Doctoral Dissertation (Shinshu University)

The Hessian matrices of generating polynomials associated to graphs and matroids

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Introduction

We say that a sequence $a_0, a_1, \ldots, a_n$ of real numbers is *unimodal*, *log-concave*, and *symmetric* if

- $a_0 \leq a_1 \leq \cdots \leq a_m \geq \cdots \geq a_n$ for some $0 \leq m \leq n$,
- $a_i^2 \geq a_{i-1}a_{i+1}$ for all $1 \leq i \leq n-1$,
- $a_i = a_{n-i}$ for all $0 \leq i \leq n$,

respectively. The log-concavity of a sequence is a stronger property than the unimodality of a sequence. We sometimes find that an important sequence coming from combinatorial objects is unimodal and symmetric. The sequence

$$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$$

of binomial coefficients is a prototypical example. By direct calculation, we can show that the sequence is unimodal and symmetric. The sequence of binomial coefficients also comes from a combinatorial objects: The Boolean lattice $B(n)$ on n elements is the poset of all subsets of n elements ordered by inclusion. The Boolean lattice is a ranked poset, whose the kth rank $B_k(n)$ consists of all subsets with k elements. Hence the cardinality of $B_k(n)$ is $\binom{n}{k}$. The sequence of binomial coefficients is realized as the rank sequence of $B(n)$, i.e., the sequence

$$\#B_0(n), \#B_1(n), \ldots, \#B_n(n)$$

of the cardinalities of each rank of the Boolean lattice $B(n)$, a combinatorial object.

We often find that such an important sequence coming from combinatorial objects also comes from algebraic objects. In the case of the prototypical example, the sequence of binomial coefficients also comes from an algebraic objects: The algebra $A = \mathbb{C}[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$ is a graded algebra. The kth homogeneous components A_k of A is spanned by square-free monomials in k elements. Hence the dimension $h_k(A)$ of A_k is $\binom{n}{k}$. The sequence of binomial coefficients is realized as the sequence

$$h_0(A), h_1(A), \ldots, h_n(A)$$

of the dimensions of each homogeneous components of the algebra A.

TABLE 1. Bases and dimensions of $H^\bullet(X)$

homogeneous spaces	bases	dimensions
$H^0(X)$	1	$\binom{n}{0}$
$H^2(X)$	$x_1, x_2, \dots, x_n$	$\binom{n}{1}$
$H^4(X)$	$x_1x_2, \dots, x_1x_n, x_2x_3, \dots$	$\binom{n}{2}$
$\vdots$	$\vdots$	$\vdots$
$H^{2n-2}(X)$	$x_1 \cdots x_{n-1}, x_1 \cdots x_{n-2}x_n, \dots, x_2 \cdots x_n$	$\binom{n}{n-1}$
$H^{2n}(X)$	$x_1 \cdots x_n$	$\binom{n}{n}$

To show that a sequence coming from algebraic objects is unimodal and symmetric, techniques from algebraic geometry is useful. In particular, the hard Lefschetz theorem is often useful. Let us recall the hard Lefschetz theorem. Let X be a compact Kähler manifold of dimension d with Kähler form ω, and $H^\bullet(X)$ the cohomology ring. The following is known as the hard Lefschetz theorem.

Theorem 1. *The linear map $\times\omega^{d-k} : H^k(X) \to H^{2d-k}(X)$ is bijective for $k = 1, 2, \dots, d$.*

For the compact Kähler manifold X of dimension d, we have the sequence

$$\dim H^0(X), \dim H^2(X), \dots, \dim H^{2d}(X)$$

of the dimensions of even parts of $H^\bullet(X)$. The bijectiveties of the linear maps $\times\omega^{d-k}$ obtained by the hard Lefschetz theorem imply

- $\dim H^0(X) \leq \dim H^2(X) \leq \cdots \leq H^{2d'}(X)$,
- $\dim H^{2d}(X) \leq \dim H^{2d-2}(X) \leq \cdots \leq H^{2d'}(X)$,
- $\dim H^k(X) = \dim H^{2d-k}(X)$ for all k,

where d' is $\lfloor \frac{d}{2} \rfloor$. In other words, the hard Lefschetz theorem induces the unimodality and symmetricity of the sequence. In the case of the prototypical example, the hard Lefschetz theorem is also useful to show the unimodality and symmetricity of the sequence. Let us see how to apply the hard Lefschetz theorem to the sequence of the binomial coefficients. Let X be the products $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ of n projective lines. Then, it is known that

$$\begin{aligned} H^\bullet(X) &= H^\bullet(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1) \\ &\cong \mathbb{C}[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2), \end{aligned}$$

where $x_i \in H^2(\mathbb{P}^1)$. Since square-free monomials in $x_1, \dots, x_n$ span $H^\bullet(X)$, the dimension $\dim H^{2k}(X)$ is $\binom{n}{k}$. See Table 1. By the hard Lefschetz theorem, we obtain the unimodality and symmetricity of the sequence of binomial coefficients.

Besides the prototypical example, there are unimodal and symmetric sequences showed from the hard Lefschetz theorem. Let us see some other examples.

The poset of the Weyl group ordered by the Bruhat order has the unimodal and symmetric rank sequence. Let W be the Weyl group of a complex semisimple algebraic group G. The Weyl group is a ranked poset by the Bruhat oreder. We consider the rank seuquence of the poset. Let X be the flag variety of W, i.e., the algebraic variety of the quotient group of a complex semisimple algebraic group G by a Borel subgroup. A basis for $H^\bullet(X)$ indexed by the Weyl group W is known. Since the dimension $H^{2k(X)}$ is the number of elements in kth rank of the Weyl group, by the hard Lefschetz theorem, we obtain the unimodality and symmetricity of the rank sequence of the poset. Moreover, in this case, since the structure of the poset W is compatible with the ring structure of $H^\bullet(X)$, the bijectivity of $\times\omega^{d-k}$ for some element ω implies the bijection between kth rank and $d-k$th rank of the Weyl group such that each elements is comparable with the corresponding element. Hence there exist δ chains such that their union is w, where δ is the maximum of the rank sequence. Generally, for a ranked poset P with the rank sequence $r_0, r_1, \ldots, r_n$, it is known that

$$\max\{\, \#A \mid A \text{ is an antichain of } P \,\}$$

is equal to the minimum $d(P)$ of numbers m such that there exists m chains whose union is P. Since each rank is an antichain, we have $\max\{\, r_0, r_1, \ldots, r_n \,\} \le d(P)$. We say that a ranked poset P has the Sperner property if $d(P) = \max\{\, r_0, r_1, \ldots, r_n \,\}$. Moreover, for k, we say that P satisfies the property S_k if

$$\begin{aligned} &\max\{\, \#(A_1 \cup \cdots \cup A_k) \mid A_i \text{ is an antichain of } P \,\} \\ &\quad = \max\{\, r_{i_1} + \cdots + r_{i_k} \mid 0 \le i_1 < \cdots < i_k \le n \,\}. \end{aligned}$$

In [**22**], Stanley applies the hard Lefschetz theorem to obtain the following.

Theorem 2. *The poset of the Weyl group defined by the Bruhat order satisfies the property S_k for all k, and the rank sequence is unimodal and symmetric.*

The next example is a sequence coming from the face poset of a polytope. Let $\mathcal{P}$ be a d-dimensional polytope, and $f_i(\mathcal{P})$ the number of faces of dimension $i-1$, where $f_0(\mathcal{P}) = 1$. The face poset, i.e., the poset of all faces of $\mathcal{P}$ ordered by inclusion, is a ranked poset with the rank sequence $f_0(\mathcal{P}), f_1(\mathcal{P}), \ldots, f_d(\mathcal{P})$. In this case, we consider not the rank sequence but the sequence

$$h_0(\mathcal{P}), h_1(\mathcal{P}), \ldots, h_d(\mathcal{P})$$

defined from the rank sequence as follows:

$$h_i(\mathcal{P}) = \sum_{j=0}^{i} \binom{d-j}{d-i} (-1)^{i-j} f_j(\mathcal{P}).$$

We can also define $h_i(\mathcal{P})$ by the following equation for the generating functions:

$$\sum_{i=0}^{d} h_i(\mathcal{P}) x^{d-i} = \sum_{i=0}^{d} f_i(\mathcal{P})(x-1)^{d-i}.$$

In [**21**], it is shown that if $\mathcal{P}$ is a simplicial convex polytope, then $(h_0(\mathcal{P}), h_1(\mathcal{P}), \ldots, h_d(\mathcal{P}))$ is unimodal and symmetric. To show it, the hard Lefschetz theorem is used. Let X be a toric variety defined by a simplicial convex polytope $\mathcal{P}$. It is shown that we have a basis for $H^\bullet(X)$ indexed by faces of $\mathcal{P}$. By the hard Lefschetz theorem, we obtain the unimodality and symmetricity of the sequence when $\mathcal{P}$ is a simplicial convex polytope. In [**6**], it is shown that if $(h_0(\mathcal{P}), h_1(\mathcal{P}), \ldots, h_d(\mathcal{P}))$ is unimodal and symmetric, then $\mathcal{P}$ is a simplicial convex polytope. It is shown in a combinatorial way. To summarize, we obtain the following, known as g-theorem.

Theorem 3. *The sequence $(h_i(\mathcal{P}))_i$ is unimodal and symmetric if and only if $\mathcal{P}$ is a simplicial convex polytope.*

The next example is the rank sequence of a vector space lattice. For a vector space V over a finite field, the vector space lattice $\mathcal{L}(V)$ is the poset of all subspaces of V ordered by inclusion. The vector space lattice $\mathcal{L}(V)$ is a ranked poset, whose the kth rank $\mathcal{L}_k(V)$ consists of all k-dimensional subspaces of V. We consider the rank sequence of the vector space lattice. In this case, we do not consider the hard Lefschetz theorem for a cohomology ring but consider the strong Lefschetz property for a graded ring, a ring theoretical abstraction of the hard Lefschetz theorem. For a graded ring $A = \bigoplus_k^d A_k$, we say that A has the strong Lefschetz property if there exist $L \in A_1$ such that the linear map $\times L^{d-2k} : A_k \to A_{d-k}$ is bijective for $k = 1, 2, \ldots, \lfloor \frac{d}{2} \rfloor$. For a graded ring $A = \bigoplus_k^d A_k$ with the strong Lefschetz property, we have the sequence

$$\dim A_0, \dim A_1, \ldots, \dim A_d$$

of the dimensions. The bijectiveties of the linear maps $\times L^{d-2k}$ obtained by the strong Lefschetz property imply

- $\dim A_0 \leq \dim A_1 \leq \cdots \leq A_{d'}$,
- $\dim A_d \leq \dim A_{d-1} \leq \cdots \leq A_{d'}$,
- $\dim A_k = \dim A_{d-k}$ for all k,

where $d' = \lfloor \frac{d}{2} \rfloor$. In other words, the strong Lefschetz property induces the unimodality and symmetricity of the sequence. The unimodality and symmetricity can be shown by using the strong Lefschetz property in the same way as the hard Lefschetz theorem. Note that in the hard Lefschetz theorem, we need a manifold or a variety and their cohomology ring. In the strong Lefschetz property, we do not need to consider any manifolds or varieties but we have to show that the ring has the strong Lefschetz property. Now we return to the rank sequence of the vector space lattice of V. We associate a vector v with the variable x_v, and consider

$$F = \sum_{B \in \mathcal{B}} \prod_{v \in B} x_v,$$

where $\mathcal{B}$ is the set of all bases for V. Let $A = \mathbb{K}[x_v | v \in V] / \operatorname{Ann}(F)$, where $\operatorname{Ann}(F)$ is the annihilator ideal generated by the polynomials that annihilate F as the partial differential operators. In [**13**], it is shown that we have a basis for the graded algebra A indexed by subspaces of the vector space V, and that A has the strong Lefschetz property. Hence, we have the following.

Theorem 4. *The rank sequence of a vector space lattice is unimodal and symmetric.*

A matroid is a combinatorial generalization of the concept of the independency of a vector space. For a matroid M on the ground set E, we can define a flat of a subset of E which is analogue of a space generated by some vectors, an independent set which is analogue of a linearly independent set of vectors, a basis for a matroid which is analogue of the basis for a vector space, and so on. The lattice of flats of a matroid M ordered by inclusion is called a geometric lattice. A vector space lattice is a special case of the geometric lattices. We try to generalize the method in the case of a vector space lattice to geometric lattice. For a matroid M on E, we can define F_M and A_M similarly to the case of a vector space lattice as follows:

$$F_M = \sum_{B \in \mathcal{B}} \prod_{v \in B} x_v,$$
$$A_M = \mathbb{K}[x_e | e \in E] / \operatorname{Ann}(F_M).$$

In [**13**], for a geometric lattice $\mathcal{L}_M$ defined by a matroid M, it is shown that $\mathcal{L}_M$ is modular if and only if we have a basis for A_M indexed by $\mathcal{L}_M$. Moreover, it is also shown that A_M has the strong Lefschetz property for a modular geometric lattice $\mathcal{L}_M$.

Theorem 5. *The rank sequence of a modular geometric lattice is unimodal and symmetric.*

If a geometric lattice $\mathcal{L}_M$ is not modular, then the rank sequence is no longer symmetric. On the other hand, A_M has the symmetric sequence $\dim A_0, \dim A_1, \ldots, \dim A_r$. Thus, if $\mathcal{L}_M$ is not modular, then A_M does not have a basis indexed by $\mathcal{L}_M$. Hence, the strong Lefschetz property for A_M does not imply the unimodality nor symmetricity for the rank sequence of $\mathcal{L}_M$ when $\mathcal{L}_M$ is not modular. From algebraic interest, however, it remains the problem whether A_M has the strong Lefschetz property. The following is conjectured in [**13**].

Conjecture 6. For any matroid M, the algebra A_M has the strong Lefschetz property.

The poset $I(M)$ of independent sets of a matroid M ordered by inclusion is a ranked poset. Note that $I(M)$ is not a lattice while the poset $\mathcal{L}_M$ of flats is a lattice. We consider the rank sequence of $I(M)$. The rank sequence is log-concave. In particular, the sequence is unimodal. This was known as Mason and Welsh conjecture, and was proved recently. This is shown in [**1, 2, 3, 7, 8**]. In [**1, 2, 3**], Anari, Gharan, and Vinzant use the theory of log-concave polynomials. In [**7, 8**], Brändén and Huh use the hard Lefschetz theorem and the theory of Lorentzian polynomials.

Theorem 7. *The rank sequence of $I(M)$ for a matroid M is log-concave.*

To show that a sequence is unimodal and symmetric, there are various ways: the hard Lefschetz theorem, the strong Lefschetz property, log-concavity, and Lorentzian polynomials. In any cases, we can find the Hessian matrix

$$H_F = \left(\frac{\partial^2}{\partial x_i \partial x_j} F\right)_{1 \leq i,j \leq n}$$

or its analogue in each theory. We illustrate them below.

First, we see the Hessian matrix appearing in the theory of the log-concavity. Recall that a sequence $a_0, a_1, \ldots, a_n$ is log-concave if $a_i \geq a_{i-1}a_{i+1}$ for $1 \leq i \leq n-1$. For a log-concave sequence $a_0, a_1, \ldots, a_n$ of positive numbers, we have

$$\log a_i \geq \frac{\log a_{i-1} + \log a_{i+1}}{2}.$$

Hence, the function which maps $i \in \{\, 0, 1, \ldots, n \,\}$ to $\log a_i \in \mathbb{R}$ is concave. We generalize the log-concavity of a sequence to a polynomial function in multi variables. Let $F \in \mathbb{K}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree d. We say that F is log-concave at $\boldsymbol{a}$ if $\log F$ is concave at $\boldsymbol{a}$, and that F is strictly log-concave at $\boldsymbol{a}$ if $\log F$ is strictly concave at $\boldsymbol{a}$. A Lorentzian polynomial is a stronger property of a log-concave polynomial. We say that F is Lorentzian if

$$\left(\frac{\partial}{\partial x_1}\right)^{k_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{k_n} F$$

is identically zero or log-concave at $\boldsymbol{a}$ for any $(k_1, \ldots, k_n) \in \mathbb{Z}^n_{\geq 0}$ with $\sum_{i=1}^n k_i \leq \deg F - 2$. Recall that for a polynomial ϕ, the log-concavity of ϕ is equivalent to the negetive definiteness of the Hessian matrix H_ϕ. Hence,

$$\begin{aligned}(H_{\log F})_{ij} &= \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\log F \\ &= \frac{\partial}{\partial x_i}\left(\frac{1}{F}\frac{\partial F}{\partial x_j}\right) \\ &= \frac{1}{F^2}\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}F - \frac{\partial F}{\partial x_i}\frac{\partial F}{\partial x_j}\right) \\ &= -\frac{1}{F^2}\left(-FH_F + (\nabla F)(\nabla F)^\top\right),\end{aligned}$$

where ∇F is the gradient vector of F. The log-concavity tells us the signature of the Hessian matrix. To show it, we note the Cauchy's interlacing theorem: Let A be a real symmetric matrix of size $n \times n$ and $v \in \mathbb{R}$. Let $B = A + vv^\top$. Let $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$ be the eigenvalues of A and B, respectively. Then we have $\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \cdots \geq \beta_n$. If we take A and v as $-FH_F$ and ∇F, respectively, then we have the following.

Theorem 8. *For a homogeneous polynomial F of degree d in n variables. Let $\boldsymbol{a} \in \mathbb{R}^n$ satisfies $F(\boldsymbol{a}) > 0$.*

(1) *If F is log-concave at $\boldsymbol{a}$, then $H_F|_{\boldsymbol{x}=\boldsymbol{a}}$ has at most one positive eigenvalue.*
(2) *If F is strictly log-concave at $\boldsymbol{a}$, then $H_F|_{\boldsymbol{x}=\boldsymbol{a}}$ has exactly $n-1$ negative eigenvalues and exactly one positive eigenvalue.*

Next, we see an analogue of the Hessian matrices appearing in the theory of the strong Lefschetz property. Let $F \in \mathbb{K}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree d. We define A_F to be

$$\mathbb{K}[x_1, \ldots, x_n]/\operatorname{Ann}(F).$$

It is known that A_F is a graded Artinian Gorenstein algebra, i.e., a Poincaré duality algebra with $\dim A_k < \infty$ for all k. Once we have some compact Kähler manifold such that the cohomology ring is isomorphic to A_F, the hard Lefschetz theorem implies the strong Lefschetz property for A_F. In general, for the algebra A_F, there might not exist such manifolds. Hence, we have to show the strong Lefschetz property for A_F by another method. We have a method using an analogue of the Hessian matrix to show the strong Lefschetz property. Let A_k be the homogeneous spaces of A_F, and Λ_k a basis for A_k. We define the matrix $H_F^{(k)}$ by

$$H_F^{(k)} = \left(e_i\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)e_j\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)F\right)_{e_i, e_j \in \Lambda_k}.$$

We call $H_F^{(k)}$ the kth Hessian matrix of F with respect to the basis Λ_k. If $\{ x_1, \ldots, x_n \}$ is a basis for A_1, then the first Hessian matrix $H_F^{(1)}$ with respect to the basis is a usual Hessian matrix H_F. The nondegeneracy of the Hessian matrices $H_F^{(k)}$ implies the strong Lefschetz property of A_F.

Theorem 9. *The algebra A_F has the strong Lefschetz property if and only if* $\det H_F^{(k)} \not\equiv 0$ *for all k.*

Through of this thesis, we consider the following polynomials:
- the generating functions F_M for basis,
- the generating functions P_M for independent sets,
- the generating functions $\overline{P}_M$ for reduced independent sets

for a matroid M. The goal of this thesis is to study the strictly log-concavity of them and the strong Lefschetz property for the graded Artinian Gorenstein algebra defined by them. We study the properties by considering the Hessian matrices of them. As an application, we show Conjecture 6 for some special case. This thesis is based on the papers [**16, 17, 25, 26**], and the results in [**16, 17, 25, 26**] are Theorems 10 to 22. We illustrate them below.

