Simplicity of p-blocks of modular adjacency algebras of association schemes

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Abstract

A criterion is given for blocks of modular adjacency algebras of association schemes to be simple.

1 Introduction

In [4], the author proved the following theorem.

Theorem 1.1. *Let (X, S) be an association scheme, and let F be a field of characteristic p. Then the adjacency algebra FS is semisimple if and only if p does not divide the Frame number $\mathcal{F}(S)$.*

For the definition of the Frame number, see §2.4.

The notion of an association scheme generalizes the notion of a finite group, so that Theorem 1.1 is a straightforward generalization of Maschke's theorem on finite groups [7, III. Theorem 1.22].

Example 1.2. Let (X, S) be a thin association scheme defined by the symmetric group $\mathfrak{S}_3$ of degree 3. Then the Frame number of S is

$$\mathcal{F}(S) = 6^6 \times \frac{1 \times 1 \times 1 \times 1 \times 1 \times 1}{1 \times 1 \times 2^4} = 2^2 \times 3^6 = 2916.$$

Thus the adjacency algebra FS is semisimple if and only if $p \neq 2, 3$. However, in this case, the adjacency algebra is just the group algebra. Therefore, it is characterized by the group order $|\mathfrak{S}_3| = 6$ in Maschke's theorem.

For a finite group G and a field F of characteristic p, we know that

$$\begin{aligned} p \nmid |G| &\iff \text{the principal block of } FG \text{ is simple} \\ &\iff FG \text{ is semisimple.} \end{aligned}$$

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For an association scheme (X, S), we still have

$$p \nmid |X| \iff \text{the principal block of } FS \text{ is simple}$$

but these two conditions are not equivalent to the fact that FS is semisimple. However, this equivalence can also be considered as a generalization of Maschke's theorem, a generalization which is different from the generalization provided by Theorem 1.1.

In this article, we will divide the Frame number to blocks and give a criterion for blocks to be simple. In §3, we define a matrix T_B for each block B of FS such that the determinant of T_B decides on the simplicity of the block B (Theorem 3.2). Also we will see that the product of $(\det T_B)^2$ for all blocks B is almost equal to the Frame number, and that provides us also with a new proof of Theorem 1.1.

2 Preliminaries

2.1 Association schemes and adjacency algebras

Let X be a finite set, and let S be a partition of $X \times X$. The pair (X, S) is said to be an *association scheme* if

(1) $1 = \{(x, x) \mid x \in X\} \in S$,

(2) for every $s \in S$, $s^* = \{(y, x) \mid (x, y) \in s\} \in S$,

(3) for every triple $(s, t, u) \in S^3$, there is a non-negative integer p_{st}^u such that $\sharp\{z \in X \mid (x, z) \in s,\ (z, y) \in t\} = p_{st}^u$ when $(x, y) \in u$.

We denote by $M_X(R)$ the full matrix ring over a commutative ring R where both rows and columns of matrices are indexed by the set X. For $s \subset X \times X$, we define the adjacency matrix $\sigma_s \in M_X(\mathbb{Z})$ whose (x, y)-entry is 1 if $(x, y) \in s$ and 0 otherwise. Using adjacency matrices, we can see that the conditions (1), (2), (3) are equivalent to the following conditions :

(1) there is a $1 \in S$ such that σ_1 is the identity matrix,

(2) for every $s \in S$, there is an $s^* \in S$ such that σ_{s^*} is the transposed matrix of σ_s,

(3) for every triple $(s, t, u) \in S^3$, there is a non-negative integer p_{st}^u such that $\sigma_s \sigma_t = \sum_{u \in S} p_{st}^u \sigma_u$.

For details, see [2] or [8].

Let (X, S) be an association scheme. By the condition (3), we can define a $\mathbb{Z}$-algebra $\mathbb{Z}S = \bigoplus_{s \in S} \mathbb{Z}\sigma_s \subset M_X(\mathbb{Z})$. For a commutative ring R with identity, we can define an R-algebra $RS = R \otimes_{\mathbb{Z}} \mathbb{Z}S \subset M_X(R)$. We call RS the *adjacency algebra* of (X, S) over R. In this article, a "representation" of (X, S) over R means a representation of RS, namely an R-algebra homomorphism from RS to the full matrix algebra $M_n(R)$ for some degree n, and a "character" means the trace function of a representation.

