INDECOMPOSABLE DECOMPOSITIONS OF MODULAR STANDARD MODULES FOR TWO FAMILIES OF ASSOCIATION SCHEMES

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ABSTRACT. We investigate the indecomposable decomposition of the modular standard modules of two families of association schemes of finite order. First, we show that, for each prime number p, the standard module over a field F of characteristic p of a residually thin scheme S of p-power order is an indecomposable FS-module. Second, we describe the indecomposable decomposition of the standard module over a field of positive characteristic of a wreath product of finitely may association schemes of rank 2.

1. INTRODUCTION

It is easy to see that algebraically isomorphic association schemes have isomorphic standard modules over algebraically closed fields of characteristic 0. In [3], the first author considered modular adjacency algebras and standard modules of cyclotomic association schemes, and gave direct sum decompositions of the standard modules. He determined indecomposable decompositions only for the case where the representation types were finite, or tame and the dimensions of algebras were small. In general, it is hard to describe indecomposability of a given module, especially if the representation type of the algebra is wild. For representation types, see [6], for example.

In this article, we provide indecomposable decompositions of modular standard modules for two families of association schemes. Let F be a field of positive characteristic p. In Section 4, we consider residually thin schemes of p-power order (called *p-schemes* in [8]). In this case, the standard modules are indecomposable (Theorem 4.2). To prove this, we consider multiple wreath products of thin schemes given by the cyclic groups of order p and show that their standard modules are indecomposable. Since residually thin schemes are fissions of such wreath products, we see that their standard modules are indecomposable. In Section 5, we consider multiple wreath products of schemes of rank 2. This is one of the simplest examples of association schemes but the structure of their standard module is not always easy to describe. The adjacency algebra is isomorphic to

$$F \oplus \cdots \oplus F \oplus F[u_1, \ldots, u_n]/(u_i u_j \mid 1 \leq i, j \leq n).$$

Therefore the representation type of the algebra is finite if $n = 0, 1$, tame if $n = 2$, and wild if $n \geq 3$. We completely determine indecomposable decompositions of

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2010 *Mathematics Subject Classification.* Primary (05E30); Secondary (16G10).

Key words and phrases. association scheme; wreath product; modular adjacency algebra; modular standard module; indecomposable decomposition.

standard modules for them (Theorem 5.1). This is the first result on indecomposable decompositions for wild adjacency algebras of association schemes.

In Section 2, we give the definition of an association scheme. In Section 3, we define an algebra V and its subalgebra W and show that V is indecomposable as a W-module (Proposition 3.3). This is a key result in this article.

Throughout this article, we denote by I_n the identity matrix of degree n, and by J_n the $n \times n$ matrix all of whose entries are one.

2. Association schemes

In this section, we will give necessary definitions for association schemes and their representations. For details, see [1, 8].

Let X be a finite set, and let S be a collection of non-empty subsets of $X \times X$. We call (X, S) an *association scheme* or a *scheme* if the following conditions hold:

(1) $X \times X = \bigcup_{s \in S} s$ and $s \cap s' = \emptyset$ for $s \neq s'$, $s, s' \in S$,
(2) $1_X = \{(x, x) \mid x \in X\} \in S$,
(3) $s^* = \{(y, x) \mid (x, y) \in s\} \in S$ for $s \in S$, and
(4) for $s, t, u \in S$, there is an integer p_{st}^u such that $\sharp\{z \in X \mid (x, z) \in s,\ (z, y) \in t\} = p_{st}^u$ whenever $(x, y) \in u$.

In this case, we also say that S is an association scheme. The number $p_{ss^*}^{1_X}$ is called the *valency* of $s \in S$ and denoted by n_s. We call the number $|X|$ the *order* of the association scheme (X, S). We denote by $M_X(R)$ the full matrix algebra over a commutative ring R with unity, where both rows and columns of whose matrices are indexed by the set X. For $s \subset X \times X$, we define the *adjacency matrix* $A_s \in M_X(\mathbb{Z})$ by $(A_s)_{xy} = 1$ if $(x, y) \in s$ and $(A_s)_{xy} = 0$ otherwise. By definition, $\mathbb{Z}S = \bigoplus_{s \in S} \mathbb{Z}A_s$ is a subalgebra of $M_X(\mathbb{Z})$. Let R be a commutative ring with unity. We define an R-subalgebra $RS = R \otimes_{\mathbb{Z}} \mathbb{Z}S$ of $M_X(R)$ and call it the *adjacency algebra* of (X, S) over R. We also write $R\mathfrak{X}$ for the adjacency algebra of $\mathfrak{X} = (X, S)$. A representation of (X, S) over R is an R-algebra homomorphism from RS to $M_n(R)$ for some degree n. Since RS is defined as a subalgebra of $M_X(R)$, the inclusion map is a representation and we call it the *standard representation* of (X, S) over R. The corresponding RS-module is called the *standard module* of (X, S) over R. The standard module has a natural basis X, so we denote it by RX.

