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# An analytic relation between the fractional parameter in the Mittag–Leffler function and the chemical potential in the Bose–Einstein distribution through the analysis of the NASA COBE monopole data

Minoru Biyajima<sup>1</sup>, Takuya Mizoguchi<sup>2</sup> and Naomichi Suzuki<sup>3</sup>

<sup>1</sup> Department of Physics, Shinshu University, Matsumoto 390-8621, Japan

<sup>2</sup> National Institute of Technology, Toba College, Toba 517-8501, Japan

<sup>3</sup> Matsumoto University, Matsumoto, 390-1295, Japan

E-mail: [suzuki@matsu.ac.jp](mailto:suzuki@matsu.ac.jp)

**Abstract.** To extend the Bose-Einstein (BE) distribution to fractional order, we turn our attention to the differential equation,  $df/dx = -f - f^2$ . It is satisfied with the stationary solution,  $f(x) = 1/(e^{x+\mu} - 1)$ , of the Kompaneets equation, where  $\mu$  is the constant chemical potential. Setting  $R = 1/f$ , we obtain a linear differential equation for  $R$ . Then, the Caputo fractional derivative of order  $p$  ( $p > 0$ ) is introduced in place of the derivative of  $x$ , and fractional BE distribution is obtained, where function  $e^x$  is replaced by the Mittag–Leffler (ML) function  $E_p(x^p)$ . Using the integral representation of the ML function, we obtain a new formula. Based on the analysis of the NASA COBE monopole data, an identity  $p \simeq e^{-\mu}$  is found.

## 1. Introduction

The COBE FIRAS experiments have shown that the cosmic microwave background (CMB) radiation spectrum is well described by the Planck distribution with temperature,  $T = 2725.0 \pm 1$  mK [1, 2]. Furthermore, a slight distortion from the Planck distribution in the photon number distribution,  $f(x)$ , is observed. It is expressed by

$$f(x) = 1/(e^{x+\mu} - 1), \quad x = h\nu/(kT), \quad (1)$$

where  $\mu$  is the dimensionless constant chemical potential and  $\nu$  is the frequency of photon. The measured value is  $\mu = (-1 \pm 4) \times 10^{-5}$  or  $|\mu| < 9 \times 10^{-5}$  with 95% confidence [1, 2].

Equation (1) is known as a stationary solution of the Kompaneets equation[3]:

$$\frac{\partial f}{\partial t} = \frac{kT_e}{m_e c^2} \frac{n_e \sigma_e}{c} x_e^{-2} \frac{\partial}{\partial x_e} x_e^4 \left( \partial f / \partial x_e + f + f^2 \right), \quad (2)$$

where  $n_e$  is the electron density,  $\sigma_e$  is the Thomson scattering cross-section and  $x_e = h\nu/(kT_e)$ . Equation (2) describes the photon distribution, which obeys the Planck distribution at the initial stage, and is affected by the elastic  $e$ - $\gamma$  scatterings in the expanding universe.



In the theories of stochastic processes, in order to take a sort of memory effect into account, fractional calculus is introduced [4, 5, 6, 7]. Based on the Caputo derivative [4], Ertik et al. proposed a generalized BE distribution [8],  $f(x) = 1/(E_p(x) - 1)$ , where  $E_p(x)$  denotes the Mittag-Leffler (ML) function defined by

$$E_p(x) = \sum_{n=0}^{\infty} x^n / \Gamma(np + 1), \quad p > 0. \quad (3)$$

To extend the BE distribution to fractional order, we turn our attention to the equation,

$$df(x)/dx = -af(x) - bf(x)^2, \quad (4)$$

where  $a$  and  $b$  are constant. If  $a = b = 1$ , Eq.(4) reduces to the equation which is adopted by Planck [9, 10] to derive the blackbody radiation law, and is satisfied with the stationary solution (1) of the Kompaneets equation (2). Putting  $f = 1/R$ , we obtain the linear differential equation,

$$dR/dx = aR + b. \quad (5)$$

In Appendix A, the Caputo fractional derivative is introduced into Eq. (5) in place of the derivative  $x$ , and a fractional BE and other distributions are obtained.