Let R be a commutative ring with identity. Since RS is defined as a subalgebra of $M_X(R)$, the map $\Gamma_R : RS \to M_X(R)$, $\Gamma_R(\sigma_s) = \sigma_s$ is a representation. We call Γ_R the *standard representation* of (X,S) over R. We denote by γ_R the *standard character* of (X,S) over R, the character of the standard representation. By definition, we can see that $\gamma_R(\sigma_1) = |X|1_R$ and $\gamma_R(\sigma_s) = 0$ for $1 \neq s \in S$. We put $n_s = p_{ss^*}^1$ for $s \in S$, and call this number the *valency* of $s \in S$. Easily we can see that every row or column of σ_s contains exactly n_s ones. Thus the map $\sigma_s \mapsto n_s 1_R$ is a representation of RS, and we call this the *trivial representation*.

Now we consider the case $R = \mathbb{C}$, the complex number field. It is known that $\mathbb{C}S$ is semisimple [8, Theorem 4.1.3 (ii)]. We denote by $\mathrm{Irr}(S)$ the set of all irreducible characters of $\mathbb{C}S$. Considering the irreducible decomposition of the standard character, we set $\gamma_{\mathbb{C}} = \sum_{\chi \in \mathrm{Irr}(S)} m_\chi \chi$. We call m_χ the *multiplicity* of $\chi \in \mathrm{Irr}(S)$.

2.2 Splitting p-modular systems and p-blocks

Let p be a rational prime number. Like in modular representation theory of finite groups, we consider a splitting p-modular system (K,R,F) of an association scheme (X,S). For details, see [7]. Now R is a complete discrete valuation ring with the maximal ideal πR, K the quotient field of R of characteristic zero, $F = R/\pi R$ the residue class field of characteristic p, and algebras KS and FS are splitting algebras. There is a natural correspondence between representations of KS and $\mathbb{C}S$. Thus we can suppose that $\mathrm{Irr}(S)$ is the set of all irreducible characters of KS.

Let $RS = B_0 \oplus \cdots \oplus B_\ell$ be the indecomposable direct sum decomposition of RS as a two-sided ideal. We call B_i a *block* or a *p-block* of (X,S) and denote by $\mathrm{Bl}(S)$ the set of all blocks. We have a decomposition $1_{RS} = e_{B_0} + \cdots + e_{B_\ell}$, $e_{B_i} \in B_i$, and call e_{B_i} the *block idempotent* of B_i. Put $FB_i = F \otimes_R B_i$ and $KB_i = K \otimes_R B_i$. Then FB_i are also indecomposable as two-sided ideals and we have direct sum decompositions $FS = FB_0 \oplus \cdots \oplus FB_\ell$ and $KS = KB_0 \oplus \cdots \oplus KB_\ell$. We also have a partition $\mathrm{Irr}(S) = \mathrm{Irr}(KB_0) \cup \cdots \cup \mathrm{Irr}(KB_\ell)$, where $\mathrm{Irr}(KB_i) = \{\chi \in \mathrm{Irr}(S) \mid \chi(e_{B_i}) \neq 0\}$.

We suppose that the trivial character is in $\mathrm{Irr}(KB_0)$ and call B_0 the *principal block*.

2.3 Decomposition and Cartan matrices

Let (X,S) be an association scheme, and let (K,R,F) be a splitting p-modular system of (X,S). For every representation Ψ of KS, we can take an R-form of Ψ, namely, taking suitable similar representation, we may assume $\Psi(\sigma_s) \in M_n(R)$ for any $s \in S$ [7, I. Theorem 1.6].

We denote by $\mathrm{Irr}(FS)$ the set of all irreducible characters of FS. Note that $\mathrm{Irr}(FS)$ is linearly independent over F. Let Ψ be an irreducible representation of KS affording $\chi \in \mathrm{Irr}(S)$. We suppose $\Psi(\sigma_s) \in M_n(R)$ for any $s \in S$. Then we can define an F-representation $\overline{\Psi}$. We denote by $d_{\chi\varphi}$ the multiplicity of $\varphi \in \mathrm{Irr}(FS)$ in the representation $\overline{\Psi}$ as an irreducible constituent. The R-form Ψ is not unique for χ, but the number $d_{\chi\varphi}$ is defined [7, I. Theorem 1.9]. We call $d_{\chi\varphi}$ the *decomposition number*. The matrix $D = (d_{\chi\varphi})_{\mathrm{Irr}(S)\times\mathrm{Irr}(FS)}$ is called the *decomposition matrix* of (X,S).