A subset T of S is said to be *closed* if $p_{st}^u = 0$ for all $s, t \in T$ and $u \notin T$. A closed subset defines subschemes and the factor scheme. An association scheme (X, S) is said to be *thin* if $n_s = 1$ for all $s \in S$. A thin association schemes are obtained by a regular permutation representation of a finite group. Thus a thin scheme is essentially a finite group. The *thin residue* $\mathbf{O}^\theta(S)$ is the smallest closed subset of S such that the factor scheme $S//\mathbf{O}^\theta(S)$ is thin. Define $(\mathbf{O}^\theta)^n(S)$ by $\mathbf{O}^\theta((\mathbf{O}^\theta)^{n-1}(S))$ inductively. An association scheme (X, S) is said to be *residually thin* if $(\mathbf{O}^\theta)^n(S) = 1$ for some n. For details, see [8].

Let $\mathfrak{X} = (X, S)$ and $\mathfrak{Y} = (X, T)$ be association schemes on a common underlying set X. When every $t \in T$ is a union of some subset of S, we say that $\mathfrak{Y}$ is a *fusion* of $\mathfrak{X}$ and $\mathfrak{X}$ is a *fission* of $\mathfrak{Y}$. In this case, the adjacency algebra RT is a subalgebra of RS.

Let $\mathfrak{X} = (X, S)$ and $\mathfrak{Y} = (Y, T)$ be association schemes with adjacency matrices $\{A_i\}_{i=0}^d$ and $\{A'_i\}_{i=0}^f$, respectively. We suppose that A_0 and A'_0 are identity matrices. For $s \in S$ and $t \in T$, we set

$$s \times t = \{((x, y), (x', y')) \mid (x, x') \in s,\ (y, y') \in t\} \subset (X \times Y) \times (X \times Y)$$

and

$$S \times T = \{s \times t \mid s \in S,\ t \in T\}.$$

Then $(X \times Y, S \times T)$ is an association scheme, called the *direct product* of $\mathfrak{X}$ and $\mathfrak{Y}$, and denoted by $\mathfrak{X} \times \mathfrak{Y}$. The adjacency matrices of $\mathfrak{X} \times \mathfrak{Y}$ are $(d+1)(f+1)$ matrices:

$$A_0 \otimes A'_0,\ \ldots,\ A_d \otimes A'_f.$$

For $s \in S$, we put

$$\tilde{s} = \{((x, y), (x', y)) \mid (x, x') \in s,\ y \in Y\} \subset (X \times Y) \times (X \times Y).$$

For $t \in T$, we put

$$\tilde{t} = \{((x, y), (x', y')) \mid x, x' \in X,\ (y, y') \in t\} \subset (X \times Y) \times (X \times Y).$$

Also we put

$$S \wr T = \{\tilde{s} \mid s \in S\} \cup \{\tilde{t} \mid t \in T \backslash \{1_Y\}\}.$$

Then $(X \times Y, S \wr T)$ is an association scheme, called the *wreath product* of $\mathfrak{X}$ and $\mathfrak{Y}$, and denoted by $\mathfrak{X} \wr \mathfrak{Y}$. The adjacency matrices of $\mathfrak{X} \wr \mathfrak{Y}$ are $d + f + 1$ matrices:

$$A_0 \otimes I_{|Y|},\ \ldots,\ A_d \otimes I_{|Y|},\ J_{|X|} \otimes A'_1,\ \ldots,\ J_{|X|} \otimes A'_f.$$

For irreducible representations of wreath products, see [4].

It is clear that the wreath product $\mathfrak{X} \wr \mathfrak{Y}$ is a fusion scheme of $\mathfrak{X} \times \mathfrak{Y}$. This means that $R(S \wr T)$ is a subalgebra of $R(S \times T)$.

Let $\mathfrak{X} = (X, S)$ be an association scheme. The cardinality of S is called the *rank* of $\mathfrak{X}$. For any finite set X with $|X| \geq 2$, there is a unique association scheme on X of rank 2, namely it is $(X, \{1_X,\ (X \times X) \setminus 1_X\})$.