In [11], we have applied the Riemann–Liouville fractional derivative to obtain a fractional BE distribution  $f(x) = 1/(E_p(x^p) - 1)$ , and we have investigated the NASA COBE monopole data using BE and fractional BE distributions. The photon spectrum given from Eq.(1) is written as

$$U^{\text{BE}}(x, \mu) = C_B / (e^{x+\mu} - 1), \quad (6)$$

where  $x = h\nu/(kT)$  and  $C_B = 2h\nu^3/c^2$ . On the other hand, the photon spectrum in the Universe, based on the fractional calculus, is given by

$$U(x, p) = C_B / (E_p(x^p) - 1). \quad (7)$$

From the analysis of NASA COBE monopole data [2], the following values of parameters are estimated [11]: from Eq.(6),  $T = 2.72501 \pm 0.00002$  K and  $\mu = (-1.1 \pm 3.2) \times 10^{-5}$ , and from Eq.(7),  $T = 2.72501 \pm 0.00003$  K and  $p - 1 = (1.1 \pm 3.5) \times 10^{-5}$ . Then, we estimated a relation between  $\mu$  and  $p$  as  $\mu \approx 1 - p$ .

In the present study, the COBE monopole data [2] is analyzed by the use of an integral representation of the ML function [5, 12]:

$$E_p(x^p) = e^x / p + \delta(x, p), \quad (8)$$

$$\delta(x, p) = -\frac{\sin(p\pi)}{\pi} \int_0^{+\infty} \frac{y^{p-1} e^{-xy}}{y^{2p} - 2y^p \cos(p\pi) + 1} dy. \quad (9)$$

Function  $\delta(p, x)$  for  $0 < p < 2$  and  $0 \leq x$  satisfies the relation,  $|\delta(p, x)| \leq |\delta(p, 0)| = |p - 1|/p$ .

## 2. Analysis of the COBE monopole data by Eqs.(8) and (9)

By the use of the integral representation of the ML function, Eq. (8), Eq. (7) is written as

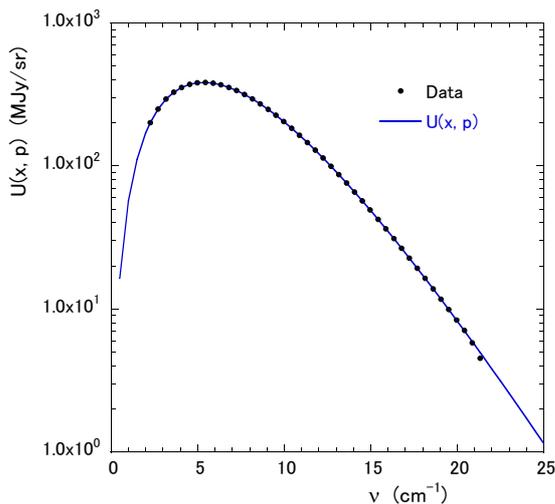
$$U(x, p) = C_B / (e^x / p - 1 + \delta(x, p)), \quad (10)$$

with two parameters,  $T$  and  $p$ . At first we analyze the COBE monopole data using Eq.(10). The results are shown in Table 1 and in Fig. 1. As is seen from Table 1, conditions that  $|p - 1| \ll 1$  and  $|\delta(p, x)| \leq |p - 1|/p \ll 1$  are satisfied. Then, we can expand Eq. (10) as

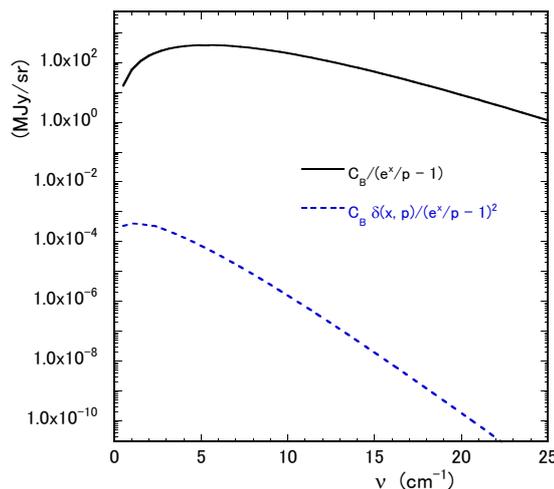
$$U(x, p) = C_B / (e^x / p - 1) - C_B \delta(x, p) / (e^x / p - 1)^2. \quad (11)$$

**Table 1.** Analysis of the NASA COBE monopole data by Eq.(10).

| $T$ (K)                        | $(p - 1)$                      | $\chi^2/N F$ |
|--------------------------------|--------------------------------|--------------|
| $2.72501 \pm 3 \times 10^{-5}$ | $(1.1 \pm 3.5) \times 10^{-5}$ | 45.0/41      |



**Figure 1.** Analysis of the COBE monopole data by Eq. (10).  $x = 0.528\nu$ .



**Figure 2.** Contribution of first and second terms on the right hand side of Eq. (11).

From the Analysis with Eq.(11), we obtain the same results with those in Table 1. Contribution from the first and second terms on the right hand side of Eq. (11) with parameter values in Table 1 are shown in Fig. 2.

