicable because of the full integer lattice sites, all the formulas of the standard BD process (6.1)–(6.6) are valid with the birth/death rates given by (5.37) and the eigenvectors by (5.38)–(5.39).

6.1.4 dual big q-Jacobi

To write down various formulas for the BD processes corresponding to the dual big q-Jacobi families § 4.2.1 is an interesting exercise. We fix one Hamiltonian, say $\mathcal{H}^{\mathrm{d}\,(+)}$, with $A_n^{(+)}$ and $C_n^{(+)}$. The sites are now parametrised by $n \in \mathbb{Z}_{\geq 0}$ and

$$L_{\mathrm{BD}}^{\mathrm{d}} \stackrel{\text{def}}{=} -\phi_0^{\mathrm{d}\,(+)} \circ \mathcal{H}^{\mathrm{d}\,(+)} \circ {\phi_0^{\mathrm{d}\,(+)}}^{-1} = -(e^{-\partial_n} - 1)A_n^{(+)} - (e^{\partial_n} - 1)C_n^{(+)}. \tag{6.18}$$

There are two series of eigenvectors $\{\hat{\phi}_x^{\mathrm{d}\,(+)}(n), \mathcal{E}^{\mathrm{d}\,(+)}(x)\}$ (4.48) and $\{\hat{\phi}_x^{\mathrm{d}\,(-)}(n), \mathcal{E}'^{\mathrm{d}\,(+)}(x)\}$ (4.50). The transition probability from site m at time $t = 0$ to site n at a later time t is

$$\begin{aligned}\mathcal{P}^{\mathrm{d}}(n,m;t) = \hat{\phi}_0^{\mathrm{d}\,(+)}(n) \sum_{x=0}^{\infty} \big(& e^{-\mathcal{E}^{\mathrm{d}\,(+)}(x)t} \hat{\phi}_x^{\mathrm{d}\,(+)}(n) \hat{\phi}_x^{\mathrm{d}\,(+)}(m) + e^{-\mathcal{E}'^{\mathrm{d}\,(+)}(x)t} \hat{\phi}_x^{\mathrm{d}\,(-)}(n) \hat{\phi}_x^{\mathrm{d}\,(-)}(m) \big) \\ & \times \hat{\phi}_0^{\mathrm{d}\,(+)}(m)^{-1} \quad (t > 0). \end{aligned} \tag{6.19}$$

It also satisfies the Chapman-Kolmogorov equation (6.6) thanks to the orthogonality relation (4.54),

$$(\hat{\phi}_x^{\mathrm{d}\,(\epsilon)}, \hat{\phi}_y^{\mathrm{d}\,(\epsilon')}) = \sum_{n=0}^{\infty} \hat{\phi}_x^{\mathrm{d}\,(\epsilon)}(n) \hat{\phi}_y^{\mathrm{d}\,(\epsilon')}(n) = \delta_{\epsilon\,\epsilon'} \delta_{xy}.$$

6.1.5 complete q-Meixner

For the exact solvability of BD processes, the completeness of the corresponding orthogonal polynomials is essential. Thus the simple form of solutions (6.3), (6.5), given in [24] for the

q-Meixner and q-Charlier is flawed. The form of the equation is the same as given in [24], *i.e.* (6.1) with the birth/death rates given by (3.2) or (3.41). The correct transition probability from site y at time $t=0$ to site x at a later time t has a similar form to that of the dual big q-Jacobi (6.19),

$$\mathcal{P}(x,y;t)=\hat{\phi}_0(x)\sum_{n=0}^{\infty}\Bigl(e^{-\mathcal{E}(n)t}\hat{\phi}_n(x)\hat{\phi}_n(y)+e^{-\mathcal{E}'(n)t}\hat{\phi}_n^{(-)}(x)\hat{\phi}_n^{(-)}(y)\Bigr)\hat{\phi}_0(y)^{-1}\quad(t>0),\quad(6.20)$$

in which the supplementary vectors $\{\hat{\phi}_n^{(-)}(x)\}$ contribute. Here $\mathcal{E}(n)=1-q^n$ and $\mathcal{E}'(n)=1+cq^n$ (4.97) for qM and $\mathcal{E}'(n)=1+aq^n$ (4.103) for qC. The eigenvectors are (4.95)–(4.96) for qM and (4.100)–(4.101) for qC. The transition probability (6.20) satisfies the Chapman-Kolmogorov equation (6.6).