A matroid is a generalization of the concept of the independency of a vector space. A matroid defined by some vectors is called a vector matroid. A vector matroid is one of important classes of matroids. We also have another important class of matroids, defined from graphs. If we think cycles in a graph as dependent sets, then a graph has the structure of a matroid. A matroid defined from a graph Γ is called the graphic matroid $M(\Gamma)$ of the graph. An independent set of the graphic matroid $M(\Gamma)$ of a graph Γ corresponds to a tree, i.e., a subgraph without cycles, in the graph Γ. For a connected graph Γ, a basis for the graphic matroid $M(\Gamma)$ corresponds to a spanning tree. In [**17, 25, 26**], the authors consider the Kirchhoff polynomial F_Γ, i.e., the generating functions $F_{M(\Gamma)}$ for the graphic matroid $M(\Gamma)$. For simplicity, we call the Hessian matrix of the Kirchhoff polynomial of a graph the Hessian matrix of the graph. In the case of the complete graphs and complete bipartite graphs, we can calculate the exact values of the eigenvalues of the Hessian matrices of the graphs at $\boldsymbol{x} = (1, 1, \ldots, 1)$, hence we have the signatures of the Hessian matrices of the graphs as Theorems 15 and 16. For any graph, we have the signature of the Hessian matrix on the positive orthant $\mathbb{R}^n_{>0}$ as Theorem 19.

To calculate the eigenvalues of the Hessian matrix of the complete graphs and complete bipartite graphs, we prepare formulas for the eigenvalues for some block matrix. If a cyclic group acts on a graph, then a matrix defined by the graph has a block decomposition such that each block is cyclic. The Hessian matrix of a graph has also such decomposition. In Theorems 10 to 12, as tools of calculation of the Hessian matrices of graphs, we give formulas for the determinants and the characteristic polynomials χ of three kinds of block matrices C, D and $M(A, \lambda, \boldsymbol{d})$. Let C be a block matrix of size $nl \times nl$ whose blocks are cyclic matrices of size $n \times n$. Since each block is a cyclic matrix of the same size $n \times n$, we have the common eigenvectors $\boldsymbol{z}_{n,k}$ for $k = 0, 1, \dots, n-1$. We obtain the matrices $\bar{C}_k$ of size $l \times l$ from the block matrix C and eigenvectors $\boldsymbol{z}_{n.k}$ of cyclic matrices. We can reduce the calculation of the determinant and the characteristic polynomial for C to the calculation of ones for smaller matrix $\bar{C}_k$ as Theorem 10. See Chapter 4 for the details.

Theorem 10 (Theorem 4.1 in Chapter 4)**.** *Let $(w_i)_{1\le i\le l} \in \mathbb{C}^l$ be an eigenvector of $\bar{C}_k$ belonging to the eigenvalue λ. Then $(w_i \boldsymbol{z}_{n,k})_{1\le i\le l} \in \mathbb{C}^{nl}$ is an eigenvector of C associated with λ. Hence*

$$\chi_C(t) = \prod_{k=0}^{n-1} \chi_{\bar{C}_k}(t),$$

$$\det C = \prod_{k=0}^{n-1} \det \bar{C}_k.$$

Let D be a block matrix $D = (D^{ij})_{1\le i,j\le l}$ such that D^{ij} is a cyclic matrix of size $2n \times 2n$ for $1 \le i, j \le l-1$, D^{ll} is a cyclic matrix of size $n \times n$, D^{il} is a vertical concatenation of a cyclic matrix of size $n \times n$ for $1 \le i \le l-1$, and D^{lj} is a horizontal concatenation of a cyclic matrix of size $n \times n$ for $1 \le j \le l-1$. Similarly to the block matrix C, we can reduce the calculation of the determinant and the characteristic polynomial for D to the calculation of ones for smaller matrix $\bar{D}_k$ obtained from the block matrix D and eigenvectors of each block as Theorem 11. See Chapter 4 for the details.

Theorem 11 (Theorem 4.2 in Chapter 4)**.** *The characteristic polynomial of D is*

$$\begin{aligned}\chi_D(t) &= \left(\prod_{k:even} \chi_{\bar{D}_k}(t)\right)\left(\prod_{k:odd} \frac{1}{t}\chi_{\bar{D}_k}(t)\right) \\ &= \frac{1}{t^n}\prod_{k=0}^{2n-1} \chi_{\bar{D}_k}(t).\end{aligned}$$

Let A be a square matrix of size l, $\boldsymbol{d} = (d_1, d_2, \dots, d_l)$, and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$. Consider a block matrices $M(A, \lambda, \boldsymbol{d})$ defined by

$$M(A, \lambda, \boldsymbol{d}) = (a_{ij} J_{d_i d_j})_{1\le i,j\le l} + \mathrm{Diag}(\lambda_1 I_{d_1}, \lambda_2 I_{d_2}, \dots, \lambda_l I_{d_l}),$$

where J_{mn} is the all one matrix of size $m \times n$, I_n is the identity matrix of size $n \times n$, $\mathrm{Diag}(A_1, \dots, A_l)$ is a block diagonal matrix with diagonal blocks A_i. In this case, we can reduce the calculation of the determinant and the characteristic polynomial for $M(A, \lambda, \boldsymbol{d})$ to the calculation of ones for smaller matrix $\bar{M}(A, \lambda, \boldsymbol{d})$ of size l defined by

$$\bar{M}(A, \lambda, \boldsymbol{d}) = \mathrm{diag}(d_1, \dots, d_l)A + \mathrm{diag}(\lambda_1, \dots, \lambda_l),$$

where $\operatorname{diag}(x_1, \ldots, x_l)$ is the diagonal matrix with entries $x_1, \ldots, x_l$. See Chapter 4 for the details.

Theorem 12 (Theorem 4.3 in Chapter 4)**.** *For a matrix A of size l, $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\boldsymbol{d} = (d_1, d_2, \ldots, d_l)$, we have*

$$\chi_{M(A,\lambda,\boldsymbol{d})}(t) = \chi_{\bar{M}(A,\lambda,\boldsymbol{d})}(t) \prod_{i=1}^{l} (t - \lambda_i)^{d_i - 1},$$

$$\det M(A, \lambda, \boldsymbol{d}) = \det \bar{M}(A, \lambda, \boldsymbol{d}) \prod_{i=1}^{l} \lambda_i^{d_i - 1}.$$

The Hessian matrices of the graphs at $\boldsymbol{x} = (1, 1, \ldots, 1)$ are matrices indexed by the edge sets of the graphs such that the entries depend on how to connect edges in the graphs. Consider the matrix $H = (h_{ee'})$ indexed by the edge set of the complete graph such that $h_{ee'} = \alpha$ if $e = e'$, $h_{ee'} = \beta$ if e and e' share one vertex, and $h_{ee'} = \gamma$ if e and e' do not share any vertices. Since the cyclic group of order n acts on the complete graph K_n with n vertices, the matrix H has a block decomposition with cyclic blocks. Thanks to Theorems 10 and 11, we can calculate the eigenvalues of the matrix H.

Theorem 13 (Theorem 4.4 in Chapter 4)**.** *The eigenvalues of H are*

$$\begin{aligned}
\lambda_1 &= \alpha + (2n-4)\beta + \frac{(n-2)(n-3)}{2}\gamma, \\
\lambda_2 &= \alpha - 2\beta + \gamma, \\
\lambda_3 &= \alpha + (n-4)\beta - (n-3)\gamma.
\end{aligned}$$

The dimensions d_λ of the eigenspaces of H associate with the eigenvalues λ are

$$d_{\lambda_1} = 1, \qquad d_{\lambda_2} = \binom{n}{2} - n, \qquad d_{\lambda_3} = n - 1.$$

Consider the matrix $H' = (h_{ee'})$ indexed by the edge set of the complete bipartite graph $K_{X,Y}$ such that $h_{ee'} = \alpha$ if $e = e'$, $h_{ee'} = \beta$ if e and e' share one vertex in X, $h_{ee'} = \gamma$ if e and e' share one vertex in Y, and $h_{ee'} = \delta$ if e and e' do not share any vertices. Similarly to the case of the complete graphs, we can decompose the matrix into blocks by a group action. Thanks to Theorem 10, we can calculate the eigenvalues of the matrix H'.

Theorem 14 (Theorem 4.5 in Chapter 4)**.** *The eigenvalues of H' are*

$$\begin{aligned}
\lambda_1 &= \alpha + (n-1)\beta + (m-1)\gamma + (m-1)(n-1)\delta, \\
\lambda_2 &= \alpha + (n-1)\beta - \gamma - (n-1)\delta, \\
\lambda_3 &= \alpha - \beta + (m-1)\gamma - (m-1)\delta, \\
\lambda_4 &= \alpha - \beta - \gamma + \delta.
\end{aligned}$$

The dimensions d_λ of the eigenspaces of H' associate with the eigenvalues λ are

$$d_{\lambda_1} = 1, \quad d_{\lambda_2} = m-1, \quad d_{\lambda_3} = n-1, \quad d_{\lambda_4} = (m-1)(n-1).$$

The diagonal entries in the Hessian matrices $H_{F_\Gamma}|_{\boldsymbol{x}=(1,1,\ldots,1)}$ are zero, and the (e,e')-entries are the numbers of spanning trees containing edges e and e'. A formula to calculate the number of spanning trees from a matrix called the Laplacian matrix of the graph is known as the Matrix-Tree theorem. In the case of the complete graph K_n, the Hessian matrix is H for $\alpha = 0$, $\beta = 3n^{n-4}$, and $\gamma = 4n^{n-4}$. Hence, we obtain the eigenvalues of the Hessian matrix of the complete graph by Theorem 13. Therefore, we have the signature of the Hessian matrix of the graph.

Theorem 15 (Corollary 5.5 in Chapter 5)**.** *The Hessian of the Kirchhoff polynomial of the complete graph K_n does not vanish for $n \geq 3$. Moreover, the matrix evaluated at $x_e = 1$ for all e has exactly one positive eigenvalue.*

In the case of the complete bipartite graphs $K_{X,Y}$ with $\#X = m$ and $\#Y = n$, the Hessian matrix is H' for $\alpha = 0$, $\beta = n(2m+n-2)$, $\gamma = m(2n+m-2)$, and $\delta = (m+n)(m+n-2)$. Hence, we obtain the eigenvalues of the Hessian matrix of the complete bipartite graph by Theorem 14. Therefore, we have the signature of the Hessian matrix of the graph.

Theorem 16 (Corollary 5.7 in Chapter 5)**.** *The Hessian of the Kirchhoff polynomial of the complete bipartite graph does not vanish for $\#X \geq 2$ and $\#Y \geq 2$. Moreover, the matrix evaluated at $x_e = 1$ for all e has exactly one positive eigenvalue.*

For the generating function $F_{K_n,k}$ for forests with k components in K_n, a generalization of the Kirchhoff polynomial of K_n, we can define the Hessian matrix of F similarly to the Hessian matrix of K_n. For the generating function $F_{K_{X,Y},k}$ for forests with k components in $K_{X,Y}$, a generalization of the Kirchhoff polynomial of $K_{X,Y}$, we can define the Hessian matrix of F similarly to the Hessian matrix of $K_{X,Y}$. The (e,e')-entries of the Hessian matrix is the number of forests containing edges e and e'. To calculate it by the Matrix-Tree theorem, Theorem 12 is used. Moreover in this case, we also calculate the eigenvalues the Hessian matrices by Theorems 13 and 14.

Theorem 17 (Theorem 5.4 in Chapter 5)**.** *Let $n \geq 3$ and $0 < k \leq n-2$. The Hessian does not vanish. Moreover, the matrix $H_{F_{K_n,k}}|_{\boldsymbol{x}=(1,1,\ldots,1)}$ has exactly one positive eigenvalue.*

Theorem 18 (Theorem 5.6 in Chapter 5)**.** *Consider sets X and Y such that $X \cap Y = \emptyset$, $\#X \geq 2$ and $\#Y \geq 2$. For $0 < k \leq \#X + \#Y - 2$,* *the Hessian does not vanish. Moreover, the matrix* $H_{F_{K_{X,Y},k}}|_{\boldsymbol{x}=(1,1,\ldots,1)}$ *has exactly one positive eigenvalue.*

For the Hessian matrices of the other graphs, it is difficult to calculate the eigenvalues of its Hessian. We have the following result. We can obtain Theorem 19 as a special case of Theorem 20. However, independently of Theorem 20, Theorem 19 is shown by theory of relative invariants of prehomogeneous vector spaces. See for Chapter 5.

Theorem 19 (Theorem 5.9 in Chapter 5)**.** *The Hessian of a graph does not vanish. Moreover, the matrix has exactly one positive eigenvalue.*

In [**16**], the authors consider the polynomials F_M, P_M, and $\overline{P}_M$ for a matroid M. The Kirchhoff polynomial of a graph is a basis generating polynomial for the graphic matroid of the graph. We consider the basis generating polynomials F_M, not only for graphic matroids, but also for all matroids M. We also consider the other two types of generating polynomials, called the independent set generating polynomials P_M and reduced independent set generating polynomials $\overline{P}_M$. Similarly to the case of the Kirchhoff polynomials, we have the following for the polynomials F_M, P_M and $\overline{P}_M$.

Theorem 20 (Theorem 6.1 in Chapter 6)**.** *Let M be a simple matroid on $[n]$ of rank $r \geq 2$. Then, we have*

(1) *The Hessian matrix of F_M evaluated $\boldsymbol{a} \in \mathbb{R}^n_{>0}$ has exactly one positive eigenvalue. Moreover, the Hessian does not vanish.*
(2) *The Hessian of P_M evaluated $(0, \boldsymbol{a}) \in \{ 0 \} \times \mathbb{R}^n_{>0}$ is zero.*
(3) *If M is not a uniform matroid, then the Hessian matrix of $\overline{P}_M$ evaluated $\boldsymbol{a} \in \mathbb{R}^{n+1}_{>0}$ has exactly one positive eigenvalue. Moreover, the Hessian does not vanish.*

We apply Theorem 20 to theory of log-concavity and the strong Lefschetz property. By Theorem 20, we obtain that the polynomial F_M and $\overline{P}_M$ is strictly log-concave on the positive orthant. To show Theorem 7, the log-concavity of F_M and $\overline{P}_M$ are shown in [**1, 2, 3, 7, 8**]. In [**16**], it is shown the strictness of the log-concavity of F_M and $\overline{P}_M$ as follows.

Theorem 21 (Theorem 6.3 in Chapter 6)**.** *Let M be a simple matroid on $[n]$ of rank $r \geq 2$. Then, we have*

(1) *The polynomial F_M is strictly log-concave at $\boldsymbol{a} \in \mathbb{R}^n_{>0}$.*
(2) *If M is not a uniform matroid, then the polynomial $\overline{P}_M$ is strictly log-concave at $\boldsymbol{a} \in \mathbb{R}^{n+1}_{>0}$.*

By Theorem 20, we obtain algebraic properties for the graded Artinian Gorenstein algebras A_{F_M} and $A_{\overline{P}_M}$ defined by F_M and $\overline{P}_M$ as Theorem 22. In particular, A_{F_M} and $A_{\overline{P}_M}$ have the strong Lefschetz property at degree one. By Theorem 22, Conjecture 6 holds for some special case.

Theorem 22 (Theorem 6.5 in Chapter 6)**.** *Let* $L = a_1x_1 + a_2x_2 + \cdots a_nx_n$ *and* $L' = a_0x_0 + L$*, where* $a_i > 0$ *for all* i*. For a simple matroid* M *on* $[n]$ *with rank* $r \geq 2$*, we have the following.*

(1) A_{F_M} *has the strong Lefschetz property at degree one with Lefschetz element* L*, and* A_{F_M} *satisfies the Hodge–Riemann relation at degree one with respect to* L*.*
(2) A_{P_M} *does not satisfy the Hodge–Riemann relation at degree one with respect to* L*.*
(3) *If* M *is not uniform matroid, then* $A_{\overline{P}_M}$ *has the strong Lefschetz property at degree one with Lefschetz element* L'*, and* $A_{\overline{P}_M}$ *satisfies the Hodge–Riemann relation at degree one with respect to* L'*.*

This thesis is organized as follows: In Chapter 1, we study the Hessian matrix relating to four topics, the log-concavity, the prehomogeneous vector spaces, the strong Lefschetz property, and the Lorentzian polynomials. In particular, we study the relation between the eigenvalues of the Hessian matrices and log-concavity, and study the form of Hessians for some functions, called the relative invariants, and study how the higher Hessians determine the strong Lefschetz property. Moreover we consider the Lorentzian polynomial, the Hodge–Riemann relation, and a relation between the strong Lefschetz property and them. In Chapter 2, first we recall definitions of graphs. We mainly study counting spanning trees or forests in a graph by using some matrices. Then, we study the generating polynomial for the spanning trees, called the Kirchhoff polynomial. In Chapter 3, first we recall definitions and example of matroids. We mainly study the generating polynomials for bases and independent sets of matroids, which are a generalization of the Kirchhoff polynomials. In Chapter 4, we study cyclic matrices, particularly, block cyclic matrices. We give formulas to calculate the eigenvalues, eigenvectors, and determinants. Moreover, we show the eigenvalues, eigenvectors, and determinants for several typical block cyclic matrices by using the formulas. Then, we calculate a block cyclic matrix arising from graphs. This section is based on the papers [**25, 26**]. The details are omitted. See [**25, 26**] for the details. In Chapter 5, we calculate the Hessian matrices of generating polynomials for forests, that polynomials are a generalization of the Kirchhoff polynomials. We use the theory of graphs, log-concavity, and prehomogeneous vector spaces. This section is based on the papers [**17, 25**]. The details are omitted. See [**17, 25**] for the details. In Chapter 6, we study the Hessian matrices of the generating polynomials of matroids defined in Chapter 3, the strictly log-concavity of the generating polynomials, and graded Artinian Gorenstein algebras defined in Chapter 1. This section is based on the paper [**16**]. The details are omitted. See [**16**] for the details.

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Contents

CHAPTER 1

Hessian matrices and higher Hessian matrices

As stated in Introduction, the goal of this thesis is to study the log-concavity of some polynomials and the strong Lefschetz property for the graded Artinian Gorenstein algebra defined by the polynomials by considering the Hessian matrices. In this chapter, we consider the Hessian matrices and their properties. Moreover, we consider the Hessian matrices and two more topics to use to study the log-concavity and the strong Lefschetz property.

Let us consider a homogeneous polynomial F of degree $r \geq 2$ in n variables with real coefficients. Let

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}, \qquad \nabla F = \begin{pmatrix} \frac{\partial}{\partial x_1} F \\ \vdots \\ \frac{\partial}{\partial x_n} F \end{pmatrix},$$

and we define

$$H_F = \nabla \nabla^\top F = \left(\frac{\partial^2}{\partial x_i \partial x_j} F \right)_{i,j}.$$

We call H_F and $\det H_F$ the *Hessian matrix* and *Hessian* of F, respectively.

In this chapter, we study Hessian matrices and higher Hessian matrices related to four topics. The polynomial $F = x_1 x_2 x_3 x_4$ is an example through this chapter as Examples 1.6, 1.9, 1.17 and 1.21.

1. Hessian matrices and log-concavity

In this section, we study relations between Hessian matrices and log-concavity. Let

$$G_F = (\nabla F)(\nabla F)^\top = \left(\frac{\partial F}{\partial x_i} \frac{\partial F}{\partial x_j} \right)_{i,j}.$$

We say that F is *log-concave* at $\boldsymbol{a} \in \mathbb{R}^n$ if

$$(-F H_F + G_F)\big|_{\boldsymbol{x}=\boldsymbol{a}}$$

is positive semidefinite. For an open convex set $X \subset \mathbb{R}^n$, we say that F is *log-concave* on X if F is log-concave for all $\boldsymbol{a} \in X$. We say that F is *strictly log-concave* at $\boldsymbol{a} \in \mathbb{R}^n$ if

$$(-F H_F + G_F)\big|_{\boldsymbol{x}=\boldsymbol{a}}$$

is positive definite. For an open convex set $X \subset \mathbb{R}^n$, we say that F is *strictly log-concave* on X if F is strictly log-concave for all $\boldsymbol{a} \in X$.

Remark 1.1. A polynomial F is log-concave if and only if $\log F$ is concave. Hence, the original definition is the following: A polynomial F is log-concave if for $v, v' \in \mathbb{R}^n_{\geq 0}$ and $\lambda \in [0,1]$, we have

$$F(\lambda v + (1-\lambda)v') \geq F(v)^{\lambda} F(v')^{(1-\lambda)}.$$

A polynomial F is strictly log-concave if for $v, v' \in \mathbb{R}^n_{\geq 0}$ and $\lambda \in (0,1)$, we have

$$F(\lambda v + (1-\lambda)v') > F(v)^{\lambda} F(v')^{(1-\lambda)}.$$

Recall that a polynomial (or a function) φ is concave if and only if the Hessian matrix H_φ of φ is negative semidefinite. Hence, a polynomial F is log-concave if and only if $H_{\log F}$ is negative semidefinite. Indeed,

$$\begin{aligned}(H_{\log F})_{ij} &= \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\log F \\ &= \frac{\partial}{\partial x_i}\left(\frac{1}{F}\frac{\partial F}{\partial x_j}\right) \\ &= \frac{1}{F^2}\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}F - \frac{\partial F}{\partial x_i}\frac{\partial F}{\partial x_j}\right).\end{aligned}$$

Hence, if $F \neq 0$, then

$$F^2(H_{\log_F}) = -\left(-FH_F + G_F\right).$$

Hence, the positive definiteness of $-FH_F + G_F$ means the negative definiteness of the Hessian matrix of $\log F$. Then so is the log-concavity of F.

