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The space of short ropes and the classifying space of the space of long knots

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We prove affirmatively the conjecture raised by J Mostovoy (Topology 41 (2002) 435–450); the space of *short ropes* is weakly homotopy equivalent to the classifying space of the topological monoid (or category) of long knots in $\mathbb{R}^3$. We make use of techniques developed by S Galatius and O Randal-Williams (Geom. Topol. 14 (2010) 1243–1302) to construct a manifold space model of the classifying space of the space of long knots, and we give an explicit map from the space of short ropes to the model in a geometric way.

57R19; 55R35, 57M25

1 Introduction

A *long j–embedding in* $\mathbb{R}^n$ is a smooth embedding $\mathbb{R}^j \hookrightarrow \mathbb{R}^n$ that coincides with the standard inclusion outside a compact set.

The space $\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n)$ of all long j–embeddings in $\mathbb{R}^n$ equipped with the C^∞–topology has been widely studied in recent years, in particular in the metastable range of dimensions. Perhaps the space of *long knots*, long 1–embeddings in $\mathbb{R}^3$, is one of the most fascinating cases, but the dimension $(n, j) = (3, 1)$ is not in the stable range and some methods for studying $\mathrm{Emb}(\mathbb{R}^j, \mathbb{R}^n)$ in high (co)dimensional cases yield only information on $K := \pi_0(\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3))$ when applied to $\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3)$. K is just a free commutative monoid (and not a group) with respect to the connected sum, and the group completion $\Omega B\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3)$ would be strictly better from homotopy-theoretic view than $\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3)$ itself. In fact, the group completion is a 2–fold loop space, since the little 2–disks operad acts on $\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3)$ (Budney [1]). Moreover, the group completion would be useful for study of (isotopy classes of) long knots since the natural map $\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3) \to \Omega B\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3)$ induces a monomorphism on π_0.

From this viewpoint the result of Mostovoy [3] is very curious, though it is also concerned with K. A *parametrized short rope* is a smooth embedding $\rho\colon [0, 1] \hookrightarrow \mathbb{R}^3$

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of length < 3 such that $\rho(i) = (i, 0, 0)$ for $i = 0, 1$. Mostovoy has proved that the fundamental group of the space B_2 of parametrized short ropes is isomorphic to $\pi_1 BK$, the group completion of K. This leads us to the question [3, Conjecture 1]: is the space B_2 the classifying space $B\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3)$ of $\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3)$? Our main result asserts that this is the case.

Theorem 1.1 (Corollary 3.7 and Theorem 3.8) *Mostovoy's space of parametrized short ropes is weakly homotopy equivalent to the classifying space of the space of long knots.*

One of the main ingredients in the proof of Theorem 1.1 is the technique of Galatius and Randal-Williams [2]. It enables us to construct a model of $B\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3)$. The model is a space of certain 1–dimensional submanifolds in $\mathbb{R}^3$ whose connected components are noncompact closed subspaces of $\mathbb{R}^3$ (see Definition 2.3). We prove Theorem 1.1 by introducing the notion of *reducible ropes* (see Definition 3.1) and by comparing the manifold space model with the space of short ropes through reducible ropes:

Theorem 1.2 (Corollary 3.7 and Theorem 3.8) *The manifold space model and Mostovoy's space of parametrized short ropes are both weakly homotopy equivalent to the space of reducible ropes.*

It is very interesting that we can realize the weak equivalence from the manifold space model to the space of reducible ropes as a "cut-off map" which is explicit and geometric. Therefore, Mostovoy's space of short ropes and the space of reducible ropes would serve as tools to study $B\mathrm{Emb}(\mathbb{R}^1, \mathbb{R}^3)$ in a geometric way.

2 Manifold space model of the classifying space of the space of long knots

2.1 Notation

Throughout this paper D^m and $\bar{D}^m$ stand respectively for the open and closed unit m–disks,

$$D^m := \{p \in \mathbb{R}^m \mid |p| < 1\}, \quad \bar{D}^m := \{p \in \mathbb{R}^m \mid |p| \le 1\}.$$

For a 1–dimensional manifold $M \subset \mathbb{R}^1 \times D^2$ and a subset $A \subset \mathbb{R}^1$, let

$$M|_A := M \cap (A \times D^2).$$

For a one-point set $A = \{T\}$, we simply write $M|_T$ for $M|_{\{T\}}$.

Definition 2.1 A 1–dimensional manifold $M \subset \mathbb{R}^1 \times D^2$ is said to be

- *reducible at* $T \in \mathbb{R}^1$ if M intersects $\{T\} \times D^2$ transversely in a one-point set;
- *strongly reducible at* $T \in \mathbb{R}^1$ if $M|_T$ is a one-point set and there exists an $\epsilon > 0$ satisfying
$$M|_{(T-\epsilon,T+\epsilon)} = (T-\epsilon, T+\epsilon) \times \{p_{23}(M|_T)\},$$
where $p_{23}\colon \mathbb{R}^1 \times D^2 \to D^2$ is the projection.

Remark 2.2 The word "reducible" indicates that the manifold looks like a "connected sum" of two 1–manifolds. But the meaning is different from that in knot theory, in that a reducible manifold does not need to split into a connected sum of nontrivial knots.

2.2 The category $\mathcal{K}$ of long knots

First we define the space ψ that we have referred to in Section 1 as the *manifold space model*.

Definition 2.3 Let ψ be the set of 1–dimensional submanifolds $M \subset \mathbb{R}^1 \times D^2$ such that

- $\partial M = \varnothing$,
- each connected component of M is a closed, noncompact subspace in $\mathbb{R}^3$, and
- there exists at least one $T \in \mathbb{R}$ such that M is reducible at T.