Let $\mathfrak{X} = (X, S)$ be an association scheme with a closed subset T. A subscheme $\mathfrak{Y}_x = (X, S)_{xT}$ (see [8, §1.5] for definition) depends on $x \in X$. Subschemes are algebraically isomorphic but not necessarily isomorphic. We will give an easy lemma without a proof.

Lemma 2.1. *Let $\mathfrak{X} = (X, S)$ be an association scheme with a closed subset T. Suppose that all subschemes $\mathfrak{Y}_x$ $(x \in X)$ of $\mathfrak{X}$ defined by T are isomorphic. Then $\mathfrak{X}$ is a fission scheme of the wreath product $\mathfrak{Y}_x \wr (S//T)$.*

3. Indecomposability of a module

In this section, for $n_i \geq 2$ $(i = 1, \ldots, r)$, we define an algebra $V = V(n_1 \ldots, n_r)$ and a subalgebra $W = W(n_1, \ldots, n_r)$ and prove that V is indecomposable as a W-module in Proposition 3.3.

Let F be a field and $n_1, \ldots, n_r$ be integers which are greater than or equal to 2. We define a finite dimensional F-algebra V by

$$V = V(n_1, \ldots, n_r) = F[t_1, \ldots, t_r]/(t_1^{n_1}, \ldots, t_r^{n_r}).$$

We write t_i in the factor ring V by the same letter t_i. The set

$$\mathbf{B} = \{t_1^{e_1} \ldots t_r^{e_r} \mid 0 \leq e_i < n_i \quad (1 \leq i \leq r)\}$$

is a basis of V. We use the lexicographical order in sequences of powers for the basis and write a base $\boldsymbol{t}^e = t_1^{e_1} \dots t_r^{e_r}$ for $e = (e_1, \dots, e_r)$. Let us put $V_{>e} = \bigoplus_{f>e} F\boldsymbol{t}^f$. For any element α of V, we can write uniquely as

$$\alpha = c_e\boldsymbol{t}^e + \beta, \quad (c_e \in F \setminus \{0\},\ \beta \in V_{>e}).$$

We consider a subalgebra $W = W(n_1, \dots, n_r)$ of V with basis

$$\mathbf{B}' = \{1\} \cup \left(\bigcup_{i=1}^{r} \{t_1^{n_1-1} \dots t_{i-1}^{n_{i-1}-1} t_i^{e_i} \mid 1 \leq e_i \leq n_i - 1\} \right).$$

We can consider V as a right W-module.

By considering products of elements in $\mathbf{B}'$, easily we have the structure of W, that is

$$W = F[u_0, u_1, \dots, u_\ell]/\mathcal{I},$$

where $\ell = \sum_{i=2}^{r}(n_i - 1)$ and $\mathcal{I}$ is the ideal generated by

$$\{u_0^{n_1}\} \cup \{u_i^2 \mid 1 \leq i \leq \ell\} \cup \{u_i u_j \mid 0 \leq i < j \leq \ell\}.$$

We remark that the algebra W has wild representation type when $\ell \geq 2$ (see [6], for example), and representation theory is difficult in this case.

We show that the endomorphism algebra $\mathrm{End}_W(V)$ is a local algebra so that we prove V_W is indecomposable.

Lemma 3.1. *For $\varphi \in \mathrm{End}_W(V)$ and $0 \neq \alpha \in \mathrm{Ker}\,\varphi$, put $\alpha = c_e\boldsymbol{t}^e + \beta$ ($c_e \in F \setminus \{0\},\ \beta \in V_{>e}$), we suppose that $e < (n_1 - 1, \dots, n_r - 1)$. Then there exist a sequence f and $0 \neq \alpha' = c_f\boldsymbol{t}^f + \gamma$ ($c_f \in F \setminus \{0\},\ \gamma \in V_{>f}$) such that $f > e$ and $\alpha' \in \mathrm{Ker}\,\varphi$.*

Proof. We put $e = (n_1 - 1, \dots, n_{i-1} - 1, e_i, \dots, e_r)$, where $e_i < n_i - 1$. We remark that $t_1^{n_1-1} \dots t_{i-1}^{n_{i-1}-1} t_i^{e_i} \in \mathbf{B}'$ is a common factor of each term of β. Since $\alpha \in \mathrm{Ker}\,\varphi$,

$$0 = \varphi(\alpha) = \varphi(c_e t_{i+1}^{e_{i+1}} \dots t_r^{e_r} + \beta') t_1^{n_1-1} \dots t_{i-1}^{n_{i-1}-1} t_i^{e_i}.$$