One might think that the definition of the log-concavity of a polynomial F using the matrix $-FH_F + G_F$ is strange. The definition is useful to consider the signature of the Hessian matrix H_F of F by using, so-called, Cauchy's interlacing theorem. Roughly speaking,

$$\begin{aligned}&\text{Cauchy's interlacing theorem} + \text{log-concavity of } F \\ &\qquad\qquad \Longrightarrow \text{signature of } H_F.\end{aligned}$$

Proposition 1.2 (Cauchy's interlacing theorem [**12**]). *Let $v \in \mathbb{R}^n$ be a column vector, and A a real symmetric square matrix of size n with eigenvalues $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$. Let $B = A + vv^\top$ with eigenvalues $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$. Then, we have*

$$\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq \beta_n.$$

By Cauchy's interlacing theorem, the log-concavity tells us the signature of the Hessian matrix.

Proposition 1.3. *Let $\boldsymbol{a} \in \mathbb{R}^n$ satisfies $F(\boldsymbol{a}) > 0$.*

(1) *If F is log-concave at $\boldsymbol{a}$, then $H_F|_{\boldsymbol{x}=\boldsymbol{a}}$ has at most one positive eigenvalue.*
(2) *If F is strictly log-concave at $\boldsymbol{a}$, then $H_F|_{\boldsymbol{x}=\boldsymbol{a}}$ has exactly $n-1$ negative eigenvalues and exactly one positive eigenvalue.*

We illustrate Proposition 1.3 with the polynomial $F = x_1x_2x_3x_4$.

Example 1.4. Let $F = x_1x_2x_3x_4$. Then, we have

$$H_F = \begin{pmatrix} 0 & x_3x_4 & x_2x_4 & x_2x_3 \\ x_3x_4 & 0 & x_1x_4 & x_1x_3 \\ x_2x_4 & x_1x_4 & 0 & x_1x_2 \\ x_2x_3 & x_1x_3 & x_1x_2 & 0 \end{pmatrix},$$

$$G_F = \begin{pmatrix} x_2^2x_3^2x_4^2 & x_1x_2x_3^2x_4^2 & x_1x_2^2x_3x_4^2 & x_1x_2^2x_3^2x_4 \\ x_1x_2x_3^2x_4^2 & x_1^2x_3^2x_4^2 & x_1^2x_2x_3x_4^2 & x_1^2x_2x_3^2x_4 \\ x_1x_2^2x_3x_4^2 & x_1^2x_2x_3x_4^2 & x_1^2x_2^2x_4^2 & x_1^2x_2^2x_3x_4 \\ x_1x_2^2x_3^2x_4 & x_1^2x_2x_3^2x_4 & x_1^2x_2^2x_3x_4 & x_1^2x_2^2x_3^2 \end{pmatrix}.$$

Hence,

$$-FH_F + G_F = \begin{pmatrix} x_2^2x_3^2x_4^2 & 0 & 0 & 0 \\ 0 & x_1^2x_3^2x_4^2 & 0 & 0 \\ 0 & 0 & x_1^2x_2^2x_4^2 & 0 \\ 0 & 0 & 0 & x_1^2x_2^2x_3^2 \end{pmatrix}.$$

For $x_i \in \mathbb{R}$, the matrix $-FH_F + G_F$ is positive semi-definite. Hence, F is log-concave on $\mathbb{R}^4$. If there exists i such that $x_i = 0$, the monomial F is not strictly log-concave.

Let $x_1 = x_2 = 0$ and $x_3 = x_4 = 1$. Then

$$H_F|_{\boldsymbol{x}=(0,0,1,1)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues are $1, 0, 0, -1$. We can verify that $F = x_1x_2x_3x_4$ is log-concave at $\boldsymbol{a} = (0,0,1,1)$, and $H_F|_{\boldsymbol{x}=(0,0,1,1)}$ has at most one positive eigenvalue.

Let $\boldsymbol{a} \in \mathbb{R}^4_{>0}$. Then, the matrix $-FH_F + G_F$ is positive definite. Hence F is strictly log-concave at $\boldsymbol{a}$. For example, we consider the case where $\boldsymbol{a} = (1,1,1,1)$. Then

$$H_F|_{\boldsymbol{x}=(1,1,1,1)} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

The eigenvalues are $3, -1, -1, -1$. Moreover, it follows from the continuity of the determinants that $H_F|_{\boldsymbol{x}=\boldsymbol{a}}$ for $\boldsymbol{a} \in \mathbb{R}^4_{>0}$ has exactly three negative eigenvalues and exactly one positive eigenvalue. We can verify that $F = x_1x_2x_3x_4$ is strictly log-concave at $\boldsymbol{a} \in \mathbb{R}^4_{>0}$, and $H_F|_{\boldsymbol{x}=\boldsymbol{a}}$ for $\boldsymbol{a} \in \mathbb{R}^4_{>0}$ has exactly one positive eigenvalue.

By Proposition 1.3, the log-concavity of F and the degeneracy of H_F imply the strictly log-concavity of F. Moreover, the calculation of $\det(-FH_F + G_F)$ reduces to calculation of $\det H_F$.

Proposition 1.5. *For a homogeneous polynomial F of degree r in n variables, we have*

$$\det\left(-FH_F + G_F\right) = \frac{(-1)^{n-1}}{r-1} F^n \det H_F.$$

The proof of Propositions 1.3 and 1.5 are in [**17**]. We illustrate Proposition 1.3 with the polynomial $F = x_1x_2x_3x_4$.

Example 1.6. Let $F = x_1x_2x_3x_4$. The degree r of F is 4, and the number n of variables of F is 4. Thus, it follows from Proposition 1.5 that

$$\det\left(-FH_F + G_F\right) = -\frac{1}{3}F^4 \det H_F.$$

In fact, we have

$$\begin{aligned}\det(-FH_F + G_F) &= \det\begin{pmatrix} x_2^2x_3^2x_4^2 & 0 & 0 & 0 \\ 0 & x_1^2x_3^2x_4^2 & 0 & 0 \\ 0 & 0 & x_1^2x_2^2x_4^2 & 0 \\ 0 & 0 & 0 & x_1^2x_2^2x_3^2 \end{pmatrix} \\ &= x_1^6x_2^6x_3^6x_4^6 \\ &= F^6.\end{aligned}$$

On the other hand, let us calculate $\det H_F$. For $H_F = (h_{ij})$ and $\sigma \in S_4$ such that $i = \sigma(i)$ for some i, we have

$$\prod_{i=1}^{4} h_{i\sigma(i)} = 0.$$

For $\sigma \in S_4$ such that $i \neq \sigma(i)$ for any i, we have

$$\begin{aligned}\prod_{i=1}^{4} h_{i\sigma(i)} &= \prod_{i=1}^{4} \frac{F}{x_i x_{\sigma(i)}} \\ &= \frac{\prod_{i=1}^4 F}{\prod_{i=1}^4 x_i \prod_{i=1}^4 x_{\sigma(i)}} \\ &= \frac{F^4}{F \cdot F} \\ &= F^2.\end{aligned}$$

The permutations $[\sigma(1), \sigma(2), \sigma(3), \sigma(4)]$ such that $i \neq \sigma(i)$ for any i are the following:

(1) $[2,1,4,3],[3,4,1,2],[4,3,2,1],$

(2) $[2,3,4,1],[2,4,1,3],[3,1,4,2],[3,4,2,1],[4,1,2,3],[4,3,1,2].$

The signature of the permutations in (1) is 1. The signature of the permutations in (2) is -1. Hence, the Hessian $\det H_F$ is equal to $-3F^2$. Therefore,

$$-\frac{1}{3}F^4 \det H_F = -\frac{1}{3}F^4(-3F^2) = F^6.$$

2. Hessian matrices and prehomogeneous vector spaces

In this section, for a special polynomial F called a relative invariant, we show the following identity

$$\det H_F = c' F^{\frac{n(r-2)}{r}}, \tag{3}$$

where c' is non-zero. In other words, the Hessian of F can be realize as a power of F. To prove it, we recall the notion of prehomogeneous vector spaces developed by Kimura and Sato [**20**] and many authors.

Let G be a connected linear algebraic group over $\mathbb{C}$, V a finite dimensional vector space over $\mathbb{C}$, and ρ a $\mathbb{C}$-rational representation of G on V. We call the triplet (G, ρ, V) a *prehomogeneous vector space* with *singular set* S if S is a proper algebraic G-invariant subset of V and $V \setminus S$ is a single G-orbit. Let (G, ρ, V) be a prehomogeneous vector space. We say that (G, ρ, V) is *irreducible* if ρ is an irreducible representation. Let F be a rational function F from V to $\mathbb{C}$. A not identically zero function F is called a *relative invariant (with respect to* χ*) of* (G, ρ, V) if there exists a rational character $\chi \in \mathrm{Hom}(G, \mathbb{C}^*)$ such that

$$F(\rho(g)\boldsymbol{x}) = \chi(g)F(\boldsymbol{x}) \quad (g \in G, \boldsymbol{x} \in V).$$

It is known that an irreducible prehomogeneous vector space (G, ρ, V) has at most one irreducible relative invariant F up to constant multiple. In particular, any relative invariant is in the form of cF^m for $c \in \mathbb{C}$ and $m \in \mathbb{Z}$. We call F the relative invariant of (G, ρ, V).

The Hessian of any relative invariant is also a relative invariant. The proof is in [**20**].

Proposition 1.7. *Let* (G, ρ, V) *be a prehomogeneous vector space of dimension* n. *If* F *is a relative invariant corresponding to a character* χ, *then* $\det H_F$ *is a relative invariant corresponding to the character* $\tilde{\chi}$, *where*

$$\begin{aligned} \tilde{\chi} : G &\to \mathbb{C}^* \\ g &\mapsto (\chi(g))^n \det(\rho(g))^{-2}. \end{aligned}$$

In other words,

$$\det H_F(\rho(g)\boldsymbol{x}) = (\chi^n \det{}^{-2})(g) \det H_F(\boldsymbol{x}).$$

We say that a prehomogeneous vector space (G, ρ, V) is *regular* if there exists a relative invariant F such that its Hessian $\det H_F$ is not identically zero on V. Then by Proposition 1.7, we have the following.

Proposition 1.8. *Let (G, ρ, V) be a regular irreducible prehomogeneous vector space of dimension n. Assume that the degree of the relative invariant F is r. Then, the Hessian of F is in the form of*

$$\det H_F = cF^{\frac{n(r-2)}{r}},$$

where $c \in \mathbb{C}^*$ is a constant.

Proposition 1.8 is the key of the proof of Theorem 5.9, which is the strictly log-concavity of the Kirchhoff polynomials of graphs.

In the following, we see a prehomogeneous vector space such that $F = x_1x_2x_3x_4$ is a relative invariant.

Example 1.9. Let $G = (\mathbb{C}^*)^4$ and $V = \mathbb{C}^4$. We define $\rho : G \to GL(4, \mathbb{C})$ by

$$\rho(a, b, c, d) = \mathrm{diag}(a, b, c, d),$$

where $\mathrm{diag}(x_1, \ldots, x_n)$ is the diagonal matrix with entries $x_1, \ldots, x_n$. For $C \subset \{1, 2, 3, 4\}$, define

$$V_C = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \;\middle|\; x_i \neq 0 \iff i \in C \right\}.$$

The G-orbits are

$$\begin{aligned}
&V_{\{1,2,3,4\}},\\
&V_{\{2,3,4\}},\ V_{\{1,3,4\}},\ V_{\{1,2,4\}},\ V_{\{1,2,3\}},\\
&V_{\{3,4\}},\ V_{\{2,4\}},\ V_{\{2,3\}},\ V_{\{1,4\}},\ V_{\{1,3\}},\ V_{\{1,2\}},\\
&V_{\{4\}},\ V_{\{3\}},\ V_{\{2\}},\ V_{\{1\}},\\
&V_{\emptyset}.
\end{aligned}$$

If $C' \subset C$, then $V_{C'} \subset \overline{V_C}$. The orbit containing

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

is $(\mathbb{C}^*)^4$, whose closure $\overline{(\mathbb{C}^*)^4}$ is $V = \mathbb{C}^4$. Hence, the triplet (G, ρ, V) is a prehomogeneous vector space with singular set

$$\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \middle| \; xyzw = 0 \right\} = V_{\{2,3,4\}} \cup V_{\{1,3,4\}} \cup V_{\{1,2,4\}} \cup V_{\{1,2,3\}}.$$

Let $F : V \to \mathbb{C}$ be a map such that

$$F(\boldsymbol{x}) = x_1 x_2 x_3 x_4.$$

Define $\chi : G \to \mathbb{C}^*$ to be $\chi(\boldsymbol{a}) = a_1 a_2 a_3 a_4$ for $\boldsymbol{a} \in (\mathbb{C}^*)^4$. Then, $\chi = \det \circ \rho \in \mathrm{Hom}(G, \mathbb{C}^*)$. For $\boldsymbol{x} \in V$ and $\boldsymbol{a} \in G$, we have

$$\begin{aligned} F(\rho(\boldsymbol{a})\boldsymbol{x}) &= (a_1 x_1)(a_2 x_2)(a_3 x_3)(a_4 x_4) \\ &= a_1 a_2 a_3 a_4 x_1 x_2 x_3 x_4 \\ &= \chi(\boldsymbol{a}) F(\boldsymbol{x}). \end{aligned}$$

Therefore, $F = x_1 x_2 x_3 x_4$ is a relative invariant with respect to χ. In this case, the triplet (G, ρ, V) is not irreducible. In spite of the fact, as in Example 1.6, the Hessian $\det H_F$ satisfies

$$\det H_F = -3F^2 = -3F^{\frac{4(4-2)}{4}},$$

which appears in Proposition 1.8.

3. Higher Hessian matrices and the strong Lefschetz property

In this section, we study higher Hessian matrices, a generalizing of Hessian matrices, regarding to the strong Lefschetz property. The strong Lefschetz property is a ring theoretical abstraction of the hard Lefschetz theorem. The details are in [**11**].

We define higher Hessian matrices from Gorenstein algebras over a field $\mathbb{K}$ of characteristic zero. The fundamental, for example the definition, of Gorenstein algebras is omitted here. See, e.g., [**15**] for the details. We, however, collect the facts which is used in our argument.

Let $A = \bigoplus_{k=0}^{s} A_k$ be a graded Artinian ring over $\mathbb{K}$. We say that A has *standard grading* if A_1 generates A as an algebra. Through this thesis, we assume that a graded algebra has standard grading. We say that A is a *Poincaré duality algebra* with socle degree s if $A_s \cong \mathbb{K}$ and the higher pairing induced by the multiplication map $A_k \times A_{s-k} \to A_s$ is nondegenerate for all k. The map $A_k \times A_{s-k} \to A_s$ is called *Poincaré duality*. The following are known facts at least for experts. See, e.g., [**13**] for the details.

Proposition 1.10. *Let $A = \bigoplus_{k=0}^{s} A_k$ be a graded Artinian ring over $\mathbb{K}$. The algebra A is a Poincaré duality algebra if and only if A is Gorenstein.*

Proposition 1.11. *Let $A = \bigoplus_{k=0}^{s} A_k$ be a graded Artinian ring over $\mathbb{K}$. The algebra A is a standard grading Gorenstein algebra with $\mathbb{K}$ of characteristic zero if and only if there exists a homogeneous polynomial F such that*

$$A \simeq \mathbb{K}[x_1, \dots, x_n]/\operatorname{Ann}(F),$$

where

$$\operatorname{Ann}(F) = \left\{ P \in \mathbb{K}[x_1, \dots, x_n] \;\middle|\; P\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) F = 0 \right\}.$$

Let $F \in \mathbb{K}[x_1, \dots, x_n]$ be a hompgeneous polynomial of degree s. We define A_F to be

$$\mathbb{K}[x_1, \dots, x_n]/\operatorname{Ann}(F).$$

Obviously, the socle degree of A_F is s. Let A_k be the homogeneous spaces of A_F, and Λ_k a basis for A_k. We define the matrix $H_F^{(k)}$ by

$$H_F^{(k)} = \left(e_i\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) e_j\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) F \right)_{e_i, e_j \in \Lambda_k}.$$

We call $H_F^{(k)}$ the *kth Hessian matrix* of F with respect to the basis Λ_k. The determinant of $H_F^{(k)}$ is called the kth *Hessian* of F.

Remark 1.12. For an algebra $A = \mathbb{K}[x_1, \dots, x_n]/\operatorname{Ann}(F)$, if we can take a basis $\{ x_1, \dots, x_n \}$ for A_1, then the first Hessian matrix $H_F^{(1)}$ with respect to the basis is a usual Hessian matrix H_F.

For a graded $\mathbb{K}$-algebra $A = \bigoplus_{k=0}^{s} A_k$ with $A_0 = \mathbb{K}$, the *Hilbert function* of A is the map

$$k \mapsto h_k := \dim_{\mathbb{K}} A_k.$$

The Hilbert function is denoted as the vector $(h_0, h_1, \dots, h_s)$. We say that the Hilbert function is *unimodal* if there exist i such that

$$h_0 \le h_1 \le \cdots \le h_i \ge h_{i+1} \ge \cdots \ge h_s.$$

We say that the Hilbert function is *symmetric* or *palindromic* if

$$h_k = h_{s-k}$$

for $k \in \left\{ 0, 1, \dots, \frac{s}{2} \right\}$. If A is Gorenstein, then the Hilbert function of A is symmetric.

Next, we recall the strong Lefschetz property for graded Artinian algebra over a field of characteristic zero. The strong Lefschetz property stems from the Hard Lefschetz theorem: Let (X, ω) be a compact Kähler manifold of $\dim_{\mathbb{C}} X = d$ with a Kähler form ω, and $H^{\bullet}(X, \mathbb{C})$ the cohomology ring of X. Then, for $k \in \{ 0, 1, \dots, d \}$, the map $[\omega]^{d-k} : H^k(M, \omega) \to H^{2d-k}(M, \omega)$ is a linear isomorphism. The strong Lefschetz property is a generalization of the concept of the cohomology ring of a compact Kähler manifold. Let $A = \bigoplus_{k=0}^{s} A_k$, $A_s \neq \mathbf{0}$, be a graded Artinian $\mathbb{K}$-algebra over a field $A_0 = \mathbb{K}$ of characteristic zero. We say that A has the *strong Lefschetz property* if there exists an element $L \in A_1$ such that the multiplication map $\times L^{s-2k} \colon A_k \to A_{s-k}$ is bijective for $k \in \left\{ 0, 1, \ldots, \frac{s}{2} \right\}$. We call L a *Lefschetz element* with this property. We say that A has the *strong Lefschetz property at degree* k if there exists an element $L \in A_1$ such that the multiplication map $\times L^{s-2k} \colon A_k \to A_{s-k}$ is bijective.

By definition of the strong Lefschetz property, we have the following.

Proposition 1.13. *If A has the strong Lefschetz property, then the Hilbert function of A is unimodal and symmetric.*

The following is a criterion for the strong Lefschetz property for a graded Artinian Gorenstein algebra by [**14, 24**]

Proposition 1.14 (Watanabe [**24**], Maeno–Watanabe [**14**])**.** *Consider a graded Artinian Gorenstein algebra A_F. The algebra A_F has the strong Lefschetz property at degree k if and only if* $\det H_F^{(k)}(\boldsymbol{a}) \neq 0$.

Remark 1.15. By Proposition 1.14, a strong Lefschetz element comes from an open dense space where the higher Hessians do not vanish. Thus, if the k-th Hessian does not vanish as a polynomial for each k, then the Artinian Gorenstein algebra A has the strong Lefschetz property. In other words, an algebra A has the strong Lefschetz property at degree k for all k if and only if A has the strong Lefschetz property.

If $\Delta_k \subset A_k$ spans A_k as a $\mathbb{K}$-vector space and

$$\det\left(e_i\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) e_j\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) F\right)_{e_i, e_j \in \Delta_k} \neq 0,$$

then $\Delta_k \subset A_k$ is a basis for A_k and A has the strong Lefschetz property at degree k. Hence, the degeneracy of the usual Hessian implies the strong Lefschetz property at degree one.

Proposition 1.16. *If* $\det H_F \neq 0$*, then A has the strong Lefschetz property at degree one.*

We illustrate the story in Section 3 with the polynomial $F = x_1x_2x_3x_4$.