6.2 Proposal for new normalisation for some orthogonal polynomials

In general, the normalisation of orthogonal polynomials is a matter of historical conventions. However, as shown in [1, 9] and recapitulated in § 2.2, there is an unambiguous and universal rule for normalisation for orthogonal polynomials of a discrete variable defined on a finite integer lattice or on a semi-infinite integer lattice (2.26),

$$\check{P}_n(0;\boldsymbol{\lambda})=P_n\bigl(\eta(0;\boldsymbol{\lambda});\boldsymbol{\lambda}\bigr)=P_n(0;\boldsymbol{\lambda})=1\quad(n=0,1,2,\ldots).$$

Most of the conventional normalisation follow the above rule, *except for* the little q-Jacobi, little q-Laguerre, Al-Salam-Carlitz II and the alternative q-Charlier (q-Bessel) polynomials. Below we present the explicit hypergeometric expressions for the above four polynomials satisfying the universal normalisation rule (2.26). They all have q^{-n} and q^{-x} among the upper indices $\{a_1,\ldots,a_r\}$ of the q-hypergeometric function ${}_r\phi_s\Bigl(\begin{smallmatrix}a_1,\,\ldots,\,a_r\\ b_1,\,\ldots,\,b_s\end{smallmatrix}\Bigm|q\,;z\Bigr)$. In I, the universal normalisation rules were applied to these four polynomials by multiplying appropriate rescaling factors to the conventional definitions, but the explicit hypergeometric expressions were not reported.

6.2.1 little q-Jacobi

The conventional form [6] is

$$p_n^{\text{conv}}(q^x;a,b|q)={}_2\phi_1\Bigl(\begin{matrix}q^{-n},\,abq^{n+1}\\ aq\end{matrix}\Bigm|q\,;q^{x+1}\Bigr).$$

The proposed new form is

$$P_n\big(\eta(x);\boldsymbol{\lambda}\big)={}_3\phi_1\Big(\genfrac{}{}{0pt}{}{q^{-n},\,abq^{n+1},\,q^{-x}}{bq}\Bigm|q\,;\frac{q^x}{a}\Big)=(-a)^{-n}q^{-\frac12 n(n+1)}\frac{(aq;q)_n}{(bq;q)_n}p_n^{\text{conv}}(q^x;a,b|q),$$
$$\eta(x)=1-q^x,\quad \mathcal{E}(n;\boldsymbol{\lambda})=(q^{-n}-1)(1-abq^{n+1}). \tag{6.21}$$

The new form is obtained by using transformation formula (III.8) in [5].

6.2.2 little q-Laguerre

The conventional form [6] is

$$p_n^{\text{conv}}(q^x;a|q)={}_2\phi_1\Big(\genfrac{}{}{0pt}{}{q^{-n},\,0}{aq}\Bigm|q\,;q^{x+1}\Big).$$

The universal form [1] is

$$P_n\big(\eta(x);\boldsymbol{\lambda}\big)={}_2\phi_0\Big(\genfrac{}{}{0pt}{}{q^{-n},\,q^{-x}}{-}\Bigm|q\,;\frac{q^x}{a}\Big)=(a^{-1}q^{-n};q)_n\,p_n^{\text{conv}}(q^x;a|q),$$
$$\eta(x)=1-q^x,\quad \mathcal{E}(n)=q^{-n}-1. \tag{6.22}$$

6.2.3 Al-Salam-Carlitz II

The conventional form [6] is

$$V_n^{(a)}(q^{-x};q)=(-a)^nq^{-\frac12 n(n-1)}{}_2\phi_0\Big(\genfrac{}{}{0pt}{}{q^{-n},\,q^{-x}}{-}\Bigm|q\,;\frac{q^n}{a}\Big).$$

The universal form [1] is

$$P_n\big(\eta(x);\boldsymbol{\lambda}\big)={}_2\phi_0\Big(\genfrac{}{}{0pt}{}{q^{-n},\,q^{-x}}{-}\Bigm|q\,;\frac{q^n}{a}\Big)=(-a)^{-n}q^{\frac12 n(n-1)}\,V_n^{(a)}(q^{-x};q),$$
$$\eta(x)=q^{-x}-1,\quad \mathcal{E}(n)=1-q^n. \tag{6.23}$$

It is obvious that the little q-Laguerre polynomial (6.22) and the Al-Salam-Carlitz II polynomial (6.23) are dual to each other, that is, they are interchanged by $x\leftrightarrow n$.