As the ratio of the second term to the first term on the right hand side of Eq. (11) becomes  $\delta(x, p)/(e^x/p - 1) < 2 \times 10^{-6}$  over the range of the COBE monopole data,  $1.20 \leq x \leq 11.26$ , we can approximate Eq. (11) as,

$$U(x, p) \simeq C_B/(e^x/p - 1) = C_B/(e^{x - \ln p} - 1). \tag{12}$$

Comparing Eq.(1) and Eq.(12), we obtain an analytic relation,  $\mu = -\ln p$ .

### 3. Concluding remarks

1) If the Caputo fractional derivative is introduced into Eq. (4), contrary to the case of Riemann-Liouville fractional derivative [11], we have fractional Bose-Einstein, Fermi-Dirac and Maxwell-Boltzmann distributions, where function  $e^x$  is replaced by the ML function,  $E_p(x^p)$ .

2) Under the condition that  $|p - 1| \ll 1$ , we can show that the analytic relation,  $\mu = -\ln p$ , is satisfied. In other words, the fractional parameter  $p$ , where a kind of memory effect of the expanding universe would be included, has a role of inverse fugacity to the dimensionless chemical potential  $\mu$ .

3) Extension of statistical distributions has already been investigated from the non-extensive statistical approach [10], where parameter  $q$  is included. We would like to study how fractional parameter  $p$  is related to  $q$  and other approaches.

**Appendix A. Application of the Caputo fractional derivative to Eq. (5)**

The Caputo fractional derivative [4, 5] of function  $f(x)$  for  $m = 1, 2, \dots$  is defined as

$${}_0^C D_x^p f(x) = \frac{1}{\Gamma(m-p)} \int_0^x (x-\tau)^{m-p-1} f^{(m)}(\tau) d\tau, \quad m-1 < p < m, \tag{A.1}$$

where  $f^{(m)}(\tau) = d^m f(\tau)/d\tau^m$ . We consider the following equation,

$${}_0^C D_x^p R(x) = aR(x) + b. \tag{A.2}$$

The Laplace transform of function  $R(x)$  is defined as,  $\tilde{R}(s) = \mathcal{L}[R(x); s] = \int_0^\infty e^{-sx} R(x) dx$ .

Applying the Laplace transform to Eq. (A.2), we obtain the following equation,

$$\tilde{R}(s) = b/\{s(s^p - a)\} + \sum_{k=0}^{m-1} R^{(m-k-1)}(0) s^{k-p} / (s^p - a). \tag{A.3}$$

Using the formula [5],

$$\mathcal{L}[x^{\beta-1} E_{\alpha,\beta}(\gamma x^\alpha); s] = s^{\alpha-\beta} / (s^\alpha - \gamma), \quad Re(s) > |\gamma|^{1/\alpha}, \tag{A.4}$$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty z^k / \Gamma(\alpha k + \beta), \quad \alpha > 0, \beta > 0, \tag{A.5}$$

where  $E_{\alpha,\beta}(z)$  is the two parameter ML function, we have

$$R(x) = b\{E_p(ax^p) - 1\} + R(0)E_p(ax^p) + \sum_{k=1}^{m-1} R^{(m-k)}(0)x^{m-k-1}E_{p,m-k+1}(ax^p). \tag{A.6}$$

Solutions  $R(x)$  according to the values of  $a, b$ , and the initial conditions are shown in Table A1.

**Table A1.** Solutions of Eq.(A.2).

| $(a, b)$  | Initial conditions                                | $R(x)$         | $f(x) = 1/R(x)$    |
|-----------|---|----------------|--------------------|
| $(1, 1)$  | $R(0) = \dots = R^{(m-1)}(0) = 0$                 | $E_p(x^p) - 1$ | $1/(E_p(x^p) - 1)$ |
| $(1, -1)$ | $R(0) = 2, R^{(1)}(0) = \dots = R^{(m-1)}(0) = 0$ | $E_p(x^p) + 1$ | $1/(E_p(x^p) + 1)$ |
| $(1, 0)$  | $R(0) = 1, R^{(1)}(0) = \dots = R^{(m-1)}(0) = 0$ | $E_p(x^p)$     | $1/E_p(x^p)$       |

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