Example 1.17. Let $F = x_1x_2x_3x_4$ and $A = \mathbb{K}[x_1, x_2, x_3, x_4]/\operatorname{Ann}(F) = A_0 \oplus A_1 \oplus A_2 \oplus A_3 \oplus A_4$. As in Example 1.6, the Hessian $\det H_F = -3F^2 \neq 0$. Thus, it follows from Proposition 1.16 that the algebra A has the strong Lefschetz property at degree one. Moreover, x_1, x_2, x_3, x_4 form basis for A_1. Since $\det H_F|_{\boldsymbol{x}=(1,1,1,1)} = -3F^2|_{\boldsymbol{x}=(1,1,1,1)} = -3 \neq 0$, it follows from Proposition 1.14 that A has the strong Lefschetz property at degree one with a Lefschetz element $L = x_1 + x_2 + x_3 + x_4$. In fact, the annihilator $\operatorname{Ann}(F)$ is generated by $\{ x_1^2, x_2^2, x_3^2, x_4^2 \}$. Therefore the square-free monomials form a basis for A as $\mathbb{K}$-vector space. Since

$$\begin{aligned} \times L^2 : A_1 &\to A_3 \\ x_1 &\mapsto 2x_1(x_2x_3 + x_2x_4 + x_3x_4), \\ x_2 &\mapsto 2x_2(x_1x_3 + x_1x_4 + x_3x_4), \\ x_3 &\mapsto 2x_3(x_1x_2 + x_1x_4 + x_2x_4), \\ x_4 &\mapsto 2x_4(x_1x_2 + x_1x_3 + x_2x_3), \end{aligned}$$

the representation matrix of the multiplication map with respect to the basis (x_1, x_2, x_3, x_4) and $(x_2x_3x_4, x_1x_3x_4, x_1x_2x_4, x_1x_2x_3)$ is

$$\begin{pmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 0 \end{pmatrix}.$$

Therefore the multiplication map is bijective. In this case, the algebra A has the strong Lefschetz property with a strong Lefschetz element $L = x_1 + x_2 + x_3 + x_4$ since

$$\begin{aligned} &\times L^4 : A_0 \to A_4,\ x \mapsto x \times L^4, \\ &\times L^2 : A_1 \to A_3,\ x \mapsto x \times L^2, \\ &\times L^0 : A_2 \to A_2,\ x \mapsto x \times L^0 = x \end{aligned}$$

are bijective. The Hilbert function of A is $(1, 4, 6, 4, 1)$.

4. Hessian matrices and Lorentzian polynomials

In this section, we study the relation between Hessian matrices and Lorentzian polynomials. The proofs of the propositions in this section are in [**16**]. Lorentzian polynomials are related to the strong Lefschetz property and the Hodge–Riemann relation.

We recall the Hodge–Riemann relations. The Hodge–Riemann relations imply the hard Lefschetz theorem. See [**10, 4**] for details. Since the strong Lefschetz property is an abstraction of the hard Lefschetz theorem, we can define the Hodge–Riemann relations for the strong Lefschetz property as follows: We consider $A = A_F = \mathbb{R}[x_1, \dots, x_n]/\operatorname{Ann}(F) = \bigoplus_{k=0}^{s} A_k$. Define

$$[D] = D\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) F$$

for $D \in A_s$. Then, $[-]$ gives the isomorphism from A_s to $\mathbb{R}$. For $L = a_1x_1 + \dots + a_nx_n$, we define the bilinear from $Q_L^k : A_k \times A_k \to \mathbb{R}$ by $Q_L^k(\xi_1, \xi_2) = (-1)^k[\xi_1 L^{s-2k} \xi_2]$. We say that A satisfies the *Hodge–Riemann relation* at degree k with respect to $L \in A_1$ if A has the strong

Lefschetz property at degree k with a Lefschetz element L and the bilinear form Q_L^k is positive definite on $\ker(\times L^{s-2k+1} : A_k \to A_{s-k+1})$.

If A_F has the Hodge–Riemann relation at degree one with some condition, then we obtain the signature of the Hessian matrix of F.

Proposition 1.18. *Let $F \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial, $L = a_1x_1 + \cdots + a_nx_n$, and $A = \mathbb{R}[x_1, \ldots, x_n]/\operatorname{Ann}(F)$. If $\frac{\partial}{\partial x_1}F, \ldots, \frac{\partial}{\partial x_n}F$ are $\mathbb{R}$-linearly independent, then the following are equivalent:*

- *The algebra A satisfies the Hodge–Riemann relation at degree one with respect to L.*
- *$H_F|_{\boldsymbol{x}=\boldsymbol{a}}$ has signature $(+, -, \ldots, -)$.*

Let F be a homogeneous polynomial of degree r in n variables with positive coefficients. We call that F is a *Lorentzian polynomial* if for any $(k_1, \ldots, k_n) \in \mathbb{Z}^n_{\geq 0}$ with $\sum_{i=1}^n k_i \leq \deg F - 2$, $(\frac{\partial}{\partial x_1})^{k_1} \cdots (\frac{\partial}{\partial x_n})^{k_n} F$ is identically zero or log-concave at any $\boldsymbol{a} \in \mathbb{R}^n_{>0}$. The Lorentzian polynomials are introduced and studied in [**8**]. The important instances of the Lorentzian polynomials are the generating functions for a matroid (cf. Proposition 3.2).

Generally, the Hodge–Riemann relations imply the strong Lefschetz property. If F is Lorentzian, then the Hodge–Riemann relation at degree one and the strong Lefschetz property at degree one are equivalent.

Proposition 1.19. *Let F be Lorentzian, and $A = \mathbb{R}[x_1, \ldots, x_n]/\operatorname{Ann}(F)$. For $L = a_1x_1 + \cdots + a_nx_n$ with $F|_{\boldsymbol{x}=\boldsymbol{a}} > 0$, the following are equivalent:*

- *the algebra A has the strong Lefschetz property at degree one with a Lefschetz element L.*
- *the algebra A satisfies the Hodge–Riemann relation at degree one with respect to L.*

Proposition 1.20 (Murai–Nagaoka–Yazawa [**16**])**.** *Let $L = a_1x_1 + \cdots + a_nx_n$ with $a_i > 0$. If F is Lorentzian, then $A = \mathbb{R}[x_1, \ldots, x_n]/\operatorname{Ann}(F)$ satisfies the the Hodge–Riemann relation at degree one with respect to L.*

Proposition 1.20 is the key of Theorem 6.3, which is one of our goals in this thesis. We illustrate Proposition 1.20 with the polynomial $F = x_1x_2x_3x_4$.

Example 1.21. Let $F = x_1x_2x_3x_4$. As in Example 1.6, the monomial F is log-concave. Let $F' = \frac{\partial}{\partial x_4}F = x_1x_2x_3$. Then

$$H_{F'} = \begin{pmatrix} 0 & x_3 & x_2 & 0 \\ x_3 & 0 & x_1 & 0 \\ x_2 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$-F'H_{F'} = \begin{pmatrix} 0 & -x_1x_2x_3^2 & -x_1x_2^2x_3 & 0 \\ -x_1x_2x_3^2 & 0 & -x_1^2x_2x_3 & 0 \\ -x_1x_2^2x_3 & -x_1^2x_2x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned} G_{F'} &= \begin{pmatrix} x_2x_3 \\ x_1x_3 \\ x_1x_2 \\ 0 \end{pmatrix} \begin{pmatrix} x_2x_3 & x_1x_3 & x_1x_2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} x_2^2x_3^2 & x_1x_2x_3^2 & x_1x_2^2x_3 & 0 \\ x_1x_2x_3^2 & x_1^2x_3^2 & x_1^2x_2x_3 & 0 \\ x_1x_2^2x_3 & x_1^2x_2x_3 & x_1^2x_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Thus, we have

$$-F'H_{F'} + G_{F'} = \begin{pmatrix} x_2^2x_3^2 & 0 & 0 & 0 \\ 0 & x_1^2x_3^2 & 0 & 0 \\ 0 & 0 & x_1^2x_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, F' is log-concave. Similarly, the monomial $\frac{\partial}{\partial x_i}F$ is also log-concave. Let $F'' = \frac{\partial^2}{\partial x_3 \partial x_4}F = x_1x_2$. Then

$$H_{F''} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$-F''H_{F''} = \begin{pmatrix} 0 & -x_1x_2 & 0 & 0 \\ -x_1x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned} G_{F''} &= \begin{pmatrix} x_2 \\ x_1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} x_2 & x_1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} x_2^2 & x_1x_2 & 0 & 0 \\ x_1x_2 & x_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Thus, we have

$$-F''H_{F''} + G_{F''} = \begin{pmatrix} x_2^2 & 0 & 0 & 0 \\ 0 & x_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, F'' is log-concave. Similarly, the monomial $\frac{\partial^2}{\partial x_i \partial x_j} F$ is also log-concave for $i \neq j$. If $i = j$, then $\frac{\partial^2}{\partial x_i \partial x_j} F$ is identically zero. Therefore, F is a Lorentzian polynomial. Let $A = \mathbb{R}[x_1, x_2, x_3, x_4]/\operatorname{Ann}(F)$, and $L = x_1 + x_2 + x_3 + x_4$. Since F is Lorentzian, it follows from Proposition 1.20 that A satisfies the the Hodge–Riemann relation at degree one with respect to L. In fact, since

$$L^3 = (x_1 + x_2 + x_3 + x_4)^3 = 6(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4),$$

we have

$$L^3x_1 = L^3x_2 = L^3x_3 = L^3x_4 = 6x_1x_2x_3x_4.$$

Hence $P_1 = x_1 - x_4$, $P_2 = x_2 - x_4$, and $P_3 = x_3 - x_4$ form a basis for $\ker(\times L^3 : A_1 \to A_4)$. Let us consider $Q_L^1 : A_1 \times A_1 \to \mathbb{R}$. Since

$$L^2 = 2(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4),$$

we have

$$L^2x_ix_j = \begin{cases} 2x_1x_2x_3x_4 = 2F, & i \neq j, \\ 0, & i = j. \end{cases}$$

Since

$$Q_L^1(x_i, x_j) = \begin{cases} -[2F] = -2, & i \neq j, \\ 0, & i = j, \end{cases}$$

the representation matrix for Q_L^1 with respect to the basis (x_1, x_2, x_3, x_4) is

$$-2\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

The eigenvalues of this matrix are $-6, 2, 2, 2$. The eigenspace associated to -6 is

$$\left\{ \begin{pmatrix} a \\ a \\ a \\ a \end{pmatrix} \middle| \; a \in \mathbb{R} \right\}.$$

This matrix is not positive semi-definite. Let us consider the restriction of Q_L^1 to $\ker(\times L^3 : A_1 \to A_4)$. Since

$$\begin{aligned} p_ip_j &= (x_i - x_4)(x_j - x_4) \\ &= \begin{cases} -2x_ix_4, & i = j, \\ x_ix_j - x_ix_4 - x_jx_4, & i \neq j, \end{cases} \end{aligned}$$

we have

$$L^2 p_i p_j = \begin{cases} -4F, & i = j, \\ -2F, & i \neq j. \end{cases}$$

Since

$$Q_L^1(p_i, p_j) = \begin{cases} -[-2F] = 2, & i \neq j, \\ -[-4F] = 4, & i = j, \end{cases}$$

the representation matrix for Q_L^1 on $\ker(\times L^3 : A_1 \to A_4)$ with respect to the basis (p_1, p_2, p_3) is

$$2 \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

The eigenvalues of this matrix are $8, 2, 2$. Since this matrix is positive definite, the monomial F satisfies the the Hodge–Riemann relation at degree one with respect to L.

CHAPTER 2

Graphs

In this chapter, we provide basic terms of graphs, and count spanning trees. Using the Laplacian matrix of a graph is an algebraic way to count them. We see how to count them, and study the Kirchhoff polynomials. The Kirchhoff polynomial is an important instance of the basis generating polynomial for a matroid, which are one of main objects in this thesis.

1. Definitions

In this section, we recall some basic terms of graphs. See, e.g., [**5, 18**] for basics of graphs.

We call the pair $\Gamma = (V(\Gamma), E(\Gamma))$ of a set $V(\Gamma)$ and $E(\Gamma) \subset \binom{V(\Gamma)}{2}$ a *(simple) graph*, where $\binom{V}{2} = \{ \{ x, y \} \mid x, y \in V, x \neq y \}$. An element in $V(\Gamma)$ is called a *vertex*, and an element in $E(\Gamma)$ is called an *edge*. For $\{ u, v \} \in E(\Gamma)$, u, v are called the *ends* of $\{ u, v \}$. A simple graph is a graph which does not allow edges from a vertex v to v, called a *loop*, and multiple edges between 2 vertices, called *multiple edges* or *parallel edges*. If $\{ x, y \} \in E(\Gamma)$, then we write $x \sim y$, and say that x and y are *adjacent*. If $\{ x, y \} \notin E(\Gamma)$, then we write $x \not\sim y$. Let Γ, Γ' be graphs. If $V(\Gamma') \subset V(\Gamma)$ and $E(\Gamma') \subset E(\Gamma)$, then Γ' is called *subgraph* of Γ, written $\Gamma' \subset \Gamma$. For a vertex $v \in V(\Gamma)$, define $d(v) = \# \{ v' \in V(\Gamma) \mid v \sim v' \}$. We call $d(v)$ the *degree* of the vertex v.

For distinct vertices $u, u_1, \dots, u_{n-1}, v$, let

$$
\begin{aligned}
V(P) &= \{ u, u_1, \dots, u_{n-1}, v \}, \\
E(P) &= \{ \{ u, u_1 \}, \{ u_1, u_2 \}, \dots, \{ u_{n-1}, v \} \}.
\end{aligned}
$$

Then P is called a *path from u to v*. We say that a graph is connected if there exists a path between any two vertices. A connected graph such that the degrees of all vertices are two is called a *cycle*. The cycle with n vertices is denoted by C_n. We say that a graph is a *forest* if any subgraph of the graph is not a cycle. A connected forest is called a *tree*. By definition, a path is a tree. We call a subgraph T of Γ a *spanning tree* in Γ if T is a tree and $V(T) = V(\Gamma)$. For a graph Γ, B_Γ denote the collection of spanning trees in Γ. If a graph Γ is not connected, then $B_\Gamma = \emptyset$. In general, a spanning tree in Γ with n vertices has $n-1$ edges. A graph satisfying $E(\Gamma) = \binom{V(\Gamma)}{2}$ is called a *complete graph*. The complete graph with n vartices is denoted by K_n. Let $V(\Gamma) = X_1 \sqcup \cdots \sqcup X_k$. We call the graph Γ a *k-partite graph* with a partition $V(\Gamma) = X_1 \sqcup \cdots \sqcup X_k$ if the following condition is satisfied for $i \in \{ 1, 2, \ldots, k \}$:

$$x, x' \in X_i \implies x \not\sim x'.$$

Let $\Gamma = (X_1 \sqcup \cdots \sqcup X_k, E(\Gamma))$ be a k-partite graph. We call Γ a *complete k-partite graph*, written $K_{X_1,\ldots,X_k}$, if the following condition is satisfied:

$$i \neq j, x_i \in X_i, x_j \in X_j \implies x_i \sim x_j.$$

The complete k-partite graph $\Gamma = (X_1 \sqcup \cdots \sqcup X_k, E(\Gamma))$ with $\#X_1 = m_1, \ldots, \#X_k = m_k$ is denoted by $K_{m_1,\ldots,m_k}$ For a graph Γ, define $V(\Gamma^*) = E(\Gamma)$, $E(\Gamma^*) = \{ \{ e, e' \} \mid e, e' \in E(\Gamma), e \neq e', e \cap e' \neq \emptyset \}$. The graph Γ^* is called the *line graph* of Γ.

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph and $w : E(\Gamma) \to \mathbb{Z}_{\leq 0}$. We say that $(V(\Gamma), E(\Gamma), w)$ is a *weighted graph*. We regard a weighted graph with weight $w(e) = 1$ for all e as a simple graph. For a weighted graph $\Gamma = (V, E, w)$, the matrix $A_\Gamma = (a_{ij})$ and the matrix $D_\Gamma = (d_{ij})$ with the index set $V \times V$ defined by

$$a_{ij} = \begin{cases} w(\{ i, j \}) & i \sim j, \\ 0 & i \not\sim j, \end{cases}$$

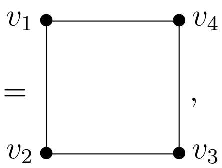
$$d_{ij} = \begin{cases} \sum_{k \sim i} w(\{ i, k \}) & i = j, \\ 0 & i \neq j \end{cases}$$

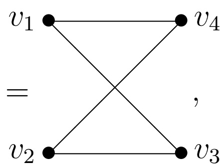
are called the *adjacency matrix* and *degree matrix* of Γ, respectively. If all weights of the edges of Γ are one, then $\sum_{k \sim i} w(\{ i, k \})$ is equal to the degree $d(i)$ of $i \in V$. Let $E \subset V \times V$. We say that (V, E) is a *directed graph*. For a weighted directed graph $\Gamma = (V, E, w)$, the matrix $J_\Gamma = (j_{v,(x,y)})$ with the index set $V \times E$ defined by

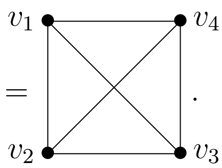
$$j_{v,(x,y)} = \begin{cases} \sqrt{w((x, y))} & v = x, \\ -\sqrt{w((x, y))} & v = y, \\ 0 & \text{otherwise} \end{cases}$$

is called the *directed incidence matrix* of Γ.

Example 2.1. Let us consider the following graphs. Assume that each weight of an edge of each graph is one.

$C_4 =$ 

, $K_{2,2} =$ 

, $K_4 =$ 

.

The following matrices are the adjacency matrices of $C_4, K_{2,2}, K_4$, respectively:

$$A_{C_4} = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

$$A_{K_{2,2}} = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

$$A_{K_4} = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

The following matrices are the degree matrices of $C_4, K_{2,2}, K_4$, respectively:

$$D_{C_4} = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

$$D_{K_{2,2}} = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

$$D_{K_4} = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

The following matrices are the directed incidence matrices of $C_4, K_{2,2}, K_4$ with some orientation, respectively:

$$J_{C_4} = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix},$$

$$J_{K_{2,2}} = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{pmatrix},$$

$$J_{K_4} = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{pmatrix} -1 & 0 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{pmatrix}.$$

Two graphs Γ and Γ' are *isomorphic* if and only if there exists a permutation matrix P such that $A_\Gamma P = PA_{\Gamma'}$. Indeed, C_4 and $K_{2,2}$ are isomorphic, and if we take P as

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

then $A_{C_4} P = PA_{K_{2,2}}$. Therefore, the eigenvalues of the adjacency matrices of two graphs which are isomorphic to each other are the same. On the other hand, there exists nonisomorphic graphs such that the eigenvalues of their adjacency matrices are the same.

2. Counting spanning trees

Consider a weighted graph Γ. Let J_Γ be the directed incidence matrix with respect to some orientation of Γ. A matrix L_Γ with the index set $V \times V$ and a matrix L'_Γ with the index set $E \times E$ defined by

$$L_\Gamma = J_\Gamma (J_\Gamma)^\top, \qquad L'_\Gamma = (J_\Gamma)^\top J_\Gamma$$

are called the *Laplacian* and *edge Laplacian* of Γ, respectively. The matrices L_Γ and L'_Γ are independent of the orientation of Γ. In fact, it is known that

$$L_\Gamma = D_\Gamma - A_\Gamma, \qquad L'_\Gamma = D_{\Gamma^*} - A_{\Gamma^*}.$$

Note that L'_Γ is a Laplacian of the line graph Γ^*. More precisely, we have $J_\Gamma^\top = J_{\Gamma^*}$.

Let Γ has n vertices. For J_Γ, define $\widetilde{J}_\Gamma$ to be the matrix forgot the last row of J_Γ. Let $w_1, w_2, \ldots, w_{n-1}$ form an orthonormal basis of the vector space spanned by the row vectors of $\widetilde{J}_\Gamma$. A matrix K_Γ with the index set $E \times E$ defined by

$$K_\Gamma = \sum_{i=1}^{n-1} (w_i)^\top w_i$$

is called the *graph correlation kernel*.

Remark 2.2. To my best of knowledge, I can not find the name of the matrix L'_Γ and K_Γ. In this thesis, we call L'_Γ and K_Γ edge Laplacian and graph correlation kernel, respectively. However, note that these terms are not common.