(See Figure 2.1). The above conditions imply that $M \in \psi$ contains exactly one connected component M_0 satisfying $M_0|_t \neq \varnothing$ for any $t \in \mathbb{R}^1$. Such a component is said to be *long*. It can also be seen that the other connected components (if they exist) are *long on exactly one side*; we say a component M_1 is *long on the left* (resp. *right*) if there exists $T \in \mathbb{R}^1$ such that $M_1|_s \neq \varnothing$ for any $s \leq T$ (resp. $s \geq T$) but $M_1|_{(T,\infty)} = \varnothing$ (resp. $M_1|_{(-\infty,T)} = \varnothing$). The set ψ is topologized as a subspace of $\psi(3,1)$ from Galatius and Randal-Williams [2, Section 3.1] (without any "tangential data").

Remark 2.4 Roughly speaking, two manifolds $M, N \in \psi$ are "close to each other if they are close in a compact subspace of $\mathbb{R}^3$". A bit more precisely, for $M \in \psi$, the set of manifolds whose intersections with some compact subspace of $\mathbb{R}^3$ is obtained by shifting M along small normal sections to M is a basic open neighborhood of M in ψ.

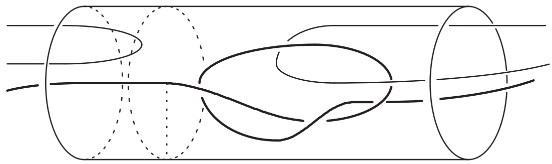

Figure 2.1: An element of ψ; the long component is drawn with a thick curve.

It is worth mentioning the following example: Let $\alpha\colon [0,1) \to \mathbb{R}_{\geq 0}$ be a monotonically increasing function with $\alpha(0)=0$ and $\lim_{t\to 1}\alpha(t)=\infty$, and $M(t)\in\psi$ for $0\leq t<1$ a continuous family satisfying $M(t)|_{[-\alpha(t),\alpha(t)]}=[-\alpha(t),\ \alpha(t)]\times\{(0,0)\}$. Then $M(t)$ converges to the trivial long knot $\mathbb{R}^1\times\{(0,0)\}$ in this topology as t tends to 1 (see also [2, Example 2.2]).

Remark 2.5 For any $M\in\psi$ there exists $T\in\mathbb{R}^1$ such that all the components of M that are long on the left (resp. right) are contained in $(-\infty,T)\times D^2$ (resp. $(T,\infty)\times D^2$).

Definition 2.6 We define the *category* $\mathcal{K}$ *of long knots* as follows. The space of objects of $\mathcal{K}$ is D^2 with the usual topology. A nonidentity morphism from p to q is a pair (T,M), where $T>0$ and $M\in\psi$ is a *long knot from* p *to* q, namely a connected 1–manifold (and hence long) that is strongly reducible at any $t\in(-\infty,\epsilon)\cup(T-\epsilon,\infty)$ for some $\epsilon>0$:

$$M|_{(-\infty,\epsilon)}=(-\infty,\epsilon)\times\{p\},\quad M|_{(T-\epsilon,\infty)}=(T-\epsilon,\infty)\times\{q\}.$$

The identity morphism id: $p\to p$ is given by $(0,\mathbb{R}^1\times\{p\})$. The total space

$$\bigcup_{p,q}\operatorname{Map}_{\mathcal{K}}(p,q)$$

of all morphisms is topologized as a subspace of $(\{0\}\sqcup\mathbb{R}^1_{>0})\times\psi$, where $\{0\}\sqcup\mathbb{R}^1_{>0}$ is a disjoint union. The composition $\circ\colon \operatorname{Map}_{\mathcal{K}}(q,r)\times\operatorname{Map}_{\mathcal{K}}(p,q)\to\operatorname{Map}_{\mathcal{K}}(p,r)$ is defined by

$$(T_1,M_1)\circ(T_0,M_0):=(T_0+T_1,M_0|_{(-\infty,T_0]}\cup(M_1|_{[0,\infty)}+T_0\boldsymbol{e}_1)),$$

where $\boldsymbol{e}_1=(1,0,0)\in\mathbb{R}^3$ and $+T\boldsymbol{e}_1$ stands for the translation by T in the direction of $\mathbb{R}^1$.

In this section we show that $B\mathcal{K}$ (see Section 2.3 for the definition) is weakly equivalent to ψ. The following posets play roles as interfaces between them:

Definition 2.7 Define a poset $\mathcal{D}$ by

$$\mathcal{D} := \{(T, M) \in \mathbb{R}^1 \times \psi \mid M \text{ is reducible at } T\}$$

and topologize $\mathcal{D}$ as a subspace of $\mathbb{R}^1 \times \psi$. Define the partial order $\leq$ on $\mathcal{D}$ so that $(T, M) < (T', M')$ if and only if $M = M'$ and $T < T'$. We regard $\mathcal{D}$ as a small category in the usual way, namely $\mathrm{Map}_{\mathcal{D}}(x, y)$ is a one-point set $\{(x, y)\}$ if $x \leq y$, and $\varnothing$ otherwise. The total space of all morphisms is topologized as a subspace of $(\Delta \sqcup (\mathbb{R}^1 \times \mathbb{R}^1 \setminus \Delta)) \times \psi$, where $\Delta := \{(x, x) \in \mathbb{R}^1 \times \mathbb{R}^1\}$ is the diagonal set.

Define $\mathcal{D}^{\perp}$ as a subposet of $\mathcal{D}$ consisting of (T, M) with M strongly reducible at T.

2.3 Classifying spaces of categories

Here we recall the general definition of classifying spaces of topological categories.