Multiplying both sides of this equation by $t_i^{n_i-e_i-1}$, since $t_1^{n_1-1} \dots t_{i-1}^{n_{i-1}-1} t_i^{n_i-1} \in \mathbf{B}'$, we have

$$0 = \varphi(c_e t_{i+1}^{e_{i+1}} \dots t_r^{e_r} + \beta') t_1^{n_1-1} \dots t_{i-1}^{n_{i-1}-1} t_i^{n_i-1} = \varphi(c_e\boldsymbol{t}^e t_i^{n_i-e_i-1} + \beta t_i^{n_i-e_i-1}).$$

If we put $\alpha' = c_e\boldsymbol{t}^e t_i^{n_i-e_i-1} + \beta t_i^{n_i-e_i-1}$, then the conditions are satisfied. □

Lemma 3.2. *For $\varphi \in \mathrm{End}_W(V)$, φ is an isomorphism if and only if $\varphi(\boldsymbol{t}^{(n_1-1,\dots,n_r-1)}) \neq 0$.*

Proof. If φ is an isomorphism, then it is clear that $\varphi(\boldsymbol{t}^{(n_1-1,\dots,n_r-1)}) \neq 0$. We assume that φ is not an isomorphism. This means that $\mathrm{Ker}\,\varphi \neq 0$. By repeatedly using Lemma 3.1, we can conclude that $\boldsymbol{t}^{(n_1-1,\dots,n_r-1)} \in \mathrm{Ker}\,\varphi$. □

Now we can prove the main result in this section.

Proposition 3.3. *The endomorphism algebra $\mathrm{End}_W(V)$ is a local algebra. Namely, V is indecomposable as a right W-module.*

Proof. If φ is a non-isomorphism and $\psi \in \mathrm{End}_W(V)$, then clearly $\varphi\psi$ and $\psi\varphi$ are non-isomorphisms. Suppose that both φ and ψ are non-isomorphisms. Then, by Lemma 3.2, $\varphi + \psi$ is not an isomorphism. Hence the set of all non-isomorphisms in $\mathrm{End}_W(V)$ is an ideal of $\mathrm{End}_W(V)$, and thus $\mathrm{End}_W(V)$ is local by [5, I, §5]. □

We will give a matrix form of Proposition 3.3. We consider a regular representation of V. Put N_n the $n \times n$ matrix of the form

$$N_n = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}.$$

Then $\Phi : t_1^{e_1} \dots t_r^{e_r} \mapsto N_{n_1}^{e_1} \otimes \cdots \otimes N_{n_r}^{e_r}$ is a regular representation of V.

Proposition 3.4. *The restriction Φ to W is an indecomposable representation of W.*

4. Residually thin schemes of prime power order

Let F be a field of characteristic p, and let (X, S) be a residually thin association scheme of p-power order, a p-scheme in [8]. In this section, we will show that the standard module FX is an indecomposable FS-module. Thus the indecomposable decomposition of the standard module FX is determined for this case.

We need one lemma.

Lemma 4.1. *Let F be a field of characteristic p, and let $\mathfrak{C}_p$ be the association scheme given by the regular permutation representation of the cyclic group order p. Then the standard module of the wreath product $\mathfrak{C}_p \wr \cdots \wr \mathfrak{C}_p$ over F is indecomposable.*

Proof. We can write $\mathfrak{C}_p = (C_p, C_p)$, where C_p is the cyclic group of order p. We know that $F\mathfrak{C}_p \cong F[t]/(t^p)$ and the all-one matrix J_p corresponds to t^{p-1}. The standard module of $\mathfrak{C}_p \wr \cdots \wr \mathfrak{C}_p$ (r-times) is $F(C_p \times \cdots \times C_p)$. The module $F(C_p \times \cdots \times C_p)$ has a natural algebra structure and it is easy to see that

$$\begin{aligned} F(C_p \times \cdots \times C_p) &\cong FC_p \otimes \cdots \otimes FC_p \\ &\cong F[t]/(t^p) \otimes \cdots \otimes F[t]/(t^p) \\ &\cong F[t_1, \dots, t_r]/(t_1^p, \dots, t_r^p) \\ &= V(p, \dots, p). \end{aligned}$$

Now the adjacency algebra $F(\mathfrak{C}_p \wr \cdots \wr \mathfrak{C}_p)$ is a subalgebra of $F(C_p \times \cdots \times C_p)$, and since J_p corresponds t^{p-1}, we have $F(\mathfrak{C}_p \wr \cdots \wr \mathfrak{C}_p) \cong W(p, \dots, p)$. Thus the standard module is indecomposable by Proposition 3.3. □

Now we show one of the main results in this article.