Let $\tau(\Gamma)$ be the number of spanning trees in a simple graph Γ. For $F \subset E(\Gamma)$, $\tau(\Gamma, F)$ denotes the number of spanning trees in a simple graph Γ containing F. The Laplacian and graph correlation kernel play an important role in counting spanning trees. The following is known as Matrix-Tree Theorem. See, e.g, [**5**].

Proposition 2.3. *For a simple graph Γ, any cofactor of the Laplacian L_Γ is equal to $\tau(\Gamma)$. In other words, for a graph $\Gamma = (V, E)$ with $\#V = n$,*

$$\tau(\Gamma) = (-1)^{i+j} \det(L_\Gamma^{(ij)})$$

for any $1 \leq i, j \leq n$.

Submatrices of the graph correlation kernels tell us the ratios of forests in the spanning trees in a graph.

Proposition 2.4. *Let $\Gamma = (V, E)$ be a simple graph and F a subset of E. Then, the ratio $\frac{\tau(\Gamma,F)}{\tau(\Gamma)}$ is $\det K_\Gamma(F)$, where $K_\Gamma(F)$ is the submatrix of K_Γ corresponding to the index set $F \times F$.*

As corollary to Propositions 2.3 and 2.4, we have the following.

Corollary 2.5. *Let Γ be a simple graph with $\#V(\Gamma) \geq 2$.*

(1) *If F is the edges of a spanning tree, then*

$$\frac{1}{\det K_\Gamma(F)} = \tau(\Gamma).$$

(2) *For $F \subset E(\Gamma)$,*

$$\det(L_\Gamma^{(11)}) \det K_\Gamma(F) = \tau(\Gamma, F).$$

We illustrate Proposition 2.3 with the complete graphs.

Example 2.6. Let $n \geq 2$. The Laplacian of K_n of size $n \times n$ is the following:

$$L_{K_n} = \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & n-1 \end{pmatrix}.$$

The $(1,1)$-cofactor of L_{K_n} of size $(n-1)\times(n-1)$ is the following:

$$L_{K_n}^{(11)} = \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & n-1 \end{pmatrix}.$$

By Proposition 2.3, we have $\tau(K_n) = n^{n-2}$

The following are the number of the spanning trees including some forests. These consequences can be obtained from the graph correlation kernels of the complete graphs.

Example 2.7. Let F be a forest of K_n with k connected components. By Corollary 2.5, the number of spanning trees in K_n containing F is $n^{k-2}\prod_{i=1}^{k} j_i$, where j_i is the number of vertices of each connected component of F.

3. Kirchhoff polynomials

In this section, we consider a weighted graph. For a weighted graph, "Matrix-Tree theorem", which is a generalization of Proposition 2.3 holds.

Proposition 2.8. *Let Γ be a weighted connected graph with weight w. We associate the weight $w(\{ i,j \}) = x_{ij} = x_{ji}$ with $\{ i,j \} \in E(\Gamma)$. Any cofactor of the weighted Laplacian L'_Γ is equal to the generating function of the spanning trees in Γ. In other words, for a graph $\Gamma = (V,E)$ with $\#V = n$,*

$$\sum_{T\in B_\Gamma} \prod_{\{ ij \}\in E(T)} x_{ij} = (-1)^{i+j}\det(L_\Gamma^{(ij)})$$

for any $1 \leq i,j \leq n$.

We call the left hand side of the equation in Proposition 2.8 the *Kirchhoff polynomial*. Since any spanning trees in Γ with $n+1$ vertices have n edges, the Kirchhoff polynomial of Γ are homogeneous polynomial of degree n. Moreover, the monomials in the Kirchhoff polynomial are square-free. For a graph which is not connected, we define the Kirchhoff polynomial for the graph to be the product of the Kirchhoff polynomial for each connected component of the graph.

Let Γ be a graph, e an edge with ends v and v' of Γ. We define the *deletion* $\Gamma \setminus e$ to be the graph $(V(\Gamma), E(\Gamma)\setminus\{ e \})$. We define the *contraction* Γ/e to be the graph obtained by removing the edge e from $E(\Gamma)$ and by putting v in v'. Note that the deletion of a simple graph is a simple graph. The contraction of a simple graph, however, may be not simple. For edges e, e' of Γ, we have

- $(\Gamma\setminus e)\setminus e' = (\Gamma\setminus e')\setminus e$.

- $(\Gamma/e)/e' = (\Gamma/e')/e$.

We write $\Gamma \setminus e, e'$ and $\Gamma/e, e'$ to denote $(\Gamma \setminus e) \setminus e'$ and $(\Gamma/e)/e'$, respectively.

Kirchhoff polynomials satisfy the deletion-contraction formula. The formula provides a way to calculate the Kirchhoff polynomial recursively.

Proposition 2.9. *Let F_Γ be the Kirchhoff polynomial of a graph Γ. Then, we have*

$$F_\Gamma = F_{\Gamma\setminus e} + x_e F_{\Gamma/e}.$$

We see the Kirchhoff polynomial of a tree with five vertices.

Example 2.10. Let us consider a tree T with 5 vertices, where

$V(T) = \{ v_1, v_2, v_3, v_4, v_5 \}$,
$E(T) = \{ e_1 = \{ v_1, v_2 \}, e_2 = \{ v_2, v_3 \}, e_3 = \{ v_3, v_4 \}, e_4 = \{ v_4, v_5 \} \}$.

The adjacency matrix A_T and degree matrix D_T are

$$A_T = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad D_T = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

respectively. Thus, the Laplacian L_T is

$$L_T = D_T - A_T = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

The $(4, 4)$-cofactor of the Laplacian L_T is

$$(-1)^{4+4} \det \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} = 1.$$

It follows from Proposition 2.3 that the number of spanning tree in T is one. In fact, itself T is the spanning tree in T. We associate x_i to the edge $e_i = \{ v_i, v_{i+1} \} \in E(T)$. The weighted adjacency matrix A'_T

and weighted degree matrix D'_T are

$$A'_T = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{pmatrix} 0 & x_1 & 0 & 0 & 0 \\ x_1 & 0 & x_2 & 0 & 0 \\ 0 & x_2 & 0 & x_3 & 0 \\ 0 & 0 & x_3 & 0 & x_4 \\ 0 & 0 & 0 & x_4 & 0 \end{pmatrix},$$

$$D'_T = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{pmatrix} x_1 & 0 & 0 & 0 & 0 \\ 0 & x_1+x_2 & 0 & 0 & 0 \\ 0 & 0 & x_2+x_3 & 0 & 0 \\ 0 & 0 & 0 & x_3+x_4 & 0 \\ 0 & 0 & 0 & 0 & x_4 \end{pmatrix},$$

respectively. Thus, the weighted Laplacian L'_T is

$$L'_T = D'_T - A'_T = \begin{pmatrix} x_1 & -x_1 & 0 & 0 & 0 \\ -x_1 & x_1+x_2 & -x_2 & 0 & 0 \\ 0 & -x_2 & x_2+x_3 & -x_3 & 0 \\ 0 & 0 & -x_3 & x_3+x_4 & -x_4 \\ 0 & 0 & 0 & -x_4 & x_4 \end{pmatrix}.$$

The $(4,4)$-cofactor of the weighted Laplacian L'_T is

$$\begin{aligned} &(-1)^{4+4} \det \begin{pmatrix} x_1 & -x_1 & 0 & 0 \\ -x_1 & x_1+x_2 & -x_2 & 0 \\ 0 & -x_2 & x_2+x_3 & -x_3 \\ 0 & 0 & -x_3 & x_3+x_4 \end{pmatrix} \\ &= x_1 \det \begin{pmatrix} x_1+x_2 & -x_2 & 0 \\ -x_2 & x_2+x_3 & -x_3 \\ 0 & -x_3 & x_3+x_4 \end{pmatrix} \\ &\qquad - (-x_1) \det \begin{pmatrix} -x_1 & 0 & 0 \\ -x_2 & x_2+x_3 & -x_3 \\ 0 & -x_3 & x_3+x_4 \end{pmatrix} \\ &= x_1x_2x_3x_4. \end{aligned}$$

It follows from Proposition 2.8 that the monomial $x_1x_2x_3x_4$ is the generating function for the spanning trees in T. In fact, since itself T is the spanning tree in T, we have

$$\sum_{T' \in B_T} \prod_{i \in E(T')} x_i = \prod_{i \in E(T)} x_i = x_1x_2x_3x_4.$$

Therefore, the monomial $x_1x_2x_3x_4$ is the Kirchhoff polynomial of a tree with 5 vertices.

CHAPTER 3

Matroids

As stated in Introduction, we mainly consider three generating polynomials, the generating functions F_M for basis, the generating functions P_M for independent sets, and the generating functions $\overline{P}_M$ for reduced independent sets, for a matroid M. In this chapter, we provide some basic terms and study the generating polynomials for a matroid. We study the Hessian matrices of them in Chapter 5.

1. Bases and independent sets

First we recall basic terms of matroids. Here, we note that matroids have several different equivalent definitions. The definition here is by bases for matroids. See [**19**] for the details.

We call a pair $(E, \mathcal{B})$ a *matroid* if a finite set E and nonempty collection $\mathcal{B}$ of subsets of E satisfies the following property, called *the basis exchange property*:

- If B_1 and B_2 are in $\mathcal{B}$ and $x \in B_1 \setminus B_2$, then there is an element $y \in B_2 \setminus B_1$ such that $\{y\} \cup (B_1 \setminus \{x\}) \in \mathcal{B}$.

We call each $B \in \mathcal{B}$ a *basis* for M. We call an element $e \in E$ a *loop* of M if $\{e\}$ is not contained by any basis for M. We call an element $e \in E$ a *coloop* of M if $\{e\}$ is contained by each basis for M.

The following is directly proved from the basis exchange property.

Proposition 3.1. *All bases of a matroid M have the same cardinality.*

We say that a matroid M has *rank* r if the number of elements of a basis of M is r. The rank of M is denoted by $\operatorname{rank} M$.

Let $M = (E, \mathcal{B})$ be a matroid. We call each subset of a basis for M an *independent set* of M and call each subset of E which is not contained in any basis a *dependent set* of M. We write $\mathcal{I}(M)$ for the set of independent sets and $I_k(M)$ for the cardinality of the set of independent sets with k elements. A minimal dependent set of M is called a *circuit* of M. A circuit with n elements is called an n-circuit. A loop is a 1-circuit. We define $\operatorname{girth}(M)$ to be the minimum cardinality of its circuit. We call $\operatorname{girth}(M)$ the *girth* of M. Equivalently, $\operatorname{girth}(M) = \min\left\{ k \;\middle|\; I_k(M) \neq \binom{n}{k} \right\}$.

We call a 2-circuit a *parallel*. Let M be a matroid on E. Let E_0 be the set of loops, and $E' = E \setminus E_0$. We define a binary relation $\parallel$ on E'

by

$$i \parallel j \iff i = j \text{ or } \{ i, j \} \text{ is a parallel}$$

for $i, j \in E'$. Then, we have the following:

- $i \parallel i$.
- If $i \parallel j$, then $j \parallel i$.
- If $i \parallel j$ and $j \parallel k$, then $i \parallel k$.

Therefore, $\parallel$ is an equivalent relation on E'. We decompose E' into the equivalent classes $E' = E_1 \sqcup \cdots \sqcup E_s$. The decomposition $E = E_0 \sqcup E_1 \sqcup \cdots \sqcup E_s$ is called *parallel class decomposition* of M. We call $E_1, \ldots, E_s$ *parallel classes* of M.

We say that two matroids $M = (E, \mathcal{B})$ and $M' = (E', \mathcal{B}')$ are isomorphic, written $M \cong M'$, if there is a bijection ψ from E to E' such that B is a basis for M if and only if $\psi(B)$ is a basis for M'.

2. Classes of matroids

Let us see important instances of matroids.

Simple matroid. We say that a matroid M is *simple* if there is neither a loop nor a parallel. For a matroid $M = (E, \mathcal{B}(M))$, we define the matroid $\overline{M}$ by deleting all loops and deleting all but one element in each parallel class, namely, choose a representative of equivalence classes, in the matroid M. We call the operation *simplification*.

Vector matroid. Let A be a matrix of size $m \times n$ over a field $\mathbb{K}$. Let E be the column index set, and $\mathcal{B}$ the set of maximal subsets B of E such that the multiset of columns labeled by B is linearly independent in the vector space $\mathbb{K}^m$. Then $M[A] = (E, \mathcal{B})$ is a matroid. We call $M[A] = (E, \mathcal{B})$ a *vector matroid*. The rank of matroid is a the rank of A. Thus, if $A \in GL(n, \mathbb{K})$, then we have $\operatorname{rank} M[A] = n$ and $B \in \mathcal{B}$ is the index set of a basis for $\mathbb{K}^m$. For example, consider the following matrix A:

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Then, $E = \{ 1, 2, 3, 4, 5 \}$ and $\mathcal{B} = \{ \{ 2, 3, 4 \}, \{ 3, 4, 5 \} \}$. We can see that $\{ 1 \}$, correspond to the 0-vector, is a loop, $\{ 2, 5 \}$ is a parallel, and $\{ 2, 3, 4, 5 \}$ is a 4-circuit. We have $\operatorname{rank} M[A] = 3$. The girth of $M[A]$ is 4.

Uniform matroid. Let E be a finite set with n elements. For $0 \leq r \leq n$, let $\mathcal{B}_r$ be the collection of subsets of E such that subsets have r elements. Then, $U_{r,n} = (E, \mathcal{B}_r)$ is a matroid of $\operatorname{rank} r$. These matroids are called *uniform matroids*. We say that $U_{n,n}$ is the *free* matroid of rank n. We call $U_{0,0}$ the *empty matroid*. For the uniform matroid $U_{r,n}$, the matroid is not simple if and only if $r \leq 1$. In the case where $r = 0$, for all $i \in E$, i is a loop. In the case where $r = 1$, for $i, j \in E$, $i \neq j$, i and j are parallel. The girth of $U_{r,n}$ is $r + 1$.

Graphic matroid. For a finite graph $\Gamma = (V(\Gamma), E(\Gamma))$, let $\mathcal{B}_\Gamma$ be the set of all maximal forests in Γ. Then $M(\Gamma) = (E(\Gamma), \mathcal{B}_\Gamma)$ is a matroid. These matroids are called *graphic matroids*. If Γ is connected, then $\mathcal{B}_\Gamma$ is the set B_Γ of spanning trees in Γ. Note that if M is a graphic matroid of non-connected graph, then there exists a connected graph Γ such that $M(\Gamma)$ is isomorphic to M. You can see that for a graphic matroid $M(\Gamma) = (E(\Gamma), \mathcal{B}_\Gamma)$, a loop of $M(\Gamma)$ corresponds to a loop of Γ, and an n-circuit of $M(\Gamma)$ corresponds to an n-cycle in Γ, and girth$(M(\Gamma))$ corresponds to the girth of Γ. Thus, $M(\Gamma)$ is simple if and only if Γ is a simple graph. Let $\omega(\Gamma)$ be the number of connected components of Γ. The rank $M(\Gamma)$ is $\#V(\Gamma) - \omega(\Gamma)$. In particular, if Γ is connected graph, then $\operatorname{rank} M(\Gamma) = \#V(\Gamma) - 1$. Note that in [**19**], $M(\Gamma)$ is called a *cycle matroid*. A matroid that is isomorphic to the cycle matroid of a graph is called a graphic matroid. In this thesis, we call both of matroids graphic matroids. Let us see some examples: Let B_n be the n-bouquet, one vertex and n loops. Let G_n be the n multiple edges graph, two vertecies and n parallel edges. Let C_n be the n-cycle. Let T_{n+1} be the tree with $n + 1$ vertices. Then, we have

$$M(B_n) \cong U_{0,n}, \quad M(G_n) \cong U_{1,n}, \quad M(C_n) \cong U_{n-1,n}, \quad M(T_{n+1}) \cong U_{n,n}.$$

A uniform matroid is not always a graphic matroid. In fact, $U_{2,n}$ is not graphic for $n \geq 4$.

Representable matroid. We say that M is a $\mathbb{K}$-*representable matroid* if M is isomorphic to the vector matroid of a matrix over a field $\mathbb{K}$. A matroid that is representable over some field called *representable*. For example, uniform matroids are representable for some field. Let $v_1, v_2, \ldots, v_n$ be vectors of $\mathbb{R}^n$. Let E be the label of the vectors, and $\mathcal{B}$ be the set of subsets of E which is corresponding to independent vectors in general position in $\mathbb{R}^r$. Then $(E, \mathcal{B})$ is a uniform matroid. Graphic matroids are also representable every field. The incidence matrix represents the graphic matroid. In fact, $M(\Gamma) \cong M[J_\Gamma]$ with arbitrary orientation. There are matroids which are not representable. See [**19**] for the details.

Submatroid. For $E' \subset E$, we define $\mathcal{B}'$ by $\mathcal{B}' = \{ B \in \mathcal{B} \mid B \subset E' \}$. Then $M' = (E', \mathcal{B}')$ is a matroid. We call M' a *submatroid* of M.

Contraction. Let $M = (E, \mathcal{B})$ be a matroid. For a non-loop element e in E, let $\mathcal{B}(M/e) = \{B \setminus \{e\} \mid e \in B \in \mathcal{B}(M)\}$. Then $M/e = (E \setminus \{e\}, \mathcal{B}(M/e))$ is a matroid. We call M/e the *contraction* of M with respect to e. The contraction of M is a submatroid of M.

Deletion. Let $M = (E, \mathcal{B})$ be a matroid. For $X \subset E$, let $\mathcal{B}(M|_X) = \{ B \in \mathcal{I}(M) \mid B \subset X, \#B = \operatorname{rank}(X) \}$. Then $M|_X = (E \setminus$

$X, \mathcal{B}(M|_X))$ is a matroid. We call $M|_X$ the *restriction* of M to X. In particular, for $e \in E$, we write $M \setminus e = M|_{E\setminus\{e\}}$ and call it the *deletion* of e from M. The deletion of M is a submatroid of M.

Truncated matroid. Let $M = (E, \mathcal{B})$ be a matroid. Let

$$\mathcal{B}(TM) = \{\, I \in \mathcal{I}(M) \mid \#I = \operatorname{rank}(M) - 1 \,\}.$$

Then, $TM = (E, \mathcal{B}(TM))$ is a matroid. We define $T^kM = T(T^{k-1}M)$ for $k > 1$, inductively. For $k = 0$, we define $T^kM = M$. We call T^kM a truncated matroid of M. The truncation of a uniform matroid is a uniform matroid. In fact, $T^kU_{r,n} = U_{r-k,n}$. Note that truncated matroids of a graphic matroid is not always graphic matroid.

3. Generating polynomials

In this section, we study the generating polynomials for a matroid, which are main objects in this thesis.

Let $[n]$ be the set $\{\, 1, 2, \dots, n \,\}$. For a matroid $M = ([n], \mathcal{B})$ of rank r, we define

$$F_M = F_M(\boldsymbol{x}) = \sum_{B\in\mathcal{B}} \prod_{i\in B} x_i \in \mathbb{Z}[x_1, \dots, x_n],$$

$$P_M = P_M(\boldsymbol{x}) = \sum_{I\in\mathcal{I}(M)} \left(\prod_{i\in I} x_i\right) x_0^{n-|I|} \in \mathbb{Z}[x_0, x_1, \dots, x_n],$$

$$\overline{P}_M = \overline{P}_M(\boldsymbol{x}) = \left(\frac{\partial}{\partial x_0}\right)^{n-r} P_M(\boldsymbol{x}) \in \mathbb{Z}[x_0, x_1, \dots, x_n].$$

We call them the *basis generating polynomial*, *independent set generating polynomial*, *reduced independent set generating polynomial* of M, respectively. By definition,

$$P_M = P_M(\boldsymbol{x}) = \sum_{i=0}^{\operatorname{rank}(M)} x_0^{n-\operatorname{rank}(T^iM)} F_{T^iM}(\boldsymbol{x}).$$

It is known that the generating polynomials are Lorentzian.

Proposition 3.2 (Anari–Gharan–Vinzant [**3, 2, 1**], Brändén–Huh [**7, 8**])**.** *The generating polynomials F_M, P_M, and $\overline{P}_M$ of a matroid M for any matroid with* $\operatorname{rank} M \geq 2$ *are Lorentzian.*

Since Lorentzian property implies the log-concavity, we have the following.

Proposition 3.3. *The generating polynomials F_M, P_M, and $\overline{P}_M$ of a matroid M for any matroid with* $\operatorname{rank} M \geq 2$ *are log-concave on the positive orthant.*

Let us see some example of the generating polynomials in Examples 3.4 to 3.6.