For a topological category $\mathcal{C}$, its *nerve* is the simplicial space whose level l space $N_l\mathcal{C}$ consists of sequences of l composable morphisms $(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_l} x_l)$ in $\mathcal{C}$ and is topologized as a subspace of the l^{th} power of the total space of all morphisms in $\mathcal{C}$. By definition, $N_0\mathcal{C}$ is the space of objects in $\mathcal{C}$. The face maps are given by the compositions, and the degeneracy maps are given by inserting the identity morphisms. The *classifying space* $B\mathcal{C}$ of $\mathcal{C}$ is defined as the geometric realization of $N_*\mathcal{C}$,

$$B\mathcal{C} := |N_*\mathcal{C}| := \Big(\bigsqcup_{l \geq 0} (N_l\mathcal{C} \times \Delta^l)\Big)/\sim,$$

where $\Delta^l := \big\{(\lambda_0, \dots, \lambda_l) \in [0,1]^{l+1} \mid \sum_i \lambda_i = 1\big\}$ is the standard l–simplex. The relation $\sim$ is defined so that, for any order-preserving map $\sigma\colon \{0, \dots, l \pm 1\} \to \{0, \dots, l\}$,

$$N_{l\pm1}\mathcal{C} \times \Delta^{l\pm1} \ni (\sigma^* f, \lambda) \sim (f, \sigma_*\lambda) \in N_l\mathcal{C} \times \Delta^l, \tag{2-1}$$

where σ_* and σ^* are the induced maps on (co)simplicial spaces.

Recall from Segal [4] a sufficient condition for a simplicial map to induce a homotopy equivalence on geometric realizations:

Definition 2.8 [4, Definition A.4] We say a simplicial space A_* is *good* if $s_i A_l \hookrightarrow A_{l+1}$ is a closed cofibration for each l and $0 \leq i \leq l$, where s_i stands for the i^{th} degeneracy map.

Lemma 2.9 [4, Proposition A.1] *Let A_* and B_* be good simplicial spaces. Suppose there exists a simplicial map $f_*\colon A_* \to B_*$ which is a levelwise homotopy equivalence,* *that is, $f_l\colon A_l \to B_l$ is a homotopy equivalence for each l. Then f induces a homotopy equivalence $|f_*|\colon |A_*| \xrightarrow{\simeq} |B_*|$ on the geometric realizations.*

2.4 The classifying space of $\mathcal{K}$

Notice that any element of $N_l\mathcal{D}$ (resp. $N_l\mathcal{D}^\perp$), with $l \geq 0$, can be expressed as a pair $(T_0 \leq \cdots \leq T_l; M)$, where $M \in \psi$ is reducible (resp. strongly reducible) at each T_i. Similarly, any element of $N_l\mathcal{K}$ for $l \geq 1$ is of the form $(0 \leq T_1 \leq \cdots \leq T_l; M)$, where M is a long knot that is strongly reducible at each T_i.

Lemma 2.10 *The simplicial spaces $N_*\mathcal{K}$, $N_*\mathcal{D}$ and $N_*\mathcal{D}^\perp$ are good.*

Proof For $0 \leq i \leq l$, $s_i N_l\mathcal{K} = \{(0 \leq T_1 \leq \cdots \leq T_{l+1}; M) \mid T_i = T_{i+1}\} \subset N_{l+1}\mathcal{K}$ (here $T_0 := 0$) is a union of connected components of sequences involving identity morphisms, and hence the pair $(N_{l+1}\mathcal{K}, s_i N_l\mathcal{K})$ has the homotopy extension property. The proofs for $N_*\mathcal{D}$ and $N_*\mathcal{D}^\perp$ are the same. □

Proposition 2.11 *There exists a zigzag of levelwise homotopy equivalences $N_*\mathcal{K} \leftarrow N_*\mathcal{D}^\perp \to N_*\mathcal{D}$. Consequently, $B\mathcal{K} \leftarrow B\mathcal{D}^\perp \to B\mathcal{D}$ are all homotopy equivalences.*

Proof The proof is the same as in Galatius and Randal-Williams [2, Theorem 3.9]. That $B\mathcal{D}^\perp \to B\mathcal{D}$, induced by the inclusion, is a homotopy equivalence follows from [2, Lemma 3.4], which states that, for any $(T_0 \leq \cdots \leq T_l; M) \in N_l\mathcal{D}$, M can be modified to be strongly reducible at T_i in a canonical way.

Define the functor $F\colon \mathcal{D}^\perp \to \mathcal{K}$ on objects by $(T, M) \mapsto M|_T$, and on morphisms by

$$F(T_0 \leq \cdots \leq T_l; M) := (0 \leq T_1 - T_0 \leq \cdots \leq T_l - T_0; \overline{M|_{[T_0,T_l]}} - T_0\boldsymbol{e}_1),$$

where $\overline{M|_{[T_0,T_l]}}$ is the *long-extension* of $M|_{[T_0,T_l]}$ (see Figure 2.2), namely

$$\overline{M|_{[T_0,T_l]}} := \big((-\infty, T_0] \times \{p_{23}(M|_{T_0})\}\big) \cup M|_{[T_0,T_l]} \cup \big([T_l, \infty) \times \{p_{23}(M|_{T_l})\}\big), \tag{2-2}$$

where $p_{23}\colon \mathbb{R} \times D^2 \to D^2$ is the second projection (the set (2-2) is the same as $(\varphi_\infty(T_0, T_l) \times \mathrm{id})^{-1}(M)$ in [2, Section 3.2]). Notice that $M|_{[T_0,T_l]}$ is a connected subspace of the long component of M (see Remark 2.5), and its long-extension is also connected. This induces a map $F\colon N_*\mathcal{D}^\perp \to N_*\mathcal{K}$ of simplicial spaces.