Let W and W' be simple FS-modules affording irreducible F-characters φ and φ', respectively. We denote by $c_{\varphi\varphi'}$ the multiplicity of W' in the projective cover of W as an irreducible constituent. We call $c_{\varphi\varphi'}$ the *Cartan invariant*. The matrix $C = (c_{\varphi\varphi'})_{\mathrm{Irr}(FS)\times\mathrm{Irr}(FS)}$ is called the *Cartan matrix* of (X, S). We have $C = {}^tDD$ same as in [7, II. Theorem 6.8].

The decomposition matrix D and the Cartan matrix C are decomposed into blocks. Namely, for distinct blocks B and B', $d_{\chi\varphi} = 0$ if $\chi \in \mathrm{Irr}(KB)$ and $\varphi \in \mathrm{Irr}(FB')$, and $c_{\varphi\varphi'} = 0$ if $\varphi \in \mathrm{Irr}(FB)$ and $\varphi' \in \mathrm{Irr}(FB')$. We write

$$D = \begin{pmatrix} D_{B_0} & & \\ & \ddots & \\ & & D_{B_\ell} \end{pmatrix}, \quad C = \begin{pmatrix} C_{B_0} & & \\ & \ddots & \\ & & C_{B_\ell} \end{pmatrix},$$

and call D_B and C_B the decomposition and the Cartan matrices of the block B. We have $C_B = {}^tD_BD_B$.

We will give an easy lemma without proof to use it later.

Lemma 2.1. *For $B \in \mathrm{Bl}(S)$, the following conditions are equivalent :*

(1) FB is a semisimple algebra.

(2) FB is a simple algebra.

(3) $D_B = (1)$.

(4) $C_B = (1)$.

2.4 Schur relations and the Frame number

Let (X, S) be an association scheme. Write $\mathrm{Irr}(S) = \{\chi_1, \ldots, \chi_r\}$ and fix irreducible representations $\Psi^{(i)}$ affording χ_i for $i = 1, \ldots, r$. For $1 \le i \le r$ and $1 \le j,\ k \le \chi_i(1)$, define a function $\Psi^{(i)}_{jk}$ which send σ_s to the (j, k)-entry of $\Psi^{(i)}(\sigma_s)$.

Theorem 2.2 (Schur relation [3, 5])**.** *Under the above notations, we have*

$$\frac{m_{\chi_i}}{|X|} \sum_{s\in S} \frac{1}{n_s} \Psi^{(i)}_{jk}(\sigma_s)\Psi^{(i')}_{j'k'}(\sigma_{s^*}) = \delta_{ii'}\delta_{jk'}\delta_{j'k}.$$

Put $I = \{(i, j, k) \mid 1 \le i \le r,\ 1 \le j,\ k \le \chi_i(1)\}$. Then $|I| = |S|$. We define an $I \times S$ square matrix $T = (\Psi^{(i)}_{jk}(\sigma_s))_{(i,j,k),s}$. We have the matrix form of Schur relation.

Theorem 2.3 (Schur relation – matrix form)**.** *We have*

$$|X|^{-1}TN^{-1}P^tTQM = E,$$

where $T_{I\times S} = (\Psi^{(i)}_{jk}(\sigma_s))$, $N_{S\times S} = \mathrm{diag}(n_s)$, $M_{I\times I} = \mathrm{diag}(m_{\chi_i})$, $P_{S\times S}$ is the permutation matrix representing $s \mapsto s^*$, $Q_{I\times I}$ is the permutation matrix representing $(i, j, k) \mapsto (i, k, j)$, and E is the identity matrix.

Considering the determinants of the equation in Theorem 2.3, we have

$$(\det T)^2 = \varepsilon |X|^{|S|} (\det N)(\det M)^{-1} = \varepsilon |X|^{|S|} \frac{\prod_{s\in S} n_s}{\prod_{\chi\in \mathrm{Irr}(S)} {m_\chi}^{\chi(1)^2}}$$

where $\varepsilon = (\det P)(\det Q) \in \{-1,1\}$. We put $\mathcal{F}(S) = |(\det T)^2|$ and call this number the *Frame number* of (X,S). It is known that $\mathcal{F}(S)$ is a rational integer.