Example 3.4. Let $M = U_{4,4} = ([4], \mathcal{B})$, namely, the matroid M is the free matroid on $[4]$ of rank 4. Then, we have $\mathcal{B} = \{\,\{\,1,2,3,4\,\}\,\}$ and $\mathcal{I}(M) = 2^{[4]}$. Hence,

$$\begin{aligned}
F_M &= \sum_{B\in\mathcal{B}} \prod_{i\in B} x_i = x_1x_2x_3x_4,\\
P_M &= \sum_{I\in\mathcal{I}(M)} \left(\prod_{i\in I} x_i\right) x_0^{4-|I|}\\
&= x_1x_2x_3x_4 + x_0x_1x_2x_3 + x_0x_1x_2x_4 + x_0x_1x_3x_4 + x_0x_2x_3x_4\\
&\quad + x_0^2x_1x_2 + x_0^2x_1x_3 + x_0^2x_1x_4 + x_0^2x_2x_3 + x_0^2x_2x_4 + x_0^2x_3x_4\\
&\quad + x_0^3x_1 + x_0^3x_2 + x_0^3x_3 + x_0^3x_4 + x_0^4,\\
\overline{P}_M &= \left(\frac{\partial}{\partial x_0}\right)^{4-4} P_M = P_M.
\end{aligned}$$

Example 3.5. If $M = U_{r,n}$ on $[n]$, then

$$\begin{aligned}
F_M &= e_r(x_1,\ldots,x_n),\\
P_M &= \sum_{k=n-r}^{n} x_0^k e_{n-k}(x_1,\ldots,x_n),\\
\overline{P}_M &= \sum_{k=n-r}^{n} x_0^{k-(n-r)} e_{n-k}(x_1,\ldots,x_n)\\
&= \sum_{k=0}^{r} x_0^k e_{r-k}(x_1,\ldots,x_n),
\end{aligned}$$

where $e_r(x_1,\ldots,x_n)$ is the elementary symmetric polynomial of degree r in n variables, and $e_0(x_1,\ldots,x_n) = 1$.

Example 3.6. Let M be the graphic matroid of a connected graph Γ with n vertices and m edges. The rank of $M(\Gamma)$ is $n-1$. Then, we have

$$\begin{aligned}
F_M &= \text{the Kirchhoff polynomial of } \Gamma \text{ (of degree } n-1\text{)},\\
P_M &= \sum_{k=0}^{n-1} \sum_{\substack{F:\text{forest}\\ \#E(F)=k}} x_0^{m-k} \prod_{i\in E(F)} x_i,\\
\overline{P}_M &= \sum_{k=0}^{n-1} \sum_{\substack{F:\text{forest}\\ \#E(F)=k}} x_0^{m-k-(m-n+1)} \prod_{i\in E(F)} x_i\\
&= \sum_{k=0}^{n-1} \sum_{\substack{F:\text{forest}\\ \#E(F)=k}} x_0^{n-k-1} \prod_{i\in E(F)} x_i,
\end{aligned}$$

where $\prod_{i\in\emptyset} x_i = 1$.

We give some properties of the generating polynomials for our goals in this thesis.

Generating polynomials satisfy the deletion-contraction formula. These formulae in Proposition 3.7 are useful to the inductive proofs.

Proposition 3.7. *For any $e \in E$ which is not a loop or a coloop, we have*

$$\begin{aligned} F_M &= F_{M\backslash e} + x_e F_{M/e}, \\ P_M &= P_{M\backslash e} + x_e P_{M/e}, \\ \overline{P}_M &= \overline{P}_{M\backslash e} + x_e \overline{P}_{M/e}. \end{aligned}$$

In particular, if matroid M_0 is obtained by deleting $e_1, \dots, e_k \in E$ from M, then we have

$$\begin{aligned} F_{M_0} &= F_M|_{x_{e_1}=\cdots=x_{e_k}=0}, \\ P_{M_0} &= P_M|_{x_{e_1}=\cdots=x_{e_k}=0}, \\ \overline{P}_{M_0} &= \overline{P}_M|_{x_{e_1}=\cdots=x_{e_k}=0}. \end{aligned}$$

As the partial derivation of the generating polynomials, we can easily find the following.

Proposition 3.8. *Let M be a matroid on $[n]$ of rank r.*

- *If $i \in [n]$ is a loop, then $\frac{\partial}{\partial x_i} F_M = \frac{\partial}{\partial x_i} P_M = 0$.*
- *If $i \in [n]$ is not a loop, then $\frac{\partial}{\partial x_i} F_M = F_{M/i}$ and $\frac{\partial}{\partial x_i} P_M = P_{M/i}$.*
- *If $i_1, i_2 \in [n]$ are parallel, then $\frac{\partial}{\partial x_{i_1}} F_M = \frac{\partial}{\partial x_{i_2}} F_M$ and $\frac{\partial}{\partial x_{i_1}} P_M = \frac{\partial}{\partial x_{i_2}} P_M$.*

Proposition 3.9. *Let M be a matroid on $[n]$. If $[n] = E_0 \sqcup E_1 \sqcup \cdots \sqcup E_s$ is the parallel class decomposition, then*

$$F_M = F_{\overline{M}}\left(\textstyle\sum_{i\in E_1} x_i, \dots, \sum_{i\in E_s} x_i\right)$$

and

$$P_M = x_0^{n-s} P_{\overline{M}}\left(x_0, \textstyle\sum_{i\in E_1} x_i, \dots, \sum_{i\in E_s} x_i\right),$$

where E_0 is the set of loops and we consider that $\overline{M}$ is a matroid on $[s]$ such that i corresponds to an element in E_i for $i = 1, 2, \dots, s$.

Proposition 3.9 will be use in the proof of Theorem 6.1.

Proposition 3.10 (Murai–Nagaoka–Yazawa [**16**]). *Let M be a simple matroid on $[n]$ of rank $r \geq 2$.*

- *$\frac{\partial}{\partial x_1} F_M, \dots, \frac{\partial}{\partial x_n} F_M$ are $\mathbb{R}$-linearly independent.*
- *If $M \neq U_{r,n}$ then $\frac{\partial}{\partial x_0} \overline{P}_M, \frac{\partial}{\partial x_1} \overline{P}_M, \dots, \frac{\partial}{\partial x_n} \overline{P}_M$ are $\mathbb{R}$-linearly independent.*

Proposition 3.10 means that the generating polynomials for a matroid satisfy the hypothesis of Proposition 1.18. This fact is the key of the proof of Theorem 6.1, which is one of our goals in this thesis.

CHAPTER 4

Cyclic matrices

In this chapter, we study cyclic matrices, in particular, block cyclic matrices. The theorems in this chapter give formulae of the eigenvalues and the determinants of some matrices. We apply the theorems to the matrices defined by graphs in Chapter 5.

A *cyclic matrix* is a matrix $C = (c_{i,j})_{1\le i\le m, 1\le j\le n}$ such that $c_{i,j} = c_{i+1,j+1}$ for $1 \le i \le m, 1 \le j \le n$. Note that we take the convention that every subscript less than 1 or greater than n should be shifted into the correct range. Let C be a cyclic matrix of size n. One can see that

$$\boldsymbol{z}_{n,k} = \begin{pmatrix} 1 \\ \zeta_n^k \\ \zeta_n^{2k} \\ \vdots \\ \zeta_n^{(n-1)k} \end{pmatrix}$$

is an eigenvector of C for $0 \le k \le n-1$, where ζ_n is the nth primitive root. Let $C^{(1)}$ be the first row of C. Then the characteristic polynomial $\chi_C(t)$ and the determinant $\det C$ of C are

$$\chi_C(t) = \prod_{k=0}^{n-1} \left(t - C^{(1)}\boldsymbol{z}_{n,k}\right),$$

$$\det C = \prod_{k=0}^{n-1} C^{(1)}\boldsymbol{z}_{n,k}.$$

1. Block cyclic matrices

We now consider the block cyclic matrices. A *block cyclic matrix* is a block matrix such that each block is cyclic. A cyclic matrix is a block cyclic matrix with one block, and an $m \times n$ matrix is a block cyclic matrix with $m \times n$ blocks, we regard each entry as a block. We study the three types of block cyclic matrices. Let I_n be the identity matrix of size n, and J_{mn} the all-one matrix of size $m \times n$. Let $l \in \mathbb{Z}$, $\boldsymbol{d} = (d_1, d_2, \dots, d_l)$, and $\delta = d_1 + d_2 + \cdots + d_l$. Let C^{ij} be a cyclic matrix of size $d_i \times d_j$ for $1 \le i, j \le l$.

Type I. The case where $\boldsymbol{d} = (n, \dots, n)$ i.e., a block matrix whose blocks are $n \times n$ cyclic matrices. Let C be $(C^{ij})_{1\le i,j\le l}$, C^{ij} an $n \times n$ matrix for each i, j, and $c_{ij}^{(k)}$ an eigenvalue of C^{ij} associated with an eigenvector $\boldsymbol{z}_{n,k}$. For C and $0 \le k \le n-1$, we define the $l \times l$ matrix $\bar{C}_k$ by

$$\bar{C}_k = \left(c_{ij}^{(k)}\right)_{1 \le i,j \le l}.$$

Theorem 4.1 (Yazawa [**25**]). *Let $(w_i)_{1 \le i \le l} \in \mathbb{C}^l$ be an eigenvector of $\bar{C}_k$ belonging to the eigenvalue λ. Then $(w_i \boldsymbol{z}_{n,k})_{1 \le i \le l} \in \mathbb{C}^{nl}$ is an eigenvector of C associated with λ. Hence*

$$\chi_C(t) = \prod_{k=0}^{n-1} \chi_{\bar{C}_k}(t),$$
$$\det C = \prod_{k=0}^{n-1} \det \bar{C}_k.$$

We apply Theorem 4.1 to the matrix defined by the complete bipartite graph in Theorem 4.5.

Type II. The case where $\boldsymbol{d} = (2n, 2n, \dots, 2n, n)$. Let D be the block matrix $D = (D^{ij})_{1 \le i,j \le l}$ defined by

$$D^{ij} = \begin{cases} \text{a } 2n \times 2n \text{ cyclic matrix} & 1 \le i, j \le l-1, \\ \text{an } n \times n \text{ cyclic matrix} & i = j = l, \\ \begin{pmatrix} X_i \\ X_i \end{pmatrix} & j = l, \\ \begin{pmatrix} Y_j & Y_j \end{pmatrix} & i = l, \end{cases}$$

where X_i and Y_j are $n \times n$ cyclic matrices.

For $0 \le k \le 2n-1$, we define the $l \times l$ matrix $\bar{D}_k = (d_{ij}^k)_{1 \le i,j \le l}$ as follows: If k is even, then we define

$$d_{ij}^k = \begin{cases} (D^{ij})^{(1)} \boldsymbol{z}_{2n,k} & \text{if } 1 \le j \le l-1, \\ (D^{ij})^{(1)} \boldsymbol{z}_{n,\frac{k}{2}} & \text{if } j = l. \end{cases}$$

If k is odd, then

$$d_{ij}^k = \begin{cases} (D^{ij})^{(1)} \boldsymbol{z}_{2n,k} & \text{if } 1 \le j \le l-1, \\ 0 & \text{if } j = l. \end{cases}$$

Theorem 4.2 (Yazawa [**25**]). *The characteristic polynomial of D is*

$$\begin{aligned} \chi_D(t) &= \left(\prod_{k:even} \chi_{\bar{D}_k}(t)\right)\left(\prod_{k:odd} \frac{1}{t}\chi_{\bar{D}_k}(t)\right) \\ &= \frac{1}{t^n} \prod_{k=0}^{2n-1} \chi_{\bar{D}_k}(t). \end{aligned}$$

We apply Theorems 4.1 and 4.2 to the matrix defined by the complete graph in Theorem 4.4.

Type III. The case where $\boldsymbol{d} = (d_1, d_2, \ldots, d_l)$. The block size are more general but the entries in each block are at most two numbers. For a square matrix A of size l, $\boldsymbol{d} = (d_1, d_2, \ldots, d_l)$, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, we define

$$T(A, \boldsymbol{d}) = (a_{ij} J_{d_i d_j})_{1 \le i,j \le l},$$
$$D(\lambda, \boldsymbol{d}) = \begin{pmatrix} \lambda_1 I_{d_1} & & & \mathbf{0} \\ & \lambda_2 I_{d_2} & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_l I_{d_l} \end{pmatrix}.$$

We define the square matrix $M(A, \lambda, \boldsymbol{d})$ of size $d_1 + \cdots + d_l$ by

$$M(A, \lambda, \boldsymbol{d}) = T(A, \boldsymbol{d}) + D(\lambda, \boldsymbol{d}).$$

We also define the square matrix $\bar{M}(A, \lambda, \boldsymbol{d})$ of size l by

$$\bar{M}(A, \lambda, \boldsymbol{d}) = \operatorname{diag}(d_1, \ldots, d_l) A + \operatorname{diag}(\lambda_1, \ldots, \lambda_l).$$

Theorem 4.3 (Yazawa [**25**]). *For a matrix A of size l, $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\boldsymbol{d} = (d_1, d_2, \ldots, d_l)$, we have*

$$\chi_{M(A,\lambda,\boldsymbol{d})}(t) = \chi_{\bar{M}(A,\lambda,\boldsymbol{d})}(t) \prod_{i=1}^{l} (t - \lambda_i)^{d_i - 1},$$
$$\det M(A, \lambda, \boldsymbol{d}) = \det \bar{M}(A, \lambda, \boldsymbol{d}) \prod_{i=1}^{l} \lambda_i^{d_i - 1}.$$

Theorems 4.1 to 4.3 claim that a block cyclic matrix can reduce to a smaller size matrix. The details and proofs of Theorems 4.1 to 4.3 are in [**25**].

2. Block cyclic matrices arising from graphs

The block cyclic matrices are often appeared in a situation that a cyclic group action on an object. We consider

(1) the cyclic group C_n of order n action on the graph K_n,
(2) $C_m \times C_n$ action on the graph $K_{m,n}$.

First, we consider (1). Let $V = \{0, 1, \ldots, n-1\}$ and E be the vertex set and the edge set of the complete graph K_n, respectively. Let $H = (h_{e,e'})_{e,e' \in E}$ be the matrix defined by

$$h_{e,e'} = \begin{cases} \alpha, & \#(e \cap e') = 2, \\ \beta, & \#(e \cap e') = 1, \\ \gamma, & \#(e \cap e') = 0. \end{cases}$$

The index set of the matrices are the edges set of graphs, and the entries are determined by how to connect index edges. Consider the following action from the cyclic group C_n generated by σ on V:

$$\sigma(i) = i + 1 \pmod{n}.$$

The action induces an action on E by the following:

$$\sigma(\{ i, j \}) = \{ i + 1, j + 1 \}.$$

The orbit decomposition of the edge set gives the block decomposition of H. Moreover, we can arrange the indexes such that each block of H is cyclic. The behavior of the action C_n on the edge set of K_n depends on the parity of n. If n is odd, then the matrix H is a block cyclic matrix of type I. If n is even, then the matrix H is a block cyclic matrix of type II. We get the same consequence in either case as Theorem 4.4. We prove it in the next section.

Theorem 4.4 (Yazawa [**26**]). *The eigenvalues of H are*

$$\begin{aligned}
\lambda_1 &= \alpha + (2n-4)\beta + \frac{(n-2)(n-3)}{2}\gamma, \\
\lambda_2 &= \alpha - 2\beta + \gamma, \\
\lambda_3 &= \alpha + (n-4)\beta - (n-3)\gamma.
\end{aligned}$$

The dimensions d_λ of the eigenspaces of H associate with the eigenvalues λ are

$$d_{\lambda_1} = 1, \qquad d_{\lambda_2} = \binom{n}{2} - n, \qquad d_{\lambda_3} = n - 1.$$

We apply Theorem 4.4 to Theorem 5.4, which is concerned with the Hessian matrix of the complete graph.

Next, we consider (2). Let $V = X \sqcup Y$ and E be the vertex set and the edge set of the complete bipartite graph $K_{X,Y}$, respectively. Let $X = \{ 1, \ldots, m \}$, and $Y = \{ \bar{1}, \ldots, \bar{n} \}$. Let $H = (h_{e,e'})_{e,e' \in E}$ be the matrix defined by

$$h_{e,e'} = \begin{cases} \alpha, & e = e', \\ \beta, & e \cap e' \in X, \\ \gamma, & e \cap e' \in Y, \\ \delta, & e \cap e' = \emptyset. \end{cases}$$

The definition of the matrix H is similar to the matrix H in (1), the index set are the edges, and the entries are determined by how to connect index edges. Consider the following action from the direct product group $C_m \times C_n$ of the cyclic groups C_m and C_n generated by σ and σ', respectively, on $V = X \sqcup Y$:

$$\sigma(i) = i + 1 \pmod{n}, \qquad \sigma'(\bar{i}) = \overline{i+1} \pmod{m}.$$

The action induces an action on E by the following:

$$(\sigma, \sigma')(\{\, i, \bar{j} \,\}) = \left\{\, i+1, \overline{j+1} \,\right\}.$$

The orbit decomposition of the edge set gives the block decomposition of H. Moreover, we can arrange the indexes such that each block of H is cyclic, and the entries in each block of H are at most two. Therefore, the matrix H is a block matrix of type III, which $\boldsymbol{d} = \{\, n, n, \dots, n \,\}$ and $l = m$. We prove it in the next section.

Theorem 4.5 (Yazawa [**26**])**.** *The eigenvalues of H are*

$$\begin{aligned}
\lambda_1 &= \alpha + (n-1)\beta + (m-1)\gamma + (m-1)(n-1)\delta,\\
\lambda_2 &= \alpha + (n-1)\beta - \gamma - (n-1)\delta,\\
\lambda_3 &= \alpha - \beta + (m-1)\gamma - (m-1)\delta,\\
\lambda_4 &= \alpha - \beta - \gamma + \delta.
\end{aligned}$$

The dimensions d_λ of the eigenspaces of H associate with the eigenvalues λ are

$$d_{\lambda_1} = 1, \quad d_{\lambda_2} = m-1, \quad d_{\lambda_3} = n-1, \quad d_{\lambda_4} = (m-1)(n-1).$$

We apply Theorem 4.5 to Theorem 5.6, which is concerned with the Hessian matrix of the complete graph.

3. Proofs of Theorems

A special case of Theorems 4.4 and 4.5 was shown in [**25**]. The case is for spanning trees. As the generalization of them to the case of forests, Theorems 4.4 and 4.5 was given in [**26**]. Since direct analogue of the proof in [**25**] works, the details of proof was omitted in [**26**]. In this section, we completes Theorems 4.4 and 4.5.

3.1. Proof of Theorem 4.4. We prove Theorem 4.4. Let $V = \{\, 0, 1, \dots, n-1 \,\}$ and E be the vertex set and the edge set of the complete graph K_n, respectively. Let C_n be the cyclic group generated by σ. The proof differs depending on whether n is odd or even.

First, let n be odd and $n = 2l+1$. Let $e_i = \{\, 0, i \,\} \in E$ for $1 \le i \le l$. We can see that

$$E = \bigsqcup_{i=1}^{l} \left\{\, \sigma^k(e_i) \;\middle|\; 0 \le k \le n-1 \,\right\}.$$

In other words, the edges e_i are a complete set of representative of E. For $1 \le i, j \le l$, we define

$$\begin{aligned}
C^{ij} &= \left(h_{\sigma^k(e_i), \sigma^{k'}(e_j)} \right)_{0 \le k, k' \le n-1},\\
C &= \left(C^{ij} \right)_{1 \le i, j \le l},
\end{aligned}$$

where $h_{\bullet,\bullet}$ are entries in H. By the way of construction of the matrix C, we can see that C is a matrix rearranged from H, and C is a block cyclic matrix of type I. For $0 \le k \le n-1$, let

$$\bar{C}_k = \left((C^{ij})^{(1)} \boldsymbol{z}_{n,k}\right)_{1 \le i,j \le l},$$

where the notations, $C^{(1)}$ and $\boldsymbol{z}_{n,k}$, are in the first of this chapter. We separate in the case $k \neq 0$ and $k = 0$.

Lemma 4.6. *Let $1 \le k \le n-1$. Then*

$$\bar{C}_k - (\alpha - 2\beta + \gamma) I_l = \left((\beta - \gamma)\xi_i \xi'_j\right)_{1 \le i,j \le l},$$

where $\xi_i = 1 + \zeta_n^{ik}$ and $\xi'_j = 1 + \zeta_n^{-jk}$ for all i, j. Moreover the rank of $\bar{C}_k - (\alpha - 2\beta + \gamma) I_l$ is one.