We have a map $G\colon N_*\mathcal{K} \to N_*\mathcal{D}^\perp$, defined in level 0 by $G(p) := (0, \mathbb{R}^1 \times \{p\})$, and by the natural inclusion in positive levels (letting $T_0 := 0$). This is just a simplicial

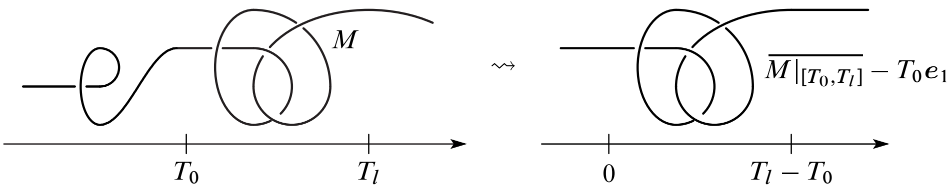

Figure 2.2: The functor F from the proof of Proposition 2.11 (cut-off and long-extension)

map up to homotopy (in levels 0 and 1), but is a levelwise homotopy inverse to F; the composite $F \circ G$ is the identity, and the other composite $G \circ F$ is given by

$$G \circ F(T_0 \leq \cdots \leq T_l; M) = (0 \leq T_1 - T_0 \leq \cdots \leq T_l - T_0; \overline{M_0|_{[T_0,T_l]}}),$$

which is isotopic to the identity via the same homotopy as the one exhibited in the last line in the proof of [2, Theorem 3.9] (see Figure 2.3). This homotopy firstly extends $M|_{(T_0-\epsilon,T_0]}$ and $M|_{[T_l,T_l+\epsilon)}$ to left and right, respectively, so that $M|_{(-\infty,T_0)}$ and $M|_{(T_l,\infty)}$ (in which all the one-sided long components are contained) escape to "$\{\mp\infty\} \times D^2$", respectively. Then they "vanish" at $s = 1$ by definition of the topology of ψ (see Remark 2.4). Simultaneously this homotopy translates the manifold by $-T_0$ in the direction of $\mathbb{R}^1$. This homotopy keeps manifolds strongly reducible at each T_i.

Therefore, $F\colon N_*\mathcal{D}^\perp \to N_*\mathcal{K}$ is a levelwise homotopy equivalence of good simplicial spaces (Lemma 2.10), and $B\mathcal{D}^\perp \to B\mathcal{K}$ is a homotopy equivalence by Lemma 2.9. □

Following Galatius and Randal-Williams [2], we write the element of $B\mathcal{D}$ represented by $((T_0 \leq \cdots \leq T_l; M), (\lambda_0, \ldots, \lambda_l)) \in N_l\mathcal{D} \times \Delta^l$ as a formal sum $\sum_{0 \leq i \leq l} \lambda_i T_i$ (this notation is compatible with the relation (2-1)).

Theorem 2.12 *The forgetful map $u\colon B\mathcal{D} \to \psi$ given by $\sum_i \lambda_i T_i \mapsto M$ is a weak homotopy equivalence. Thus, $B\mathcal{K}$ is weakly equivalent to ψ.*

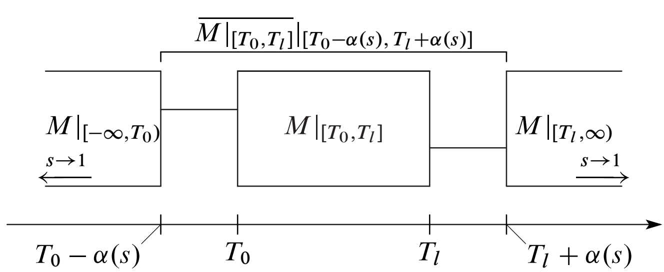

Figure 2.3: The homotopy in the proof of Proposition 2.11 from $G \circ F$ to the identity, where $\alpha(s) \to \infty$ as $s \nearrow 1$.

Proof The proof is the same as that of [2, Theorem 3.10]: Given the commutative diagram of strict arrows

$$\begin{tikzcd}
\partial\bar{D}^m \arrow[r, "\hat{f}"] \arrow[d, hook] & B\mathcal{D} \arrow[d, "u"] \\
\bar{D}^m \arrow[r, "f"] \arrow[ur, dotted, "g"] & \psi
\end{tikzcd}$$

we find a dotted $g\colon \bar{D}^m \to B\mathcal{D}$ that makes the diagram commutative. This means that the relative homotopy group $\pi_m(\psi', B\mathcal{D})$ (ψ' is the mapping cylinder of u) vanishes for all m, and u induces an isomorphism of homotopy groups in any dimension.

For $a \in \mathbb{R}$ let $U_a := \{x \in \bar{D}^m \mid f(x) \in \psi \text{ is reducible at } a\}$. This is an open subspace of $\bar{D}^m$ and $\{U_a\}_{a\in\mathbb{R}}$ is an open covering of $\bar{D}^m$ because, by definition, such an a exists for any $M \in \psi$. So, by compactness, we can pick finitely many $a_0 < \cdots < a_k$ such that $\{U_{a_i}\}_{0\le i\le k}$ covers $\bar{D}^m$. Pick a partition of unity $\{\lambda_i \mid \bar{D}^m \to [0,1]\}_{0\le i\le k}$ subordinate to the cover. Using λ_i as a formal coefficient of a_i gives a map

$$\hat{g}\colon \bar{D}^m \to B\mathcal{D}, \quad \hat{g}(x) := \sum_{0\le i\le k} \lambda_i(x)a_i$$

(represented by elements in $N_k\mathcal{D} \times \Delta^k$) which lifts f, namely $u \circ \hat{g} = f$. Now we produce a homotopy $h\colon [0,1] \times \partial\bar{D}^m \to B\mathcal{D}$ such that $h(0,-) = \hat{g}|_{\partial\bar{D}^m}(-)$, $h(1,-) = \hat{f}(-)$ and $h(s,-)$ lifts $f|_{\partial\bar{D}^m}$ for all s; if such an h exists, then we can define the desired map g by

$$g(x) := \begin{cases} \hat{g}(2x) & \text{if } |x| \le \frac{1}{2}, \\ h(2|x|-1, x/|x|) & \text{if } |x| \ge \frac{1}{2}. \end{cases}$$