6.2.4 alternative q-Charlier (q-Bessel)

The conventional form [6] is

$$K_n(q^x;a;q)={}_2\phi_1\Big(\genfrac{}{}{0pt}{}{q^{-n},\,-aq^n}{0}\Bigm|q\,;q^{x+1}\Big).$$

The proposed new form is

$$P_n\big(\eta(x);\boldsymbol{\lambda}\big)={}_3\phi_0\Big(\genfrac{}{}{0pt}{}{q^{-n},\,-aq^n,\,q^{-x}}{-}\Bigm|q\,;-\frac{q^x}{a}\Big)=q^{nx}{}_2\phi_1\Big(\genfrac{}{}{0pt}{}{q^{-n},\,q^{-x}}{0}\Bigm|q\,;-\frac{q^{1-n}}{a}\Big)$$

$$= (-aq^n)^{-n}\, K_n(q^x;a;q),$$
$$\eta(x) = 1 - q^x, \quad \mathcal{E}(n;\boldsymbol{\lambda}) = (q^{-n}-1)(1+aq^n). \tag{6.24}$$

The new form is obtained by using transformation formula (III.8) (with $c \to 0$) in [5].

Let us stress, however, the universal normalisation cannot be applied to those having Jackson integral measures. They have two component eigenpolynomials $P_n^{(\pm)}$ and their fixed ratios due to the self-adjointness do not allow n-dependent rescaling for each of $P_n^{(\pm)}$.

7 Summary and Comments

A brief summary of the project "orthogonal polynomials from hermitian matrices" is given in section two. In essence the project offers unified understanding of various properties of orthogonal polynomials with purely discrete orthogonality measures through the framework of eigenvalue problems of certain Jacobi matrices, to be called the Hamiltonians. It is pointed out that for the orthogonal polynomials defined on the semi-infinite integer lattice, the completeness relation or the dual orthogonality relation does not necessarily hold. In section three the failure of the orthogonality relation for the proposed dual q-Meixner and dual q-Charlier polynomials in [1] is attributed to the breakdown of the self-adjointness due to the asymptotic behaviours of the potential functions and the polynomials. This also means that the q-Meixner (q-Charlier) polynomials do not form a complete set of orthogonal vectors in the ℓ^2 Hilbert space. The self-adjointness is recovered by the introduction of another Hamiltonian system which is obtained by rescaling the sinusoidal coordinate. In section four, the two component Hamiltonian formalism with two types of polynomials and the corresponding potentials is applied to the polynomials belonging to the big q-Jacobi polynomials family. It is shown that the recovery of the self-adjointness leads naturally to the Jackson integral measures for these polynomials. The striking features of the difference equations governing the families of dual big q-Jacobi polynomials are also explored in section four. The infinite dimensional Jacobi matrix (Hamiltonian) corresponding to the dual difference equation, or the three term recurrence relation of the rescaled original polynomial, has a spectrum consisting of two infinite series sharing one accumulation point. In §4.4 the complete set of orthogonal vectors involving the q-Meixner (q-Charlier) polynomials is presented. In section five, the orthogonality of the discrete q-Hermite II polynomial is explained within the framework of the extended two component Hamiltonian formalism.

Since the system is defined on the full integer lattice $x \in \mathbb{Z}$, the corresponding dual polynomials cannot be defined by the interchange $\mathbb{Z}_{\geq 0} \ni n \leftrightarrow x \in \mathbb{Z}$. As another example of orthogonal polynomials defined on the full integer lattice, the q-Laguerre polynomial and its properties are explored in §5.1. The universal Rodrigues formulas for those having Jackson integral measures are presented in §5.2. The birth and death processes corresponding to the polynomials having Jackson integral measures, *i.e.* the big q-Jacobi family and the discrete q-Hermite II are explored in § 6.1. The corresponding stationary stochastic process has two states ($\pm$) at each site interacting with the nearest neighbours of the same type only. In § 6.2 we propose that the universal normalisation condition should be applied to four classical orthogonal polynomials, the little q-Jacobi, little q-Laguerre, Al-Salam-Carlitz II and the alternative q-Charlier (q-Bessel) polynomials. Their explicit forms are presented. By adopting the universal normalisation condition (2.26) with the corresponding simple q-hypergeometric expressions, the identification of the dual polynomial is automatic by the interchange $x \leftrightarrow n$. In Appendix, an alternative solution method based on closure relations is applied to those polynomials discussed in the main text. The Heisenberg operator solutions for the sinusoidal coordinates, the creation/annihilation operators, the self-consistent equations determining the eigenvalues, etc. are derived.