Proof. Let us fix k and compute $(C^{ij})^{(1)} \boldsymbol{z}_{n,k}$. In this case,

$$(C_{ij})(1) = \left(h_{e_i,\sigma^0(e_j)}, h_{e_i,\sigma^1(e_j)}, \dots, h_{e_i,\sigma^{n-1}(e_j)}\right).$$

First we consider the case where $e_i \neq e_j$. The edges e_i and $\sigma^l(e_j)$ share their vertices if and only if $l = 0$, $l = i$, $j + l = 0$, and $j + l = i$. Since $e_i \neq e_j$, we have $e_i \neq \sigma^l(e_j)$ for any l. Hence if $l \in \{\, 0, i, -j, i-j \,\}$, then

$$h_{e_i,\sigma^l(e_j)} = \beta,$$

and if $l \notin \{\, 0, i, -j, i-j \,\}$, then

$$h_{e_i,\sigma^l(e_j)} = \gamma.$$

Therefore

$$\begin{aligned}
(C^{ij})^{(1)} \boldsymbol{z}_{n,k} &= \beta\Big(\sum_{l \in \{\, 0,i,-j,i-j \,\}} \zeta_n^{kl}\Big) + \gamma\Big(\sum_{l \notin \{\, 0,i,-j,i-j \,\}} \zeta_n^{kl}\Big) \\
&= (\beta - \gamma)(1 + \zeta_n^{ki} + \zeta_n^{-kj} + \zeta_n^{k(i-j)}) \\
&= (\beta - \gamma)\xi_i \xi'_j.
\end{aligned}$$

Next we consider the case where $e_i = e_j$. The edges e_i and $\sigma^l(e_i)$ share their vertices if and only if $l = 0, l = i$ and $l + i = 0$. If $l = 0$, then

$$h_{e_i,e_i} = \alpha,$$

if $l = i$ or $l = -i$, then

$$h_{e_i,\sigma^l(e_i)} = \beta.$$

Hence if $l \notin \{\, 0, i, -i \,\}$, then

$$h_{e_i,\sigma^l(e_i)} = \gamma.$$

Therefore

$$
\begin{aligned}
(C^{ij})^{(1)}\boldsymbol{z}_{n,k} &= \alpha + \beta(\sum_{l\in\{\,i,-i\,\}} \zeta_n^{kl}) + \gamma(\sum_{l\notin\{\,0,i,-i\,\}} \zeta_n^{kl}) \\
&= (\alpha-\gamma) + (\beta-\gamma)(\zeta_n^{ki}+\zeta_n^{-ki}) \\
&= (\alpha-\gamma) + (\beta-\gamma)(\zeta_n^{ki}+\zeta_n^{-ki}) + 2(\beta-\gamma) - 2(\beta-\gamma) \\
&= (\alpha-\gamma) - 2(\beta-\gamma) + (\beta-\gamma)(2+\zeta_n^{ki}+\zeta_n^{-ki}) \\
&= (\alpha-2\beta+\gamma) + (\beta-\gamma)\xi_i\xi_i'.
\end{aligned}
$$

We have

$$
\begin{aligned}
\bar{C}_k - (\alpha-2\beta+\gamma)I_l &= \big((\beta-\gamma)\xi_i\xi_j'\big)_{1\le i,j\le l} \\
&= (\beta-\gamma)\begin{pmatrix}\xi_1\\ \vdots \\ \xi_l\end{pmatrix}\begin{pmatrix}\xi_1' & \cdots & \xi_l'\end{pmatrix}.
\end{aligned}
$$

Hence the rank of $\bar{C}_k - (\alpha-2\beta+\gamma)I_l$ is one. □

Proposition 4.7. *For $1 \le k \le n-1$, the eigenvalues of $\bar{C}_k$ are $\lambda_1 = \alpha-2\beta+\gamma$ and $\lambda_2 = \alpha+(n-4)\beta-(n-3)\gamma$. The dimensions d_λ of the eigenspaces of $\bar{C}_k$ associate with the eigenvalues λ are $d_{\lambda_1} = l-1$ and $d_{\lambda_2} = 1$.*

Proof. The trace of $\bar{C}_k$ is

$$
\begin{aligned}
\sum_{i=1}^{l}((\alpha-2\beta+\gamma)+(\beta-\gamma)\xi_i\xi_i') &= l(\alpha-2\beta+\gamma)+(\beta-\gamma)\sum_{i=1}^{l}\xi_i\xi_i' \\
&= l(\alpha-2\beta+\gamma)+(\beta-\gamma)(2l-1) \\
&= l\alpha-\beta-(l-1)\gamma.
\end{aligned}
$$

Note that $l = \frac{n-1}{2}$. The other eigenvalue of $\bar{C}_k$ is

$$
l\alpha-\beta-(l-1)\gamma-(l-1)(\alpha-2\beta+\gamma) = \alpha+(n-4)\beta-(n-3)\gamma.
$$

Therefore it follows from Lemma 4.6 that the eigenvalues of $\bar{C}_k$ are $\lambda_1 = \alpha-2\beta+\gamma$ and $\lambda_2 = \alpha+(n-4)\beta-(n-3)\gamma$. The dimensions d_λ of the eigenspaces of $\bar{C}_k$ associate with the eigenvalues λ are $d_{\lambda_1} = l-1$ and $d_{\lambda_2} = 1$. □

Similarly, we obtain the result in the case where $k = 0$.

Proposition 4.8. *The eigenvalues of $\bar{C}_0$ are $\lambda_1 = \alpha-2\beta+\gamma$ and $\lambda_2 = \alpha+(2n-4)\beta+\frac{(n-2)(n-3)}{2}\gamma$. The dimensions d_λ of the eigenspaces of $\bar{C}_0$ associate with the eigenvalues λ are $d_{\lambda_1} = l-1$ and $d_{\lambda_2} = 1$.*

Since the matrix C is a block cyclic matrix of type I, we obtain Theorem 4.4 in the case where n is odd by Theorem 4.1 and Propositions 4.7 and 4.8.

Next we consider the case where n is even. Let $n = 2l$. Let $e_i = \{ 0, i \}$ for $1 \leq i \leq l$. We can see that

$$E = \left(\bigsqcup_{i=1}^{l-1} \{ \sigma^k(e_i) \mid 0 \leq k \leq n-1 \} \right) \sqcup \{ \sigma^k(e_l) \mid 0 \leq k \leq l \} .$$

In other words, the edges e_i are a complete set of representative of E. We define the matrix D^{ij} by

$$D^{ij} = \begin{cases} \left(h_{\sigma^k(e_i),\sigma^{k'}(e_j)} \right)_{0 \leq k,k' \leq n-1} & \text{for } 1 \leq i,j \leq l-1, \\ \left(h_{\sigma^k(e_i),\sigma^{k'}(e_j)} \right)_{\substack{0 \leq k \leq n-1 \\ 0 \leq k' \leq l}} & \text{for } 1 \leq i \leq l-1, j = l, \\ \left(h_{\sigma^k(e_i),\sigma^{k'}(e_j)} \right)_{\substack{0 \leq k \leq l \\ 0 \leq k' \leq n-1}} & \text{for } i = l, 1 \leq j \leq l-1, \\ \left(h_{\sigma^k(e_i),\sigma^{k'}(e_j)} \right)_{0 \leq k,k' \leq l} & \text{for } i = j = l, \end{cases}$$

$$D = \left(D^{ij} \right)_{1 \leq i,j \leq l} .$$

By the way of construction of the matrix D, we can see that D is a matrix rearranged from H, and D is a block cyclic matrix of type II. we define the $l \times l$ matrix $\bar{D}_k = (d_{ij}^k)_{1 \leq i,j \leq l}$ as follows: If k is even, then we define

$$d_{ij}^k = \begin{cases} (D^{ij})^{(1)} \boldsymbol{z}_{n,k} & \text{if } 1 \leq j \leq l-1, \\ (D^{ij})^{(1)} \boldsymbol{z}_{l,\frac{k}{2}} & \text{if } j = l. \end{cases}$$

If k is odd, then we define

$$d_{ij}^k = \begin{cases} (D^{ij})^{(1)} \boldsymbol{z}_{n,k} & \text{if } 1 \leq j \leq l-1, \\ 0 & \text{if } j = l. \end{cases}$$

We separate in the cases $k \neq 0$ and $k = 0$.

Lemma 4.9. *Let $1 \leq k \leq 2n-1$. Then*

$$\bar{D}_k - (\alpha - 2\beta + \gamma) I_l = \left(-\xi_i \xi'_j \right)_{1 \leq i,j \leq l} ,$$

where

$$\xi_i = 1 + \zeta_n^{ik}, \qquad \xi'_j = \begin{cases} 1 + \zeta_n^{-jk} & \text{if } 1 \leq j \leq l-1, \\ \frac{1}{2}(1 + \zeta_n^{-lk}) & \text{if } j = l. \end{cases}$$

for all i, j. Moreover the rank of $\bar{D}_k - (\alpha - 2\beta + \gamma) I_l$ is one.

Proposition 4.10. *Let $1 \leq k \leq 2n-1$ and k be odd. The eigenvalues of $\bar{D}_k$ are $\lambda_1 = \alpha - 2\beta + \gamma$, $\lambda_2 = \alpha + (2n-4)\beta + \frac{(n-2)(n-3)}{2}\gamma$ and $\lambda_3 = 0$. The dimensions d_λ of the eigenspaces of $\bar{D}_k$ associate with the eigenvalues λ are $d_{\lambda_1} = l - 2$, $d_{\lambda_1} = 1$ and $d_{\lambda_1} = 1$.*

Let $1 \leq k \leq 2n-1$ and k be even. The eigenvalues of $\bar{D}_k$ are $\lambda_1 = \alpha - 2\beta + \gamma$ and $\lambda_2 = \alpha + (2n-4)\beta + \frac{(n-2)(n-3)}{2}\gamma$. The dimensions d_λ

of the eigenspaces of $\bar{D}_k$ associate with the eigenvalues λ are $d_{\lambda_1} = l-1$ and $d_{\lambda_2} = 1$.

We can show Lemma 4.9 and Proposition 4.10 as same as Lemma 4.6 and Proposition 4.7, respectively. We can also show Proposition 4.11 as same as Proposition 4.8.

Proposition 4.11. *The eigenvalues of $\bar{D}_0$ are $\lambda_1 = \alpha - 2\beta + \gamma$ and $\lambda_2 = \alpha + (2n-4)\beta + \frac{(n-2)(n-3)}{2}\gamma$. The dimensions d_λ of the eigenspaces of $\bar{D}_0$ associate with the eigenvalues λ are $d_{\lambda_1} = l-1$ and $d_{\lambda_2} = 1$.*

Since the matrix D is a block cyclic matrix of type II, we obtain Theorem 4.4 in the case where n is even by Theorem 4.2 and Propositions 4.10 and 4.11. We complete to show Theorem 4.4.

3.2. Proof of Theorem 4.5. We prove Theorem 4.5. Let $V = X \sqcup Y$ and E be the vertex set and the edge set of the complete bipartite graph $K_{X,Y}$, respectively. Let $X = \{ 1, \ldots, m \}$, and $Y = \{ \bar{1}, \ldots, \bar{n} \}$. Let $C_m \times C_n$ be the direct product group of the cyclic groups C_m and C_n of order m and n generated by σ and σ', respectively. Let $e_i = \{ i, \bar{1} \}$ for $1 \leq i \leq m$. We can see that

$$E = \bigsqcup_{i=1}^{m} \left\{ \sigma'^k(e_i) \;\middle|\; 0 \leq k \leq n-1 \right\}.$$

In other words, the edges e_i are a complete set of representative of E. We define

$$\begin{aligned} C^{ij} &= \left(h_{\sigma'^k(e_i),\sigma'^{k'}(e_j)}\right)_{0\leq k,k'\leq n-1}, \\ C &= \left(C^{ij}\right)_{1\leq i.j\leq m}. \end{aligned}$$

Then,

$$\begin{aligned} C^{ii} &= (\alpha-\beta)I_m + \beta J_{mm}, \\ C^{ij} &= (\gamma-\delta)I_m + \delta J_{mm} \end{aligned}$$

for all i and $i \neq j$. For $0 \leq k \leq n-1$, let

$$\bar{C}_k = \left((C^{ij})^{(1)} \boldsymbol{z}_{n,k}\right)_{1\leq i,j\leq l}.$$

Then,

$$\begin{aligned} \bar{C}_0 &= (\alpha-\gamma)I_m + (\gamma + (m-1)\beta)J_{mm}, \\ \bar{C}_k &= (\alpha-\beta-\gamma+\delta)I_m + (\gamma-\delta)J_{mm} \end{aligned}$$

By Theorem 4.1 and the following well-known fact, we obtain Theorem 4.5.

Lemma 4.12. *Let C be the cyclic matrix of size n defined by*

$$C^{ii} = (a-b)I_n + bJ_{nn}.$$

Then, the eigenvalues are $a+(n-1)b$ *and* $a-b$. *The following vectors*

$$\begin{pmatrix}1\\1\\\vdots\\\vdots\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\\\vdots\\\vdots\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\0\\\vdots\\0\end{pmatrix}, \cdots, \begin{pmatrix}1\\0\\\vdots\\\vdots\\0\\-1\end{pmatrix}$$

are eigenvectors associate with $a+(n-1)b$, $a-b, \ldots,$ *and* $a-b$, *respectively. Hence, the dimensions of the eigenspaces of* C *associate with* $a+(n-1)b$ *and* $a-b$ *are* 1 *and* $n-1$, *respectively.*

CHAPTER 5

Hessian matrices of graphs

In this chapter, we consider the generating function for forests in a graph. Then, we compute the Hessian matrix of the generating function. Surprisingly, any Hessian matricies of graphs has exactly one positive eigenvalue, and theirs determinants do not vanish with some values.

Let Γ be a finite connected simple graph. For $1 \leq k \leq \#V(\Gamma) - 1$, we define $\mathcal{F}_\Gamma^k$ to be the collection of the edges of the k components forests in Γ. In other words,

$$\mathcal{F}_\Gamma^k = \{\, E(F) \mid F \text{ is a forest in } \Gamma \text{ with } k \text{ components} \,\}.$$

We define a polynomial $F_{\Gamma,k} \in \mathbb{R}[x_{ij} | \{\, i,j \,\} \in E(\Gamma)]/(x_{ij} - x_{ji})$ of $\mathcal{F}_\Gamma^k$ by

$$F_{\Gamma,k} = \sum_{E(F) \in \mathcal{F}_\Gamma^k} \prod_{\{\, i,j \,\} \in E(F)} x_{ij}.$$

One can see that in the case where $k = 1$, an element in $\mathcal{F}_\Gamma^k$ is the edges of a spanning tree in Γ, and the $F_{\Gamma,k}$ is the Kirchhoff polynomial of Γ. Let $M(\Gamma)$ be the graphic matroid on $E(\Gamma)$ of a finite connected graph Γ. Then, $(E(\Gamma), \mathcal{F}_\Gamma^k)$ is a truncated matroid $T^k M(\Gamma)$ of $M(\Gamma)$. Therefore, the polynomial $F_{\Gamma,k}$ is the basis generating polynomial of $T^k M(\Gamma)$. Note that a truncated matroid of a graphic matroid is not always a graphic matroid. Obviously, the polynomial $F_{\Gamma,k}$ is a homogeneous polynomial of degree $\#V(\Gamma) - k$, and each term of $F_{\Gamma,k}$ is square-free.

We define the Hessian matrix $H_{F_{\Gamma,k}}$ of $F_{\Gamma,k}$ by

$$H_{F_{\Gamma,k}} = \left(\frac{\partial}{\partial x_e} \frac{\partial}{\partial x_{e'}} F_{\Gamma,k} \right)_{e,e' \in E(\Gamma)}.$$

The matrices $H_{F_{\Gamma,k}}$ are called *the Hessian matrices of* Γ, and the determinant $\det H_{F_{\Gamma,k}}$ is called *the Hessian of* Γ. The matrix $H_{F_{\Gamma,\#V(\Gamma)-1}}$ is always the zero matrix since $F_{\Gamma,\#V(\Gamma)-1}$ is a homogeneous polynomial of degree one. We define $\widetilde{H}_{F_{\Gamma,k}}$ to be the special value of $H_{F_{\Gamma,k}}$ at $x_e = 1$ for all e. The matrix $\widetilde{H}_{F_{\Gamma,k}}$ gives us combinatorial way to compute $H_{F_{\Gamma,k}}$ and information whether the Hessian of a graph vanishes or not. The (e, e')-entry in $\widetilde{H}_{F_{\Gamma,k}}$ is the number of k components forests including the edges e and e'.

It is too complicated to compute the Hessian matrices of graphs directly. There, however, are some graphs can calculate the Hessian matrices directly. In Sections 1 to 3, we compute directly the Hessian matrices of trees, the complete and complete bipartite graphs. In Section 4, there are a theoretical results for the Hessian matrices of all graphs.

1. Trees

We consider the Hessian matrix of the tree T_n with $n \geq 2$ vertices. Note that trees with the same number of vertices have the same generating polynomial $F_{T,k}$. Let $\{ 1, 2, \dots, n \}$ be the edge set. Then, we have

$$F_{T_n, n-k+1} = e_k(x_1, x_2, \dots, x_n)$$

for $2 \leq k \leq n+1$, where $e_k(x_1, x_2, \dots, x_n)$ is the symmetric polynomial of degree k in n variables. Since

$$\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} e_k(x_1, x_2, \dots, x_n) \bigg|_{x_m=1} = \binom{n-2}{k-2}$$

for $i \neq j$. Therefore, we have

$$\widetilde{H}_{F_{T_n, n-k+1}} = \binom{n-2}{k-2} (J_{nn} - I_n).$$

By Lemma 4.12, the eigenvalues of $\widetilde{H}_{F_{T_n, n-k+1}}$ are

$$(n-1)\binom{n-2}{k-2}, -\binom{n-2}{k-2}, \dots, -\binom{n-2}{k-2}.$$

To summarize it, we obtain the following.

Proposition 5.1. *Let $n \geq 2$ and $2 \leq k \leq n+1$. The matrix $\widetilde{H}_{F_{T_n, n-k+1}}$ has exactly one positive eigenvalue. Moreover, the Hessian does not vanish.*

Since $F_{T_n, 1}$ is the Kirchhoff polynomial of T_n, we have the following.

Corollary 5.2. *The Hessian of the Kirchhoff polynomial of T_n does not vanish for $n \geq 2$. Moreover, the matrix evaluated at x_e for all e has exactly one positive eigenvalue.*

Remark 5.3. Proposition 5.1 is shown in [**13**] in more general situation, the higher Hessians of the elementary symmetric polynomial do not vanish.

2. Complete graphs

Here, we compute the Hessian matricies of the complete graph K_n. Let us take a look at some examples. Consider the complete graph K_4 with the vertex set $V = \{ 1, 2, 3, 4 \}$. Then,

$$\begin{aligned}
F_{K_4,3} &= x_{12} + x_{13} + x_{14} + x_{23} + x_{24} + x_{34},\\
F_{K_4,2} &= x_{12}x_{13} + x_{12}x_{14} + x_{12}x_{23} + x_{12}x_{24} + x_{12}x_{34}\\
&\quad + x_{13}x_{14} + x_{13}x_{23} + x_{13}x_{24} + x_{13}x_{34} + x_{14}x_{23}\\
&\quad + x_{14}x_{24} + x_{14}x_{34} + x_{23}x_{24} + x_{23}x_{34} + x_{24}x_{34},\\
F_{K_4,1} &= x_{12}x_{13}x_{24} + x_{12}x_{13}x_{14} + x_{13}x_{14}x_{23} + x_{12}x_{14}x_{23}\\
&\quad + x_{14}x_{23}x_{24} + x_{12}x_{14}x_{34} + x_{13}x_{23}x_{34} + x_{13}x_{23}x_{24}\\
&\quad + x_{12}x_{23}x_{24} + x_{14}x_{23}x_{34} + x_{12}x_{13}x_{34} + x_{13}x_{14}x_{24}\\
&\quad + x_{13}x_{24}x_{34} + x_{12}x_{24}x_{34} + x_{13}x_{24}x_{34} + x_{14}x_{24}x_{34}.
\end{aligned}$$

In the case of $F_{K_4,3}$, the Hessian matrix is zero matrix since the degree of $F_{K_4,3}$ is one. In the case of $F_{K_4,2}$, we have

$$\widetilde{H}_{F_{K_4,2}} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1\\ 1 & 0 & 1 & 1 & 1 & 1\\ 1 & 1 & 0 & 1 & 1 & 1\\ 1 & 1 & 1 & 0 & 1 & 1\\ 1 & 1 & 1 & 1 & 0 & 1\\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

The eigenvalues of $\widetilde{H}_{F_{K_4,2}}$ are $5, -1, -1, -1, -1, -1$. In the case of $F_{K_4,1}$, we have

$$\widetilde{H}_{F_{K_4,1}} = \begin{pmatrix} 0 & 3 & 4 & 3 & 3 & 3\\ 3 & 0 & 3 & 4 & 3 & 3\\ 4 & 3 & 0 & 3 & 3 & 3\\ 3 & 4 & 3 & 0 & 3 & 3\\ 3 & 3 & 3 & 3 & 0 & 4\\ 3 & 3 & 3 & 3 & 4 & 0 \end{pmatrix}.$$

The eigenvalues of $\widetilde{H}_{F_{K_4,1}}$ are $16, -2, -2, -4, -4, -4$. We can see that the Hessian matrices have exactly one positive eigenvalue and are non-degenerate.