Since $\hat{f}$ is also a lift of $f|_{\partial\bar{D}^m}$, we may suppose that $\hat{f}$ is of the form

$$\hat{f}(x) = \sum_{0\le i\le l} \mu_i(x)b_i$$

for some $\mu_0,\ldots,\mu_l \ge 0$ with $\sum_i \mu_i(x) = 1$ and $b_0 < \cdots < b_l$ (the underlying manifolds $f(x)$ and $u(\hat{f}(x))$ are the same). Let $c_0 < \cdots < c_n$ be the reordering of the set $\{a_i\}_i \cup \{b_j\}_j$ in ascending order. Using the relation (2-1) we can write $\hat{g}|_{\partial\bar{D}^m}$ and $\hat{f}$ as

$$\hat{g}|_{\partial\bar{D}^m}(x) = \sum_{0\le i\le n} \alpha_i(x)c_i \quad \text{for some } \alpha_0,\ldots,\alpha_n \ge 0, \sum_i \alpha_i = 1,$$
$$\hat{f}(x) = \sum_{0\le i\le n} \beta_i(x)c_i \quad \text{for some } \beta_0,\ldots,\beta_n \ge 0, \sum_i \beta_i = 1$$

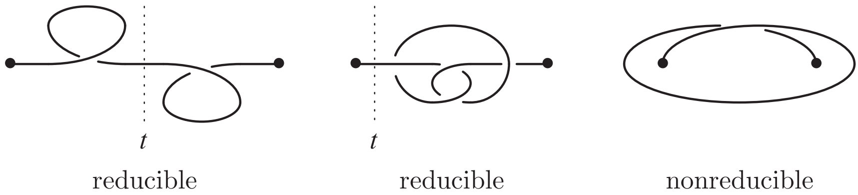

Figure 3.1: Reducible and nonreducible ropes

(represented by elements in $N_n\mathcal{D} \times \Delta^n$). We define h using the affine structure on the fibers of u:

$$h(s,x) := s\hat{g}|_{\partial\bar{D}^m}(x) + (1-s)\hat{f}(x) := \sum_{0 \le i \le n} (s\alpha_i(x) + (1-s)\beta_i(x))c_i. \qquad \square$$

Remark 2.13 We have topologized the spaces of morphisms of various categories so that the identity morphisms form disjoint components, as was also done in [2]. We may instead topologize the total space of morphisms in $\mathcal{K}$ (resp. $\mathcal{D}$) as a subspace of $[0,\infty) \times \psi$ (resp. $\mathbb{R} \times \mathbb{R} \times \psi$) and with the latter topology we can prove similar results to the above. An advantage of the former topology is that it makes the proof of goodness of the nerves easier.

3 The space of reduced ropes

In this section we show that the conjecture of Mostovoy is true. We first characterize the weak homotopy type of ψ as that of the space of *reducible ropes*, and then prove that the space of reducible ropes is weakly equivalent to the space of Mostovoy's short ropes.

3.1 $B\mathcal{K}$ and the space of reducible ropes

Definition 3.1 (Mostovoy [3]) A *rope* is a compact, connected 1–dimensional submanifold $r \subset \mathbb{R}^1 \times D^2$ with nonempty boundary $\partial r = \{\partial_0 r, \partial_1 r\}$, with $\partial_i r \in \{i\} \times D^2$. Let R be the set of all ropes that are reducible at some $t \in (0,1)$ (see Figure 3.1), topologized as a subspace of $\mathrm{Emb}([0,1], \mathbb{R} \times D^2)/\mathrm{Diff}^+([0,1])$.

The function $f(t) := \tan \pi\left(t - \frac{1}{2}\right)$ gives an orientation-preserving diffeomorphism $f\colon (0,1) \xrightarrow{\cong} \mathbb{R}$. Define the "cut-off" map $c\colon R \to \psi$ by

$$c(r) := (f \times \mathrm{id}_{D^2})(r|_{(0,1)}).$$

This map is defined since, for any reducible rope r, $c(r)$ has exactly one long component.

Our aim is to show that c is a weak equivalence, and for this we introduce the following posets as interfaces between R and ψ:

Definition 3.2 Define a poset $\mathcal{E}$ by

$$\mathcal{E} := \{(t,r) \in (0,1) \times R \mid r \text{ is reducible at } t\}.$$

Define the partial order $\leq$ on $\mathcal{E}$ so that $(t,r) < (t',r')$ if and only if $r = r'$ and $t < t'$. We regard $\mathcal{E}$ as a small category in the same way as $\mathcal{D}$. The total space of all morphisms is topologized as a subspace of $\bigl(\Delta \sqcup ((0,1)\times(0,1)\setminus\Delta)\bigr)\times R$, where Δ is the diagonal set.

Define $\mathcal{E}^{\perp}$ as a subposet of $\mathcal{E}$ consisting of (t,r) with r strongly reducible at t.

Lemma 3.3 *The simplicial spaces $N_*\mathcal{E}$ and $N_*\mathcal{E}^{\perp}$ are good.*

Proof The same as the proof of Lemma 2.10. □

Any element in $N_l\mathcal{E}$ can be expressed as a pair $(t_0 \leq \cdots \leq t_l; r)$, where $0 < t_i < 1$ and $r \in R$ is reducible at each t_i.

Proposition 3.4 *There exists a zigzag of levelwise homotopy equivalences $N_*\mathcal{E} \leftarrow N_*\mathcal{E}^{\perp} \to N_*\mathcal{D}^{\perp}$. Consequently, $B\mathcal{E}$ is weakly homotopy equivalent to $B\mathcal{D}$.*

Proof That the inclusion $\mathcal{E}^{\perp} \to \mathcal{E}$ induces a homotopy equivalence $B\mathcal{E}^{\perp} \xrightarrow{\simeq} B\mathcal{E}$ follows in the same way as [2, Theorem 3.9], using [2, Lemma 3.4].