3 Simplicity of p-blocks

Let (X,S) be an association scheme, and let (K,R,F) be a splitting p-modular system of (X,S). As in §2.4, write $\mathrm{Irr}(S) = \{\chi_1,\dots,\chi_r\}$ and fix irreducible representations $\Psi^{(i)}$ affording χ_i for $i=1,\dots,r$. We suppose that $\Psi^{(i)}(\sigma_s) \in M_{\chi_i(1)}(R)$ for any $s\in S$. Recall that $I = \{(i,j,k) \mid 1\le i\le r,\ 1\le j,\ k\le \chi_i(1)\}$. Put $I_B = \{(i,j,k) \mid \chi_i \in \mathrm{Irr}(KB),\ 1\le j,\ k\le \chi_i(1)\}$. Then $I = \bigcup_{B\in \mathrm{Bl}(S)} I_B$ is a partition of I. As before, we define $\Psi^{(i)}_{jk}$ and the matrix $T_{I\times S} = (\Psi^{(i)}_{jk}(\sigma_s))$. Also we define an $I_B \times S$ matrix $T'_B = (\Psi^{(i)}_{jk}(\sigma_s))$ for $B \in \mathrm{Bl}(S)$. We can write

$$T = \begin{pmatrix} T'_{B_0} \\ \vdots \\ T'_{B_\ell} \end{pmatrix}.$$

We also consider a partition of S. We can apply [7, III. Lemma 11.1] and get the following lemma.

Lemma 3.1. *There is a partition $S = \bigcup_{B\in\mathrm{Bl}(S)} S_B$ such that $\bigcup_{B\in\mathrm{Bl}(S)} \{e_B\sigma_s \mid s\in S_B\}$ is an R-basis of RS.*

We call the partition in Lemma 3.1 a *block decomposition* of S. We note that the decomposition is not unique. For $s\in S$, we write B_s for the block such that $s \in S_{B_s}$. Now we define an $I\times S$ matrix $T' = (\Psi^{(i)}_{jk}(e_{B_s}\sigma_s))$. Since $\Psi^{(i)}_{jk}(e_{B_s}\sigma_s) = 0$ if $\chi_i \notin \mathrm{Irr}(KB_s)$, we can write

$$T' = \begin{pmatrix} T_{B_0} & & 0 \\ & \ddots & \\ 0 & & T_{B_\ell} \end{pmatrix},$$

where T_B is an $I_B \times S_B$ square matrix over R.

Now we can state our main result.

Theorem 3.2. *Let (X,S) be an association scheme, and $B\in\mathrm{Bl}(S)$. Then FB is simple if and only if $\det T_B$ is a unit in R.*

Proof. For $B\in\mathrm{Bl}(S)$, we show that the following conditions are equivalent.

(i) FB is semisimple.

(ii) FB is simple.

(iii) The F-representation $\overline{\Psi^{(i)}}$ is surjective for every $\chi_i \in \mathrm{Irr}(KB)$.

(iv) T_B is invertible in $M_{|S_B|}(R)$.

(v) $\det T_B$ is a unit in R.

Equivalences (i) $\Longleftrightarrow$ (ii) and (iii) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v) are clear. Also it is easy to see (iii) $\Longrightarrow$ (ii). If (ii) holds, then (iii) holds by Lemma 2.1. □

Theorem 1.1 is proved again as a corollary to Theorem 3.2.

Corollary 3.3 ([4, Theorem 4.2])**.** *Let (X, S) be an association scheme, and let F be a field of characteristic p. Then the adjacency algebra FS is semisimple if and only if p does not divide the Frame number $\mathcal{F}(S)$.*

Proof. Since the adjacency algebra is defined over the prime field $\mathbb{F}_p$ and $\mathbb{F}_p$ is perfect, we may assume that the field F is arbitrary large. Also it is easy to see that $\mathcal{F}(S) = u\prod_{B\in \mathrm{Bl}(S)}(\det T_B)^2$ for some unit u in R. Therefore the assertion is clear by Theorem 3.2. □

We have shown in Theorem 3.2 that the determinant of T_B decides the simplicity of the block. However the matrix T_B is depending on the choice of representations $\Psi^{(i)}$ and a block decomposition of S. We give the following proposition.