The self-adjointness of the systems having Jackson integral measures has been discussed in [7] from a different point of view.

In these days infinitely many examples of orthogonal polynomials satisfying second order differential/difference equations are known [27]–[32]. Since they have 'holes' in the degrees or the lowest degree is greater than 1, they do not satisfy the three term recurrence relations. They are not classical. It is an interesting challenge to try and construct the multi-indexed deformations of various classical orthogonal polynomials defined on the semi-infinite and full infinite integer lattices. As shown for the other classical orthogonal polynomials [32, 33, 34, 35], the method (multiple Darboux transformations) is well established. One only needs to find appropriate seed solutions.

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Appendix: Closure Relation

Here we discuss another symmetry of classical orthogonal polynomials called *closure relation*. While shape invariance leads to exact solvability in the Schrödinger picture, closure relation means exact solvability in the Heisenberg picture. That is, the Heisenberg operator solution of the sinusoidal coordinate $e^{i\mathcal{H}t}\eta(x)e^{-i\mathcal{H}t}$ can be obtained explicitly and its positive/negative frequency parts provide the annihilation/creation operators, which enable algebraic determination of all eigenvalues and eigenvectors. It has been fully discussed for the classical orthogonal polynomials of a discrete variable in I, except for i) those having Jackson integral measures, *i.e.* the big q-Jacobi family, ii) duals of the big q-Jacobi family, ii') the q-Meixner and q-Charlier, iii) the discrete q-Hermite II, and iii') the q-Laguerre.

The closure relation is the commutation relation between the Hamiltonian $\mathcal{H}$ and the sinusoidal coordinate $\eta=\eta(x)$,

$$\big[\mathcal{H},[\mathcal{H},\eta]\big]=\eta R_0(\mathcal{H})+[\mathcal{H},\eta]R_1(\mathcal{H})+R_{-1}(\mathcal{H}), \tag{A.1}$$

$$\text{or}\quad \big[\widetilde{\mathcal{H}},[\widetilde{\mathcal{H}},\eta]\big]=\eta R_0(\widetilde{\mathcal{H}})+[\widetilde{\mathcal{H}},\eta]R_1(\widetilde{\mathcal{H}})+R_{-1}(\widetilde{\mathcal{H}}), \tag{A.2}$$

in which $R_i(z)$ are polynomials in z,

$$R_1(z)=r_1^{(1)}z+r_1^{(0)},\quad R_0(z)=r_0^{(2)}z^2+r_0^{(1)}z+r_0^{(0)},\quad R_{-1}(z)=r_{-1}^{(2)}z^2+r_{-1}^{(1)}z+r_{-1}^{(0)}, \tag{A.3}$$

and $r_k^{(j)}$ are real constants. The structure of the closure relation is unchanged by an affine transformation of η, $\eta^{\text{new}}(x)=a\eta(x)+b$ (a,b: constant, $a\neq 0$),

$$\begin{aligned}&\big[\mathcal{H},[\mathcal{H},\eta^{\text{new}}]\big]=\eta^{\text{new}}R_0^{\text{new}}(\mathcal{H})+[\mathcal{H},\eta^{\text{new}}]R_1^{\text{new}}(\mathcal{H})+R_{-1}^{\text{new}}(\mathcal{H}),\\ &R_1^{\text{new}}(z)=R_1(z),\quad R_0^{\text{new}}(z)=R_0(z),\quad R_{-1}^{\text{new}}(z)=aR_{-1}(z)-bR_0(z).\end{aligned}$$

The Heisenberg operator solution for $\eta(x)$ and the creation/annihilation operator $a^{(\pm)}$ are given by [1, 14],

$$e^{it\mathcal{H}}\eta(x)e^{-it\mathcal{H}}=a^{(+)}e^{i\alpha_+(\mathcal{H})t}+a^{(-)}e^{i\alpha_-(\mathcal{H})t}-R_{-1}(\mathcal{H})R_0(\mathcal{H})^{-1}, \tag{A.4}$$

$$\alpha_\pm(z)\stackrel{\text{def}}{=}\tfrac12\big(R_1(z)\pm\sqrt{R_1(z)^2+4R_0(z)}\,\big), \tag{A.5}$$

$$R_1(z)=\alpha_+(z)+\alpha_-(z),\quad R_0(z)=-\alpha_+(z)\alpha_-(z), \tag{A.6}$$

$$\begin{aligned}a^{(\pm)}&\stackrel{\text{def}}{=}\pm\Big([\mathcal{H},\eta(x)]-\big(\eta(x)+R_{-1}(\mathcal{H})R_0(\mathcal{H})^{-1}\big)\alpha_\mp(\mathcal{H})\Big)\big(\alpha_+(\mathcal{H})-\alpha_-(\mathcal{H})\big)^{-1}\\ &=\pm\big(\alpha_+(\mathcal{H})-\alpha_-(\mathcal{H})\big)^{-1}\Big([\mathcal{H},\eta(x)]+\alpha_\pm(\mathcal{H})\big(\eta(x)+R_{-1}(\mathcal{H})R_0(\mathcal{H})^{-1}\big)\Big).\end{aligned} \tag{A.7}$$