Define a functor $\Phi\colon \mathcal{E}^{\perp} \to \mathcal{D}^{\perp}$ that induces a simplicial map $\Phi\colon N_*\mathcal{E}^{\perp} \to N_*\mathcal{D}^{\perp}$ by

$$\Phi(t;r) := (f(t); c(r))$$

(see Figure 3.2). Define the map in the reverse direction $\Gamma\colon N_l\mathcal{D}^{\perp} \to N_l\mathcal{E}^{\perp}$ by

$$\Gamma(T_0 \leq \cdots \leq T_l; M) := (t_0 \leq \cdots \leq t_l; (f^{-1} \times \mathrm{id}_{D^2})(\overline{M|_{[T_0,T_l]}})),$$

where $\overline{M|_{[T_0,T_l]}}$ is the long-extension of $M|_{[T_0,T_l]}$ (see (2-2)) and $t_i := f^{-1}(T_i) \in (0,1)$ (see Figure 3.2). Notice that $(f^{-1} \times \mathrm{id}_{D^2})(M)$ is not necessarily a tame (or regular) submanifold of $(0,1) \times D^2$ for some $M \in \psi$ (for example, a manifold M that is "knotted" outside arbitrary compact set of $\mathbb{R}^3$), but $(f^{-1} \times \mathrm{id}_{\mathbb{R}^2})(\overline{M|_{[T_0,T_l]}})$ is indeed a tame submanifold in $(0,1) \times D^2$ since $\overline{M|_{[T_0,T_l]}}$ is a union of two straight half-lines outside $[T_0, T_l] \times D^2$.

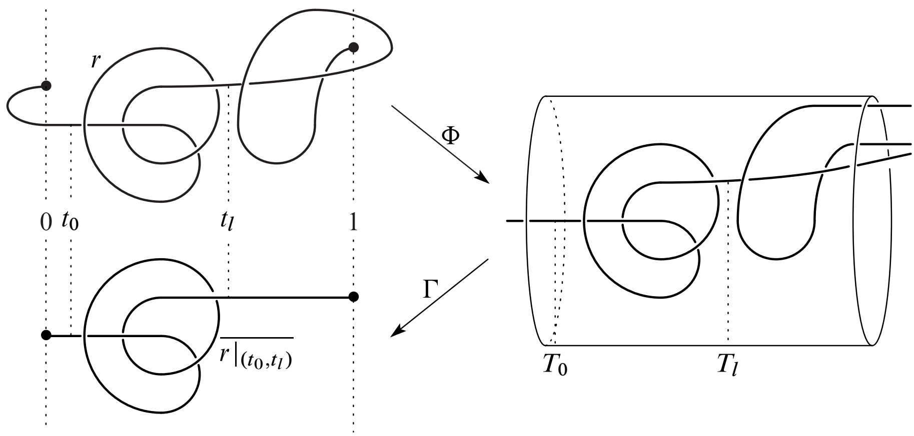

Figure 3.2: The maps Φ and Γ

We show that Φ is a levelwise homotopy equivalence, with a homotopy inverse Γ. The composite $\Phi\circ\Gamma$ is given by

$$\Phi\circ\Gamma(T_0\le\cdots\le T_l;M)=(T_0\le\cdots\le T_l;\overline{M|_{[T_0,T_l]}})$$

and a similar isotopy from the proof of Proposition 2.11 proves that $\Phi\circ\Gamma\simeq\mathrm{id}$.

The other composite $\Gamma\circ\Phi$ is given by

$$\Gamma\circ\Phi(t_0\le\cdots\le t_l;r):=(t_0\le\cdots\le t_l;\overline{r|_{(t_0,t_l)}}),$$

where

$$\overline{r|_{(t_0,t_l)}}:=\big([0,t_0]\times\{p_{23}(r|_{t_0})\}\big)\cup r|_{(t_0,t_l)}\cup\big([t_l,1]\times\{p_{23}(r|_{t_l})\}\big)\in R$$

is the "long-extension" of $r|_{(t_0,t_l)}$. The rope $\overline{r|_{(t_0,t_l)}}$ can be obtained from r by "unknotting" the edge parts $r|_{(-\infty,t_0)}\sqcup r|_{(t_l,\infty)}$ (see Figure 3.2). This unknotting can be realized by applying Lemma 3.5 and its analogue to $r_{[t_l,\infty)}$ and $r_{(-\infty,t_0]}$, respectively, keeping $r|_{[t_0,t_l]}$ unchanged (and hence keeping r strongly reducible at each t_i). Thus, $\Gamma\circ\Phi\simeq\mathrm{id}$. □

Lemma 3.5 (Mostovoy [3, Lemma 10]) *Let W be the subspace of R consisting of r that is "strongly reducible" at 0, which means $r|_{(-\epsilon,\epsilon)}=r|_{[0,\epsilon)}=[0,\epsilon)\times\{p_{23}(\partial_0 r)\}$ for some $\epsilon>0$. Then W is contractible. In other words, there exists a canonical homotopy for any $r\in W$ that transforms r to the trivial rope $[0,1]\times\{(0,0)\}$ keeping r strongly reducible at 0.*

Proof Let $W' \subset W$ be the subspace consisting of $r \in W$ with $\partial_i r = (i,0,0)$ for $i = 0,1$. We show that the inclusion $W' \hookrightarrow W$ is a homotopy equivalence. This completes the proof since W' is homeomorphic to the space W_L^0 from [3, Lemma 10] via the diffeomorphism $\mathbb{R}^1 \times D^2 \xrightarrow{\approx} \mathbb{R}^3 = \mathbb{R}^1 \times \mathbb{R}^2$ defined by $(x,u) \mapsto \bigl(x, \tan\bigl(\frac{1}{2}\pi|u|\bigr)\cdot u\bigr)$, and W_L^0 has been shown to be contractible. In the proof of [3, Lemma 10] the contracting homotopy (denoted by D_T'') keeps ropes strongly reducible at 0.