Proposition 3.4. *The determinant of T_B is unique up to unit factors in R.*

Proof. We consider a similarity of a representation. If we change $\Psi^{(i)}(\sigma_s)$ to $P^{-1}\Psi^{(i)}(\sigma_s)P$ for some invertible matrix P over K, T_B becomes

$$\begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & P^{-1}\otimes {}^tP & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} T_B$$

and $\det(P^{-1}\otimes {}^tP) = 1$. Thus the determinant of T_B is not changed.

We consider choices of block decompositions of S. Let $S = \bigcup_{B\in\mathrm{Bl}(S)} S_B = \bigcup_{B\in\mathrm{Bl}(S)} S'_B$ be two block decompositions. If we change the basis, T_B becomes T_BQ for some unimodular matrix Q over R since $\{e_B\sigma_s \mid s\in S_B\}$ and $\{e_B\sigma_s \mid s \in S'_B\}$ are R-bases of B. Therefore $\det T_BQ = (\det T_B)(\det Q)$ and $\det Q$ is a unit in R. □

Example 3.5. Let (X, S) be the unique non-commutative association scheme with

$|X| = 15$. The matrix T is as follows.

	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5
$\Psi_{11}^{(1)}$	1	1	1	4	4	4
$\Psi_{11}^{(2)}$	1	1	1	-1	-1	-1
$\Psi_{11}^{(3)}$	1	0	-1	2	0	-2
$\Psi_{21}^{(3)}$	0	-1	1	-2	2	0
$\Psi_{12}^{(3)}$	0	1	-1	0	2	-2
$\Psi_{22}^{(3)}$	1	-1	0	-2	0	2

We consider the case $p = 2$. Then $\mathrm{Bl}(S) = \{B_0, B_1, B_2\}$ and we have

$$S_{B_0} = \{\sigma_0\}, \quad S_{B_1} = \{\sigma_3\}, \quad S_{B_2} = \{\sigma_1, \sigma_2, \sigma_4, \sigma_5\}$$

and

$$\mathrm{Irr}(KB_0) = \{\chi_1\}, \quad \mathrm{Irr}(KB_1) = \{\chi_2\}, \quad \mathrm{Irr}(KB_2) = \{\chi_3\}.$$

The matrix T' is as follows.

	$e_{B_0}\sigma_0$	$e_{B_1}\sigma_3$	$e_{B_2}\sigma_1$	$e_{B_2}\sigma_2$	$e_{B_2}\sigma_4$	$e_{B_2}\sigma_5$
$\Psi_{11}^{(1)}$	1					
$\Psi_{11}^{(2)}$		-1				
$\Psi_{11}^{(3)}$			0	-1	0	-2
$\Psi_{21}^{(3)}$			-1	1	2	0
$\Psi_{12}^{(3)}$			1	-1	2	-2
$\Psi_{22}^{(3)}$			-1	0	0	2

We have $\det T_{B_0} = 1$, $\det T_{B_1} = -1$, $\det T_{B_2} = -12$. Therefore, we can see that FB_0 and FB_1 are simple and FB_2 is not simple.

Remark. Let A be a $\mathbb{Z}$-free $\mathbb{Z}$-algebra of finite rank such that the $\mathbb{C}$-algebra $\mathbb{C} \otimes_{\mathbb{Z}} A$ is semisimple. For example, adjacency algebras of coherent configurations [6] and integral standard generalized table algebras [1] satisfy this condition. Then we can apply arguments in this article, except in §2.4, especially the Frame numbers are not defined for them, in general. We can define T_B and Theorem 3.2 holds for them.

Example 3.6. Let X be a finite poset such that $|X| \geq 2$, and let n be a nonzero rational integer. Define a $\mathbb{Z}$-subalgebra A of $M_X(\mathbb{Z})$ by $A_{xy} = \mathbb{Z}$ if $x \leq y$ and $A_{xy} = n\mathbb{Z}$ otherwise. Then $\mathbb{C} \otimes_{\mathbb{Z}} A \cong M_X(\mathbb{C})$ is semisimple and Theorem 3.2 holds for A. We can define the matrix T and we have $|\det T| = n^a$ where $a = \sharp\{(x, y) \in X \times X \mid x \not\leq y\}$. Thus A/pA is a semisimple $\mathbb{F}_p$-algebra if and only if $p \nmid n$.

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