Theorem 4.2. *Let F be a field of characteristic p, and let (X, S) be a residually thin association scheme of p-power order. Then the standard module FX is an indecomposable FS-module.*

Proof. By definition, there is a series of closed subsets

$$S = S_0 \supset S_1 \supset \cdots \supset S_r = 1$$

such that $S_{i-1}//S_i \cong C_p$ for $i = 1, \dots, r$. Therefore, (X, S) is a fission scheme of $\mathfrak{C}_p \wr \cdots \wr \mathfrak{C}_p$ by Lemma 2.1 and the adjacency algebra $F(\mathfrak{C}_p \wr \cdots \wr \mathfrak{C}_p)$ is a subalgebra of FS. We can assume that the standard modules FX are common. The standard module is indecomposable as an $F(\mathfrak{C}_p \wr \cdots \wr \mathfrak{C}_p)$-module by Lemma 4.1, and thus it is indecomposable as an FS-module. □

We give an easy application as a corollary to Theorem 4.2, though it is not so hard to prove it by combinatorial argument [8, Corollary 2.4.7]. An association scheme (X, S) is said to be *p'-valenced* if $p \nmid n_s$ for all $s \in S$.

Corollary 4.3. *Let (X, S) be a p'-valenced residually thin association scheme of p-power order. Then (X, S) is thin.*

Proof. We fix $x \in X$. The map $f : FS \to FX$, $f(A_s) = xA_s$ is an FS-module monomorphism. Since (X, S) is p'-valenced, FS is a symmetric algebra by [2, Corollary 4.3] and thus FS is an injective FS-module. Therefore the map f splits, namely FS is isomorphic to a direct summand of FX. However, by Theorem 4.2, FX is indecomposable. This is possible only if $|S| = |X|$, namely (X, S) is thin. □

5. Wreath product of schemes of rank 2

Let $X_1, \dots, X_r$ be finite sets. For each element $i \in \{1, ..., r\}$, let S_i be an association scheme of order q_i (≥ 2) and rank 2 on X_i . Define $X = X_1 \times \cdots \times X_r$ and $S = S_1 \wr \cdots \wr S_r$. Then S is an association scheme on X. Association schemes of this type have been characterized in [7, Theorem A]. The adjacency matrices are described as follows

$$A_i = J_{q_1} \otimes \cdots \otimes J_{q_{i-1}} \otimes (J_{q_i} - I_{q_i}) \otimes I_{q_{i+1}} \otimes \cdots \otimes I_{q_r} \quad (i = 0, 1, \dots, r).$$

We replace the basis $\{A_i \mid i = 0, 1, \dots, r\}$ of the adjacency algebra with $\{B_i \mid i = 0, 1, \dots, r\}$ where

$$B_i = \sum_{j=0}^{i} A_j = J_{q_1} \otimes \cdots \otimes J_{q_i} \otimes I_{q_{i+1}} \otimes \cdots \otimes I_{q_r} \quad (i = 0, 1, \dots, r).$$

Let F be a field of characteristic p. The adjacency algebra is a subalgebra of $M_{X_1}(F) \otimes \cdots \otimes M_{X_r}(F)$ with basis $\{B_i \mid i = 0, 1, \dots, r\}$ and the standard module is $F(X_1 \times \cdots \times X_r) \cong FX_1 \otimes \cdots \otimes FX_r$.

The space FX_i has a natural basis $X_i = \{x_1, \dots, x_{q_i}\}$. For each $i \in \{1, \dots, r\}$, we replace this basis with $\{y_1, \dots, y_{q_i}\}$ where

$$y_1 = x_1, \quad y_2 = \sum_{j=1}^{q_i} x_j, \quad y_k = x_k - x_1 \quad (k = 3, 4, \dots, q_i),$$

if $p \mid q_i$, and

$$y_1 = \sum_{j=1}^{q_i} x_j, \quad y_k = x_k - x_1 \quad (k = 2, 3, \cdots, q_i).$$

if $p \nmid q_i$. The representing matrix of J_{q_i} with respect to the basis $\{y_1, \dots, y_{q_i}\}$ is

$$\begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & \cdots & \cdots & \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \text{ if } p \mid q_i, \text{ and } \begin{pmatrix} q_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & \cdots & \cdots & \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \text{ if } p \nmid q_i.$$

We set

- $\Delta = \{i \mid 1 \leq i \leq r, \ p \mid q_i\}$,
- $T_i = \{1, 2\}$ if $i \in \Delta$ and $T_i = \{1\}$ if $i \notin \Delta$,
- $U_i = \bigoplus_{j \in T_i} Fy_j$, and
- $\delta(k) = |\Delta \cap \{1, \dots, k\}|$.