A homotopy inverse $W \to W'$ can be realized as follows. For $p \in \mathbb{R}^2$ let $\xi_p \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the scaling by $\frac{1}{2}$ centered at p, namely $\xi_p(x) := \frac{1}{2}(x+p)$. Notice that if $p \in D^2$ then $\xi_p(D^2) \subset D^2$. Let $b \colon \mathbb{R}^1 \to \mathbb{R}^1$ be a monotonically increasing C^∞ function satisfying $b(x) = 0$ for $x < \frac{1}{3}$ and $b(x) = 1$ for $x > \frac{2}{3}$. For $r \in W$, define $\Xi_r \colon \mathbb{R}^1 \times D^2 \to \mathbb{R}^1 \times D^2$ by

$$\Xi_r(x,(y,z)) := (x, \xi_{-(1-b(x))p_{23}(\partial_0 r) - b(x) p_{23}(\partial_1 r)}(y,z)). \tag{3-1}$$

Then $\Xi_r(r) \subset \mathbb{R}^1 \times D^2$ and $\Xi_r(\partial_i r) = \bigl(i, \xi_{-p_{23}(\partial_i r)}(p_{23}(\partial_i r))\bigr) = (i,0,0)$. Moreover, $\Xi_r(r)$ is strongly reducible at 0 because for a small $0 < \epsilon < \frac{1}{3}$ such that $r|_{(-\epsilon,\epsilon)} = [0,\epsilon) \times \{p_{23}(\partial_0 r)\}$ we have

$$\begin{aligned}\Xi_r(r)|_{(-\epsilon,\epsilon)} = \Xi_r\bigl([0,\epsilon) \times \{p_{23}(\partial_0 r)\}\bigr) &= [0,\epsilon) \times \{\xi_{-p_{23}(\partial_0 r)}(p_{23}(\partial_0 r))\} \\ &= [0,\epsilon) \times \{(0,0)\}.\end{aligned}$$

Thus, we have a continuous map $\Xi_\bullet \colon W \to W'$. The composite $W' \hookrightarrow W \xrightarrow{\Xi_\bullet} W'$ is the scaling by $\frac{1}{2}$ in the (y,z)–direction and is homotopic to $\mathrm{id}_{W'}$. The other composite $W \xrightarrow{\Xi_\bullet} W' \hookrightarrow W$ is also homotopic to id_W because ξ_p is homotopic to id_{D^2} for any $p \in D^2$. □

Theorem 3.6 *The forgetful map induces a weak equivalence $v \colon B\mathcal{E} \to R$.*

Proof Replace $\mathcal{D}$ with $\mathcal{E}$ and take a from $(0,1)$ in the proof of Theorem 2.12. □

Corollary 3.7 *There exists a commutative diagram consisting of (weak) equivalences*

$$\begin{tikzcd} R \arrow[r, "c", "\sim"'] & \psi & \\ B\mathcal{E}^\perp \arrow[u, "v'", "\sim"'] \arrow[r, "\Phi", "\simeq"'] & B\mathcal{D}^\perp \arrow[u, "u'", "\sim"'] \arrow[r, "F", "\sim"'] & B\mathcal{K} \end{tikzcd}$$

where u' and v' are the composites of u and v with the inclusions.

3.2 Reducible ropes and Mostovoy's parametrized short ropes

In Corollary 3.7 we have seen that $B\mathcal{K}$ is weakly equivalent to R. The following theorem solves affirmatively the conjecture of Mostovoy. For a rope r let $l(r)$ denote the length of r.

Theorem 3.8 *Let B_2 be the space of embeddings $\rho\colon [0,1] \hookrightarrow \mathbb{R}^3$ satisfying $\rho(i) = (i,0,0)$ for $i = 0,1$ and $l(\rho([0,1])) < 3$ (Mostovoy's (parametrized)* short ropes *[3]). Then B_2 is weakly equivalent to R.*

The rest of this paper is devoted to the proof of Theorem 3.8.

It is not difficult to see that the image of any $\rho \in B_2$ is in $\mathbb{R}^1 \times D^2(2\sqrt{2})$, where $D^2(\tau)$ is the open 2–disk centered at the origin and of radius τ. Thus, we may write B_2 as

$$B_2 = \{\rho\colon [0,1] \hookrightarrow \mathbb{R}^1 \times D^2(2\sqrt{2}) \mid \rho(i) = (i,0,0) \text{ for } i = 0,1 \text{ and } l(\rho([0,1])) < 3\}.$$

Let $B_2^{\mathrm{u}} := B_2/\mathrm{Diff}^+([0,1])$ ("u" indicates "unparametrized"), namely B_2^{u} is the space of ropes in $\mathbb{R}^1 \times D^2(2\sqrt{2})$ with $\partial r = \{\partial_0 r, \partial_1 r\}$, $\partial_i r = (i,0,0)$ and $l(r) < 3$. The following holds since $\mathrm{Diff}^+([0,1])$ is contractible:

Lemma 3.9 *$B_2 \to B_2^{\mathrm{u}}$ is a homotopy equivalence.*

We notice that $l(r) < 3$ implies that r is a reducible rope, and hence we may regard B_2^{u} as a subspace of $R(2\sqrt{2})$, where $R(\tau)$ is the space of reducible ropes in $\mathbb{R}^1 \times D^2(\tau)$.

Let $R^{\mathrm{s}}(\tau) \subset R(\tau)$ be the subspace consisting of $r \in R(\tau)$ with $l(r) < 3$ ("s" indicates "short"). By definition, $B_2^{\mathrm{u}} \subset R^{\mathrm{s}}(2\sqrt{2})$.