The necessary and sufficient condition for (A.2) is (I.4.62)–(I.4.66), in which $\eta(0)=0$ is not assumed. From these conditions the following relations are derived (I.4.69), (I.4.70),

$$r_0^{(2)}=r_1^{(1)},\quad r_0^{(1)}=2r_1^{(0)}, \tag{A.8}$$

$$\eta(x+2)-(2+r_1^{(1)})\eta(x+1)+\eta(x)=r_{-1}^{(2)}. \tag{A.9}$$

The closure relation is intimately related to the three term recurrence relation. For the orthogonal polynomials $P_n(\eta)$ in I ($P_n(0)=1$, $\eta(0)=0$), the three term recurrence relation has the following form:

$$\eta P_n(\eta)=A_nP_{n+1}(\eta)-(A_n+C_n)P_n(\eta)+C_nP_{n-1}(\eta). \tag{A.10}$$

Since eigenvectors have the form $\phi_n(x)=\phi_0(x)P_n(\eta(x))$, this gives

$$\alpha_\pm\big(\mathcal{E}(n)\big)=\mathcal{E}(n\pm 1)-\mathcal{E}(n), \tag{A.11}$$

$$R_{-1}\big(\mathcal{E}(n)\big)R_0\big(\mathcal{E}(n)\big)^{-1}=A_n+C_n, \tag{A.12}$$

$$a^{(+)}\phi_n(x)=A_n\phi_{n+1}(x),\quad a^{(-)}\phi_n(x)=C_n\phi_{n-1}(x). \tag{A.13}$$

A.1 Big q-Jacobi family

The polynomials $P_n(\eta)$, (4.12) for bqJ, (4.61) for bqL and (4.78) for ASC I ($a=-1$: dqH I), satisfy $P_n(1)=1$ and the three term recurrence relation has the following form (4.16)

$$(1-\eta)P_n(\eta)=A_nP_{n+1}(\eta)-(A_n+C_n)P_n(\eta)+C_nP_{n-1}(\eta), \tag{A.14}$$

where A_n and C_n are given in (4.17)–(4.18), (4.62) and (4.79). From this form, we take the sinusoidal coordinate for the closure relation (A.1) as $\eta(x)=1-\eta^{(\pm)}(x)$. Since $B^{(\pm)}(x)$ and $D^{(\pm)}(x)$ are obtained from $B^{\mathrm{J}}(x)$ and $D^{\mathrm{J}}(x)$ as (3.32), the two Hamiltonians $\mathcal{H}^{(\pm)}$ satisfy the closure relation (A.1) with the same $R_i(z)$. We can verify (A.11), (A.12) and

$$a^{(+)}\phi_n^{(\pm)}(x)=A_n\phi_{n+1}^{(\pm)}(x),\quad a^{(-)}\phi_n^{(\pm)}(x)=C_n\phi_{n-1}^{(\pm)}(x). \tag{A.15}$$

It should be emphasised that when z is replaced by the actual spectrum $\mathcal{E}(n)$ ($\mathcal{E}(n)=(q^{-n}-1)(1-abq^{n+1})$ for bqJ, $\mathcal{E}(n)=q^{-n}-1$ for the rest), $R_1(z)^2+4R_0(z)$ in (A.5) becomes a complete square, as in all the other cases discussed in I.