Now we show the main result in this section.

Theorem 5.1. *Let (X_i, S_i), $i = 1, 2, \dots, r$, be association schemes of rank 2. We have an indecomposable decomposition of the standard module of $(X, S) = (X_1 \times \cdots \times X_r,\ S_1 \wr \cdots \wr S_r)$:*

$$FX = FX_1 \otimes \cdots \otimes FX_r = \bigoplus_{i=0}^{r} \bigoplus U_1 \otimes \cdots \otimes U_i \otimes Fy_{\ell_{i+1}} \otimes \cdots \otimes Fy_{\ell_r},$$

where the second direct sum runs over all $(\ell_{i+1}, \dots, \ell_r)$ such that $\ell_{i+1} \notin T_{i+1}$ and $1 \le \ell_k \le q_k$ $(k = i+2, \dots, r)$. Moreover, the indecomposable direct summands $U_1 \otimes \cdots \otimes U_i \otimes Fy_{\ell_{i+1}} \otimes \cdots \otimes Fy_{\ell_r}$ and $U_1 \otimes \cdots \otimes U_{i'} \otimes Fy_{\ell'_{i'+1}} \otimes \cdots \otimes Fy_{\ell'_r}$ are isomorphic as FS-modules if and only if $i = i'$.

Proof. It is easy to see that the sum is direct and every direct summand is an FS-submodule of FX.

We consider a direct summand $U = U_1 \otimes \cdots \otimes U_i \otimes Fy_{\ell_{i+1}} \otimes \cdots \otimes Fy_{\ell_r}$. We set

$$M_k = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ if } k \in \Delta, \text{ and } M_k = \begin{pmatrix} q_k \end{pmatrix} \text{ if } k \notin \Delta.$$

Then the action of B_j on U is

$$M_1 \otimes \cdots \otimes M_j \otimes I \otimes \cdots \otimes I$$

for $0 \le j \le i$, where I is the identity matrix of degree 1 or 2, and 0 for $j > i$. By the action of $\{B_i\}$ on the summands, we can characterize isomorphic direct summands. We need to show that U is indecomposable. We consider a subalgebra W of FS generated by $\{B_j \mid 1 \le j \le i,\ j \in \Delta\}$. We can see that $W \cong W(2, \dots, 2)$ ($\delta(i)$-times), defined in section 3, and the action of W on U is just the representation in Proposition 3.4 up to non-zero scalar factors. Thus FX is an indecomposable W-module, and so is an indecomposable FS-module. □

We set

$$V_i = U_1 \otimes \cdots \otimes U_i \otimes Fy_{\ell_{i+1}} \otimes \cdots \otimes Fy_{\ell_r}$$

where $\ell_{i+1} \notin T_{i+1}$. The dimension of V_i is $2^{\delta(i)}$ and V_i appears $(q_{i+1} - \mu) \prod_{k=i+2}^{r} q_k$ times in FX, where $\mu = 2$ if $p \mid q_{i+1}$ and $\mu = 1$ otherwise.

If $p \nmid q_j$ for $1 \le j \le m-1$ and $p \mid q_m$, then $\{V_0, \cdots, V_m\}$ is the complete set of isomorphism classes of simple FS-modules, V_k is not a simple FS-module if $k > m$, and all composition factors of V_k are V_m if $k > m$.

The structure of the adjacency algebra FS is

$$FS \cong \underbrace{F \oplus \cdots \oplus F}_{m\text{-times}} \oplus F[u_1, \dots, u_{r-m}]/(u_i u_j \mid 1 \le i, j \le r-m).$$

Therefore the representation type of FS is finite if $0 \le r - m \le 1$, tame if $r - m = 2$, and wild if $r - m \ge 3$.

6. Acknowledgment

The authors would like to thank the anonymous referees for their helpful comments. This work was supported by JSPS KAKENHI Grant Number JP25400011.

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