Lemma 3.10 *The inclusion $B_2^{\mathrm{u}} \hookrightarrow R^{\mathrm{s}}(2\sqrt{2})$ is a homotopy equivalence.*

Proof For $r \in R^{\mathrm{s}}(2\sqrt{2})$, let $\Xi_r\colon \mathbb{R}^1 \times D^2(2\sqrt{2}) \to \mathbb{R}^1 \times D^2(2\sqrt{2})$ be the map defined in (3-1) (notice that if $p \in D^2(\tau)$ then $\xi_p(D^2(\tau)) \subset D^2(\tau)$). Then $l(\Xi_r(r)) < 3$ because Ξ_r is a shrinking map in the (y,z)–direction and hence does not increase the length, and $\Xi_r(\partial_i r) = (i, \xi_{-\partial_i r}(\partial_i r)) = (i,0,0)$. Thus, we have a continuous map $\Xi_\bullet\colon R^{\mathrm{s}}(2\sqrt{2}) \to B_2^{\mathrm{u}}$. The composite $B_2^{\mathrm{u}} \hookrightarrow R^{\mathrm{s}}(2\sqrt{2}) \xrightarrow{\Xi_\bullet} B_2^{\mathrm{u}}$ is the scaling by $\frac{1}{2}$ in the (y,z)–direction and is homotopic to $\mathrm{id}_{B_2^{\mathrm{u}}}$. The other composite

$$R^{\mathrm{s}}(2\sqrt{2}) \xrightarrow{\Xi_\bullet} B_2^{\mathrm{u}} \hookrightarrow R^{\mathrm{u}}(2\sqrt{2})$$

is also homotopic to $\mathrm{id}_{R^{\mathrm{s}}(2\sqrt{2})}$ because ξ_p is homotopic to $\mathrm{id}_{D^2(\tau)}$ for any $p \in D^2(\tau)$. □

Next let $\mathcal{E}(\tau)$ be the poset consisting of those (t,r) with $t \in (0,1)$ and $r \in R(\tau)$ such that r is reducible at t. The partial order is defined in the same way as in Definition 3.2. Define $\mathcal{E}^{\mathrm{s}}(\tau)$ be a subposet of $\mathcal{E}(\tau)$ consisting of those (t,r) with $l(r) < 3$. Then we have a commutative diagram

$$\begin{tikzcd}
B\mathcal{E}^{\mathrm{s}}(2\sqrt{2}) \arrow[r, "v", "\sim"'] \arrow[d] & R^{\mathrm{s}}(2\sqrt{2}) \arrow[d, hook] & B_2 \arrow[l, "\simeq"', "\text{Lemmas 3.9 and 3.10}"] \\
B\mathcal{E}(2\sqrt{2}) \arrow[r, "\simeq", "\text{Theorem 3.6}"'] & R(2\sqrt{2}) \arrow[r, "\approx"] & R
\end{tikzcd} \tag{3-2}$$

where $B\mathcal{E}^{\mathrm{s}}(2\sqrt{2}) \to B\mathcal{E}(2\sqrt{2})$ and v are induced by the inclusion and the forgetful map (see Theorem 2.12), respectively. That v is a weak equivalence follows from the same argument as in the proof of Theorem 3.6. The homeomorphism $R = R(1) \xrightarrow{\approx} R(\tau)$ is given by $r \mapsto (\mathrm{id}_{\mathbb{R}^1} \times \overline{\tau})(r)$, where $\overline{\tau}\colon D^2 \xrightarrow{\approx} D^2(\tau)$ is the scalar multiplication by τ. The diagram (3-2) together with the following lemma completes the proof of Theorem 3.8.

Lemma 3.11 *$B\mathcal{E}^{\mathrm{s}}(\tau) \to B\mathcal{E}(\tau)$ is a homotopy equivalence.*

Proof Let $\mathcal{E}^{\perp}(\tau)$ be the subposet of $\mathcal{E}(\tau)$ consisting of those (t,r) with r strongly reducible at t, and $\mathcal{E}^{\perp\mathrm{s}}(\tau) := \mathcal{E}^{\perp}(\tau) \cap \mathcal{E}^{\mathrm{s}}(\tau)$. Then the inclusion $\mathcal{E}^{\perp\mathrm{s}}(\tau) \hookrightarrow \mathcal{E}^{\mathrm{s}}(\tau)$ induces a homotopy equivalence $B\mathcal{E}^{\perp\mathrm{s}}(\tau) \xrightarrow{\simeq} B\mathcal{E}^{\mathrm{s}}(\tau)$. This follows in the same way as Galatius and Randal-Williams [2, Theorem 3.9], using [2, Lemma 3.4]; modifying r to be strongly reducible at each t can be done keeping the length less than 3.

We show that $\mathcal{E}^{\perp\mathrm{s}}(\tau) \hookrightarrow \mathcal{E}^{\perp}(\tau)$ induces a levelwise homotopy equivalence $N_*\mathcal{E}^{\perp\mathrm{s}}(\tau) \to N_*\mathcal{E}^{\perp}(\tau)$. A homotopy inverse $N_l\mathcal{E}^{\perp}(\tau) \to N_l\mathcal{E}^{\perp\mathrm{s}}(\tau)$ is given as follows: firstly unknot $r|_{(-\infty,t_0]} \sqcup r|_{[t_l,\infty)}$ similarly to the proof of Lemma 3.5 to obtain $\overline{r|_{(t_0,t_l)}}$, then shrink $\overline{r|_{(t_0,t_l)}}$ to

$$\Theta(t,r) := \theta_{t,r}(\overline{r|_{(t_0,t_l)}}) \cup \big([l(\overline{r|_{(t_0,t_l)}})^{-1}, 1] \times \{p_{23}(r|_{t_l})/l(\overline{r|_{(t_0,t_l)}})\}\big),$$

where $\theta_{t,r}\colon \mathbb{R}^3 \to \mathbb{R}^3$ is given by $\theta_{t,r}(x) := x/l(\overline{r|_{(t_0,t_l)}})$. It can be seen that $l(\Theta(t,r)) < 3$ since $l(\theta_{t,r}(\overline{r|_{(t_0,t_l)}})) = 1$. The map $N_l\mathcal{E}^{\perp}(\tau) \to N_l\mathcal{E}^{\perp\mathrm{s}}(\tau)$,

$$(t_0 \le \cdots \le t_l; r) \mapsto (t_0/l(\overline{r|_{(t_0,t_l)}}), \dots, t_l/l(\overline{r|_{(t_0,t_l)}}); \Theta(t,r)),$$

gives a levelwise homotopy inverse. □

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