The data of $R_i(x)$ are

big q-Jacobi :

$$R_1(z)=(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2(z+1+abq),$$

$$R_0(z)=(q^{\frac12}-q^{-\frac12})^2\bigl((z+1+abq)^2-(1+q)^2ab\bigr),\tag{A.16}$$
$$R_{-1}(z)=-(q^{\frac12}-q^{-\frac12})^2\Bigl(z^2+\bigl(2-(a+c-ab+ac)q\bigr)z+(1-ab)(1-aq)(1-cq)\Bigr),$$

big q-Laguerre :

$$R_1(z)=(q^{\frac12}-q^{-\frac12})^2(z+1),$$
$$R_0(z)=(q^{\frac12}-q^{-\frac12})^2(z+1)^2\quad\Bigl(\Rightarrow\sqrt{R_1(z)^2+4R_0(z)}=(q^{-1}-q)|1+z|\Bigr),\tag{A.17}$$
$$R_{-1}(z)=-(q^{\frac12}-q^{-\frac12})^2\Bigl(z^2+\bigl(2-(a+b+ab)q\bigr)z+(1-aq)(1-bq)\Bigr),$$

Al-Salam-Carlitz I :

$$R_1(z)=(q^{\frac12}-q^{-\frac12})^2(z+1),$$
$$R_0(z)=(q^{\frac12}-q^{-\frac12})^2(z+1)^2\quad\Bigl(\Rightarrow\sqrt{R_1(z)^2+4R_0(z)}=(q^{-1}-q)|1+z|\Bigr),\tag{A.18}$$
$$R_{-1}(z)=-(q^{\frac12}-q^{-\frac12})^2(z+1)(z-a).$$

A.2 Dual big q-Jacobi family

The polynomials $Q_x^{(\pm)}(\mathcal{E})$ (4.35)–(4.36) for bqJ, (4.69)–(4.70) for bqL and (4.86)–(4.87) for ASC I ($a=-1$: dqH I), satisfy $Q_x^{(\pm)}(0)=1$ and the three term recurrence relation has the following form (4.38)

$$\mathcal{E}Q_x^{(\pm)}(\mathcal{E})=-B^{(\pm)}(x)Q_{x+1}^{(\pm)}(\mathcal{E})+\bigl(B^{(\pm)}(x)+D^{(\pm)}(x)\bigr)Q_x^{(\pm)}(\mathcal{E})-D^{(\pm)}(x)Q_{x-1}^{(\pm)}(\mathcal{E}),\tag{A.19}$$

where $B^{(\pm)}(x)$ and $D^{(\pm)}(x)$ are given in (4.6)–(4.7), (4.59)–(4.60) and (4.76)–(4.77). From this form, we take the sinusoidal coordinate as $\mathcal{E}(n)$ satisfying $\mathcal{E}(0)=0$, $\mathcal{E}(n)=(q^{-n}-1)(1-abq^{n+1})$ for bqJ and $\mathcal{E}(n)=q^{-n}-1$ for the rest. The closure relation for $\mathcal{H}^{\mathrm{d}\,(+)}$ reads

$$\bigl[\mathcal{H}^{\mathrm{d}\,(+)},[\mathcal{H}^{\mathrm{d}\,(+)},\mathcal{E}(n)]\bigr]=\mathcal{E}(n)R_0\bigl(\mathcal{H}^{\mathrm{d}\,(+)}\bigr)+[\mathcal{H}^{\mathrm{d}\,(+)},\mathcal{E}(n)]R_1\bigl(\mathcal{H}^{\mathrm{d}\,(+)}\bigr)+R_{-1}\bigl(\mathcal{H}^{\mathrm{d}\,(+)}\bigr).\tag{A.20}$$

The system has two types of the eigenvectors, the ordinary $\{\phi_x^{\mathrm{d}\,(+)}(n)\}$ and the supplementary $\{\phi_x^{\mathrm{d}\,(-)}(n)\}$, (4.47), (4.72) and (4.89), with the corresponding eigenvalues $\{\mathcal{E}^{\mathrm{d}\,(+)}(x)\}$ and $\{\mathcal{E}'^{\mathrm{d}\,(+)}(x)\}$ (4.43), (4.50), (4.73) and (4.91), respectively. The latter is a monotonously decreasing function of x and it lies above the former $\mathcal{E}'^{\mathrm{d}\,(+)}(x')>\mathcal{E}^{\mathrm{d}\,(+)}(x)$. We can verify that corresponding to (A.11)–(A.13), the ordinary sector is controlled by the data of the closure relation as follows:

$$\alpha_\pm\bigl(\mathcal{E}^{\mathrm{d}\,(+)}(x)\bigr)=\mathcal{E}^{\mathrm{d}\,(+)}(x\pm1)-\mathcal{E}^{\mathrm{d}\,(+)}(x),\tag{A.21}$$

$$R_{-1}\big(\mathcal{E}^{\mathrm{d}\,(+)}(x)\big)R_0\big(\mathcal{E}^{\mathrm{d}\,(+)}(x)\big)^{-1}=-B^{(+)}(x)-D^{(+)}(x), \tag{A.22}$$

$$a^{(+)}\phi_x^{\mathrm{d}\,(+)}(n)=-B^{(+)}(x)\phi_{x+1}^{\mathrm{d}\,(+)}(n),\quad a^{(-)}\phi_x^{\mathrm{d}\,(+)}(n)=-D^{(+)}(x)\phi_{x-1}^{\mathrm{d}\,(+)}(n), \tag{A.23}$$

whereas for the supplementary sector, the order of x is reversed:

$$\alpha_\pm\big(\mathcal{E}'^{\,\mathrm{d}\,(+)}(x)\big)=\mathcal{E}'^{\,\mathrm{d}\,(+)}(x\mp 1)-\mathcal{E}'^{\,\mathrm{d}\,(+)}(x), \tag{A.24}$$

$$R_{-1}\big(\mathcal{E}'^{\,\mathrm{d}\,(+)}(x)\big)R_0\big(\mathcal{E}'^{\,\mathrm{d}\,(+)}(x)\big)^{-1}=-B^{(-)}(x)-D^{(-)}(x), \tag{A.25}$$

$$a^{(+)}\phi_x^{\mathrm{d}\,(-)}(n)=-D^{(-)}(x)\phi_{x-1}^{\mathrm{d}\,(-)}(n),\quad a^{(-)}\phi_x^{\mathrm{d}\,(-)}(n)=-B^{(-)}(x)\phi_{x+1}^{\mathrm{d}\,(-)}(n). \tag{A.26}$$

The data of $R_i(x)$ are

dual big q-Jacobi :

$$\begin{aligned}
R_1(z)&=(q^{\frac12}-q^{-\frac12})^2(z-aq),\\
R_0(z)&=(q^{\frac12}-q^{-\frac12})^2(z-aq)^2\quad\Big(\Rightarrow\sqrt{R_1(z)^2+4R_0(z)}=(q^{-1}-q)|aq-z|\Big),\\
R_{-1}(z)&=(q^{\frac12}-q^{-\frac12})^2\big((1+abq)z^2+aq(b+c+a^{-1}c-1-2abq)z\\
&\qquad\qquad+acq(1-aq)(1-abc^{-1}q)\big),
\end{aligned} \tag{A.27}$$

dual big q-Laguerre :

$$\begin{aligned}
R_1(z)&=(q^{\frac12}-q^{-\frac12})^2(z-aq),\\
R_0(z)&=(q^{\frac12}-q^{-\frac12})^2(z-aq)^2\quad\Big(\Rightarrow\sqrt{R_1(z)^2+4R_0(z)}=(q^{-1}-q)|aq-z|\Big),\\
R_{-1}(z)&=(q^{\frac12}-q^{-\frac12})^2\big(z^2+aq(b+a^{-1}b-1)z+abq(1-aq)\big),
\end{aligned} \tag{A.28}$$

dual Al-Salam-Carlitz I :

$$\begin{aligned}
R_1(z)&=(q^{\frac12}-q^{-\frac12})^2(z-1),\\
R_0(z)&=(q^{\frac12}-q^{-\frac12})^2(z-1)^2\quad\Big(\Rightarrow\sqrt{R_1(z)^2+4R_0(z)}=(q^{-1}-q)|1-z|\Big),\\
R_{-1}(z)&=(q^{\frac12}-q^{-\frac12})^2\big(z^2+(a-1)z+aq^{-1}\big).
\end{aligned} \tag{A.29}$$

A.3 q-Meixner and q-Charlier

The polynomials $P_n(\eta)$ (3.4) for qM and (3.42) for qC, satisfy $P_n(0)=1$ and the three term recurrence relation has the following form

$$\eta P_n(\eta)=A_nP_{n+1}(\eta)-(A_n+C_n)P_n(\eta)+C_nP_{n-1}(\eta), \tag{A.30}$$

where A_n and C_n are given in (3.7) and (3.45). From this form, we adopt the sinusoidal coordinate for the closure relation (A.1) as $\eta(x) = q^{-x} - 1$, which satisfies $\eta(0) = 0$. This situation is the same as I.

The system has two types of eigenvectors, the ordinary $\{\phi_n(x)\}$ and the supplementary $\{\phi_n^{(-)}(x)\}$ (4.95)–(4.96) and (4.100)–(4.101), with the corresponding eigenvalues $\{\mathcal{E}(n) = 1 - q^n\}$ and $\{\mathcal{E}'(n)\}$ (4.97) and (4.103), respectively. The latter is a monotonously decreasing function of n and it lies above the former $\mathcal{E}'(n') > \mathcal{E}(n)$. We can confirm that (A.11)–(A.13) hold for the ordinary sector, whereas for the supplementary sector the order of n is reversed:

$$\alpha_\pm\bigl(\mathcal{E}'(n)\bigr) = \mathcal{E}'(n \mp 1) - \mathcal{E}'(n), \tag{A.31}$$

$$R_{-1}\bigl(\mathcal{E}'(n)\bigr)R_0\bigl(\mathcal{E}'(n)\bigr)^{-1} = A_n^{(-)} + C_n^{(-)}, \tag{A.32}$$

$$a^{(+)}\phi_n^{(-)}(x) = C_n^{(-)}\phi_{n-1}^{(-)}(x), \quad a^{(-)}\phi_n^{(-)}(x) = A_n^{(-)}\phi_{n+1}^{(-)}(x), \tag{A.33}$$

where $A_n^{(-)}$ and $C_n^{(-)}$ are given in (4.94) for qM and (3.45) with the replacement $a \to a^{-1}$ for qC.

The data of $R_i(x)$ are

q-Meixner :

$$\begin{aligned}
R_1(z) &= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2(z-1),\\
R_0(z) &= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2(z-1)^2 \quad \Bigl(\Rightarrow \sqrt{R_1(z)^2 + 4R_0(z)} = (q^{-1} - q)|1-z|\Bigr),\\
R_{-1}(z) &= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2\bigl(z^2 - (1 + bc + c)z + bc - q^{-1}c\bigr),
\end{aligned} \tag{A.34}$$

q-Charlier :

$$\begin{aligned}
R_1(z) &= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2(z-1),\\
R_0(z) &= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2(z-1)^2 \quad \Bigl(\Rightarrow \sqrt{R_1(z)^2 + 4R_0(z)} = (q^{-1} - q)|1-z|\Bigr),\\
R_{-1}(z) &= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2\bigl(z^2 - (1 + a)z - q^{-1}a\bigr).
\end{aligned} \tag{A.35}$$

A.4 Discrete q-Hermite II

The polynomials $P_n(\eta)$ (5.4) satisfy the three term recurrence relation of the following form

$$\eta P_n(\eta) = A_n P_{n+1}(\eta) + B_n P_n(\eta) + C_n P_{n-1}(\eta), \quad A_n = q^{-n}, \quad B_n = 0, \quad C_n = q^{-n} - 1. \tag{A.36}$$

From this form, we adopt the sinusoidal coordinate for the closure relation (A.1) as $\eta(x) = \eta^{(\pm)}(x) = \pm c q^x$. The data of $R_i(z)$ do not depend on $(\pm)$:

$$R_1(z) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2(z-1),$$

$$R_0(z) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2(z-1)^2 \quad \Bigl(\Rightarrow \sqrt{R_1(z)^2 + 4R_0(z)} = (q^{-1} - q)|1-z|\Bigr), \tag{A.37}$$
$$R_{-1}(z) = 0.$$

The following relations can be easily verified:

$$\alpha_\pm\bigl(\mathcal{E}(n)\bigr) = \mathcal{E}(n \pm 1) - \mathcal{E}(n), \tag{A.38}$$
$$R_{-1}\bigl(\mathcal{E}(n)\bigr)R_0\bigl(\mathcal{E}(n)\bigr)^{-1} = B_n = 0, \tag{A.39}$$
$$a^{(+)}\phi_n^{(\pm)}(x) = A_n\phi_{n+1}^{(\pm)}(x), \quad a^{(-)}\phi_n^{(\pm)}(x) = C_n\phi_{n-1}^{(\pm)}(x). \tag{A.40}$$

A.5 q-Laguerre

From the recurrence relation (5.33), we adopt as the sinusoidal coordinate for the closure relation $1 + \eta(x; \boldsymbol{\lambda})$. The data $R_i(z)$ are

$$R_1(z) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2(z-1),$$
$$R_0(z) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2(z-1)^2 \quad \Bigl(\Rightarrow \sqrt{R_1(z)^2 + 4R_0(z)} = (q^{-1} - q)|1-z|\Bigr), \tag{A.41}$$
$$R_{-1}(z) = -(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2\bigl(z^2 + (a^{-1} - 1)z + a^{-1}q^{-1}\bigr).$$

It is straightforward to verify the fundamental relations (A.11)–(A.13).

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