Theorem 5.4 (Yazawa [**26**]). *Let $n \geq 3$ and $0 < k \leq n-2$. The matrix $\widetilde{H}_{F_{K_n,k}}$ has exactly one positive eigenvalue. Moreover, the Hessian does not vanish.*

Since $F_{K_n,1}$ is the Kirchhoff polynomial of K_n, we have the following.

Corollary 5.5 (Yazawa [**25**]). *The Hessian of the Kirchhoff polynomial of the complete graph K_n does not vanish for $n \geq 3$. Moreover, the matrix evaluated at $x_e = 1$ for all e has exactly one positive eigenvalue.*

Let us prove Theorem 5.4. We calculate eigenvalues of $\widetilde{H}_{F_{K_n,k}}$. For $e \neq e'$, the (e, e')-entry in $\widetilde{H}_{F_{K_n,k}}$ is the number of k components forests including the edges e and e'. The diagonals are zero since each term of $F_{K_n,k}$ is square-free. Let $\widetilde{H}_{F_{K_n,k}} = (h_{e,e'})_{e,e' \in E}$. By Example 2.7, we have the following:

$$h_{e,e'} = \begin{cases} 0, & \#(e \cap e') = 2, \\ 3n^{n-4}, & \#(e \cap e') = 1, \\ 4n^{n-4}, & \#(e \cap e') = 0. \end{cases}$$

This matrix is the same of the matrix H in Section 2 of Theorem 4.4 where $\alpha = 0, \beta = 3n^{n-4}$, and $\gamma = 4n^{n-4}$. By Theorem 4.4, we obtain Theorem 5.4. See [**26, 25**] for the concrete values of the eigenvalues.

3. Complete bipartite graphs

We consider the Hessian matrix of the complete bipartite graph. Let us take a look at some examples. Consider the complete graph K_4 with the vertex set $V = \{ 1, 2, 3, 4 \}$. Let $X = \{ 1, 2 \}$ and $Y = \{ \bar{1}, \bar{2} \}$. Let $K_{X,Y}$ be the complete bipartite graph with the vertex sets X and Y. Then,

$$\begin{aligned} F_{K_{X,Y},3} &= x_{1\bar{1}} + x_{1\bar{2}} + x_{2\bar{1}} + x_{2\bar{2}}, \\ F_{K_{X,Y},2} &= x_{1\bar{1}}x_{1\bar{2}} + x_{1\bar{1}}x_{2\bar{1}} + x_{1\bar{1}}x_{2\bar{2}} + x_{1\bar{2}}x_{2\bar{1}} + x_{1\bar{2}}x_{2\bar{2}} + x_{2\bar{1}}x_{2\bar{2}}, \\ F_{K_{X,Y},1} &= x_{1\bar{1}}x_{1\bar{2}}x_{2\bar{1}} + x_{1\bar{1}}x_{1\bar{2}}x_{2\bar{2}} + x_{1\bar{1}}x_{2\bar{1}}x_{2\bar{2}} + x_{1\bar{2}}x_{2\bar{1}}x_{2\bar{2}}. \end{aligned}$$

In the case of $F_{K_{X,Y},3}$, the Hessian matrix is zero matrix since the degree of $F_{K_{X,Y},3}$ is one. In the case of $F_{K_{X,Y},2}$, we have

$$\widetilde{H}_{F_{K_{X,Y},2}} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

The eigenvalues of $\widetilde{H}_{F_{K_{X,Y},2}}$ are $3, -1, -1, -1$. In the case of $F_{K_{X,Y}1}$, we have

$$\widetilde{H}_{F_{K_{X,Y},1}} = \begin{pmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 0 \end{pmatrix}.$$

The eigenvalues of $\widetilde{H}_{F_{K_{X,Y},1}}$ are $6, -2, -2, -2$. We can see that the Hessian matrices have exactly one positive eigenvalue and are non-degenerate.

Theorem 5.6 (Yazawa [**26**]). *Consider sets X and Y such that $X \cap Y = \emptyset$, $\#X \geq 2$ and $\#Y \geq 2$. For $0 < k \leq \#X + \#Y - 2$, the matrix*

$\widetilde{H}_{F_{K_{X,Y},k}}$ has exactly one positive eigenvalue. Moreover, the Hessian does not vanish.

Since $F_{K_{X,Y},1}$ is the Kirchhoff polynomial of $K_{X,Y}$, we have the following.

Corollary 5.7 (Yazawa [**25**]). *The Hessian of the Kirchhoff polynomial of the complete bipartite graph does not vanish for $\#X \geq 2$ and $\#Y \geq 2$. Moreover, the matrix evaluated at $x_e = 1$ for all e has exactly one positive eigenvalue.*

Let us prove Theorem 5.6. Fix

$$\begin{aligned} X &= \{ 1, 2, \ldots, m \}, \\ Y &= \{ \bar{1}, \bar{2}, \ldots, \bar{n} \}, \end{aligned}$$

and $m, n \geq 2$. We calculate eigenvalues of $\widetilde{H}_{F_{K_{X,Y},k}}$. For $e \neq e'$, the (e, e')-entry in $\widetilde{H}_{F_{K_n,k}}$ is the number of k components forests including the edges e and e'. To the best of my knowledge, there is no explicit formula as Example 2.7 for the complete bipartite graph. Thus, we just put letters to these numbers. Define

$$\begin{aligned} P &= \left\{ F \in \mathcal{F}^k_{X,Y} \;\middle|\; \{ 1, \bar{1} \}, \{ 1, \bar{2} \} \in E(F) \right\}, \\ Q &= \left\{ F \in \mathcal{F}^k_{X,Y} \;\middle|\; \{ 1, \bar{1} \}, \{ \bar{1}, 2 \} \in E(F) \right\}, \\ R &= \left\{ F \in \mathcal{F}^k_{X,Y} \;\middle|\; \{ 1, \bar{1} \}, \{ 2, \bar{2} \} \in E(F) \right\}, \end{aligned}$$

and

$$p = \#P, \qquad q = \#Q, \qquad r = \#R.$$

Let $\widetilde{H}_{F_{X,Y,k}} = (h_{e,e'})$. Since there is an automorphism of $K_{X,Y}$, we have

$$h_{e,e'} = \begin{cases} 0, & e = e', \\ p, & e \cap e' \in X, \\ q, & e \cap e' \in Y, \\ r, & e \cap e' = \emptyset. \end{cases}$$

This matrix is the same of the matrix H in Section 2 of Theorem 4.5, where $\alpha = 0, \beta = p, \gamma = q$, and $\delta = r$. By Theorem 4.5, we obtain the eigenvalues of $K_{X,Y}$. Then, we need to calculate the signature of the eigenvalues. See [**26, 25**] for the concrete values of the eigenvalues. We will find that Theorem 5.6 holds in [**26, 25**].

4. The other graphs

For general graphs, we obtain similar results to Proposition 5.1, Corollaries 5.2, 5.5 and 5.7, and Theorems 5.4 and 5.6.

Theorem 5.8 (Murai–Nagaoka–Yazawa [**16**]). *Let Γ be a graph. The matrix $\widetilde{H}_{F_{\Gamma,k}}$ has exactly one positive eigenvalue. Moreover, the Hessian does not vanish.*

Since $F_{\Gamma,1}$ is the Kirchhoff polynomial of Γ, we have the following.

Theorem 5.9 (Nagaoka–Yazawa [**17**]). *The Hessian of the Kirchhoff polynomial of a graph does not vanish. Moreover, the matrix evaluated at $x_e = 1$ for all e has exactly one positive eigenvalue.*

We study more precisely Theorem 5.8 in Chapter 6.

Theorem 5.8 implies Theorem 5.9 directly. In [**17**], Theorem 5.9, however, was shown by completely different way. In the rest of this section, we see crucial idea of the proof. See [**17**] for the details.

We know that if the Kirchhoff polynomial is strictly log-concave on the positive orthant, then the Hessian matrix of the Kirchhoff polynomial evaluated positive real numbers has exactly one positive eigenvalue and does not vanish (Proposition 1.3). By Propositions 1.5 and 3.3, we only have to prove that $H_{F_{\Gamma,1}}$ evaluated positive real numbers does not have zero eigenvalues. The following are key idea for the proof:

- We can reduce general graph to the complete graph since any graph is a subgraph of the complete graph with the same number of vertices.
- The Kirchhoff polynomial of the complete graph is the relative invariant of a regular irreducible prehomogeneous vector space.

Here, we prove only the latter. Consider the complete graph K_{n+1} and assign each edge $\{\, i, j \,\}$ to a variable x_{ij}. Note that $x_{ij} = x_{ji}$. the entries in Laplacian $L_{K_{n+1}} = (\ell_{ij})_{1 \le i,j \le r+1}$ is

$$\ell_{ij} = \begin{cases} \left(\sum_{k=1}^{n+1} x_{ik}\right) - x_{ii} & (\text{if } \ i = j), \\ -x_{ij} & (\text{otherwise}). \end{cases}$$

Hence we have

$$\left\{ \, L^{(11)}_{K_{n+1}} \;\middle|\; x_{ij} \in \mathbb{C} \, \right\} = \mathrm{Sym}(n, \mathbb{C}),$$

where $\mathrm{Sym}(n, \mathbb{C})$ is the set of symmetric matrix of size $n \times n$ over $\mathbb{C}$. By definition of the Kirchhoff polynomial, we have

$$F_{K_{n+1}} = \det : \mathrm{Sym}(n, \mathbb{C}) \to \mathbb{C}.$$

Let ρ be the representation of $GL_n(\mathbb{C})$ on $\mathrm{Sym}(n, \mathbb{C})$ such that

$$\rho(P)X = PXP^{\top}$$

for $P \in GL_n(\mathbb{C})$ and $X \in \mathrm{Sym}(n, \mathbb{C})$.

Proposition 5.10 (Sato–Kimura [**20**]). *The triplet $(GL_n(\mathbb{C}), \rho, Sym(n, \mathbb{C}))$ is a regular irreducible prehomogeneous vector space. Moreover, the relative invariant is given by* $\det : \mathrm{Sym}(n, \mathbb{C}) \to \mathbb{C}^*$.

By Proposition 5.10, we obtain

$$\det H_{F_{K_{n+1},1}} = c(F_{K_{n+1},1})^a,$$

where $c \neq 0$ and $a = \binom{n+1}{2} - n - 1$. Since the Kirchhoff polynomial has positive coefficients, the matrix $H_{F_{\Gamma,1}}$ evaluated positive real numbers does not have zero eigenvalues.

CHAPTER 6

Hessian matrices and the strong Lefschetz property of matroids

In Chapter 5, we defined the forest generating polynomial $F_{\Gamma,k}$. Recall that the bases of a graphic matroid of $M(\Gamma)$ is the edge set of spanning trees in Γ. Thus, the polynomial $F_{\Gamma,1}$, the Kirchhoff polynomial, is the basis generating polynomial of $M(\Gamma)$, and the polynomial $F_{\Gamma,1}$ for $k > 1$ is the basis generating polynomial of the truncated matroid $T^{k-1}M(\Gamma)$. We see that the Hessian matrix of $F_{\Gamma,k}$ evaluated $x_e = 1$ has exactly one positive eigenvalues and the Hessian does not vanish as Proposition 5.1, Corollaries 5.2, 5.5 and 5.7, and Theorems 5.4, 5.6, 5.8 and 5.9. This means that the Hessian matrices evaluated $x_e = 1$ of the basis generating polynomials F_M of the truncated matroids M of graphic matroids has exactly one positive eigenvalues and the Hessian does not vanish. In this chapter, we consider the basis generating polynomial F_M, independent set generating polynomial P_M, and reduced independent set generating polynomial $\overline{P}_M$ for all matroid M.

1. Hessian matrices of matroids

In this section, we consider the Hessian matrices of the basis generating polynomial F_M, independent set generating polynomial P_M, and reduced independent set generating polynomial $\overline{P}_M$.

For a simple matroid, the Hessian matrices of F_M and $\overline{P}_M$ are similar results to Proposition 5.1, Corollaries 5.2, 5.5 and 5.7, and Theorems 5.4, 5.6, 5.8 and 5.9, namely, the matrices has exactly one positive eigenvalue, and the Hessians do not vanish. Note that the Hessian vanishes for a non simple matroid since each of the Hessian matrices of F_M, P_M, and $\overline{P}_M$ has the same rows corresponding to parallel edges.

Theorem 6.1 (Murai–Nagaoka–Yazawa [**16**]). *Let M be a simple matroid on $[n]$ of rank $r \geq 2$. Then, we have*

(1) *The Hessian matrix of F_M evaluated $\boldsymbol{a} \in \mathbb{R}^n_{>0}$ has exactly one positive eigenvalue. Moreover, the Hessian does not vanish.*
(2) *The Hessian of P_M evaluated $(0, \boldsymbol{a}) \in \{ 0 \} \times \mathbb{R}^n_{>0}$ is zero.*
(3) *If M is not a uniform matroid, then the Hessian matrix of $\overline{P}_M$ evaluated $\boldsymbol{a} \in \mathbb{R}^{n+1}_{>0}$ has exactly one positive eigenvalue. Moreover, the Hessian does not vanish.*

We see crucial ideas of the proof of Theorem 6.1. See [**16**] for the details. The key for the proof is the relations among Lorentzian polynomials, the strong Lefschetz property, and the Hodge–Riemann relations.

First, we see crucial ideas the proof of (1) and (3). We prove by induction on the rank of matroids. Since F_M and P_M satisfy the deletion-contraction formula (Proposition 3.7), the induction on works. We consider the first step of the induction. By Proposition 3.10, F_M and $\overline{P}_M$ satisfy the hypothesis of Proposition 1.18. If F satisfies the Hodge–Riemann relation at degree one with respect to L, then the Hessian matrix has signature $(+.-, \ldots, -)$. We know that F_M and $\overline{P}_M$ are Lorentzian (Proposition 3.2). For a Lorentzian polynomial F, A_F has the strong Lefschetz property at degree one with the Lefschetz element L is equivalent to A_F satisfies the Hodge–Riemann relation at degree one with respect to L (Proposition 1.19).

By Proposition 3.9, we can see that (2) holds.

We illustrate Theorem 6.1 with the graphic matroid of the complete graph on four vartices.

Example 6.2. Let us consider the graphic matroid $M = M(K_4)$. Assume that $[6]$ is the edge set of K_4. Then, we have

$$\begin{aligned}
F_M &= x_1x_5x_6 + x_1x_5x_4 + x_5x_4x_2 + x_1x_4x_2 + x_4x_2x_6 + x_1x_4x_3 \\
&\quad + x_5x_2x_3 + x_5x_2x_6 + x_1x_2x_6 + x_4x_2x_3 + x_1x_5x_3 + x_5x_4x_6 \\
&\quad + x_5x_6x_3 + x_1x_6x_3 + x_1x_2x_3 + x_4x_6x_3, \\
P_M &= x_0^3F_M + x_0^4(x_1x_2 + x_1x_3 + x_1x_4 + x_1x_5 + x_1x_6 + x_2x_3 + x_2x_4 \\
&\quad + x_2x_5 + x_2x_6 + x_3x_4 + x_3x_5 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6) \\
&\quad + x_0^5(x_1 + x_2 + x_3 + x_4 + x_5 + x_6) + x_0^6, \\
\overline{P}_M &= F_M + x_0(x_1x_2 + x_1x_3 + x_1x_4 + x_1x_5 + x_1x_6 + x_2x_3 + x_2x_4 \\
&\quad + x_2x_5 + x_2x_6 + x_3x_4 + x_3x_5 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6) \\
&\quad + x_0^2(x_1 + x_2 + x_3 + x_4 + x_5 + x_6) + x_0^3.
\end{aligned}$$

By Theorem 5.9, the signature of the Hessian matrix H_{F_M} evaluated positive real numbers is $(+, -, \ldots, -)$. By Proposition 1.14, A_{F_M} has the strong Lefschetz property at degree one with Lefschetz element $L = a_1x_1 + a_2x_2 + \cdots a_nx_n$, where $a_i > 0$ for all i.

Since P_M is divisible by x_0^3, the Hessian matrix of P_M substituting $x_0 = 0$ is the zero matrix.

The Hessian matrix of $\overline{P}_M$ is

$$H_{\overline{P}_M} = \begin{pmatrix} a_{00} & a_{01} & & \cdots & & a_{06} \\ a_{10} & & & & & \\ \vdots & & & H_{F_M} + x_0J_{6,6} - \operatorname{diag}(x_0, \ldots, x_0) & & \\ a_{60} & & & & & \end{pmatrix},$$

where $a_{00} = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$ and $a_{0i} = a_{i0} = x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 - x_i$. Then, the Hessian matrix of $\overline{P}_M$ evaluated *one* for all x_i is

$$\begin{pmatrix} 7 & 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 0 & 4 & 5 & 4 & 4 & 4 \\ 6 & 4 & 0 & 4 & 5 & 4 & 4 \\ 6 & 5 & 4 & 0 & 4 & 4 & 4 \\ 6 & 4 & 5 & 4 & 0 & 4 & 4 \\ 6 & 4 & 4 & 4 & 4 & 0 & 5 \\ 6 & 4 & 4 & 4 & 4 & 5 & 0 \end{pmatrix}.$$

The eigenvalues are $\sqrt{265} + 14, -3, -3, -5, -5, -5$, and $-\sqrt{265} + 14$. We used the mathematical software Sage [**23**] to calculate the eigenvalues. You can see that the matrix has exactly one positive eigenvalue and 6 negative eigenvalues.

2. The log-concavity of the generating polynomials for a matroid

In this section, we discuss the strictly log-concavity of the generating polynomials F_M and $\overline{P}_M$. This section is one of our goals.

By Propositions 3.2 and 3.3, the generating polynomials F_M and $\overline{P}_M$ are Lorentzian, and hence, F_M and $\overline{P}_M$ are log-concave. By Theorem 6.1, the Hessians of F_M and $\overline{P}_M$ are non-degenerate. By Proposition 1.5, we obtain the strictly log-concavity of F_M and $\overline{P}_M$. See [**16**] for the details.

Theorem 6.3 (Murai–Nagaoka–Yazawa [**16**])**.** *Let M be a simple matroid on $[n]$ of rank $r \geq 2$. Then, we have*

(1) *The polynomial F_M is strictly log-concave at $\boldsymbol{a} \in \mathbb{R}^n_{>0}$.*
(2) *If M is not a uniform matroid, then the polynomial $\overline{P}_M$ is strictly log-concave at $\boldsymbol{a} \in \mathbb{R}^{n+1}_{>0}$.*

Actually, the reduced independent set generating polynomials for the uniform matroids are also strictly log-concave on the positive orthant. Since the reduced independent set generating polynomial for the uniform matroid $U_{r,n}$ is

$$\sum_{k=0}^{r} x_0^k e_{r-k}(x_1, \ldots, x_n) = e_r(x_0, x_1, \ldots, x_n),$$

and the elementary symmetric polynomials are the basis generating polynomials for the uniform matroids, it follows from (1) of Theorem 6.3 that the strictly log-concavity of the reduced independent set generating polynomials for the uniform matroids.

Corollary 6.4. *Let M be a simple matroid on $[n]$ of rank $r \geq 2$. Then the polynomial $\overline{P}_M$ is strictly log-concave at $\boldsymbol{a} \in \mathbb{R}^{n+1}_{>0}$.*

Note that the proofs of Theorems 6.1 and 6.3 in [**16**] exclude to consider the reduced independent set generating polynomials for the uniform matroids for a technical reason.

3. The strong Lefschetz property of matroids

In this section, we consider the strong Lefschetz property and the Hodge–Riemann relation at degree one for the graded Artinian Gorenstein algebras defined by F_M, P_M and $\overline{P}_M$.

For a matroid M on $[n]$, we defined algebras as follows:

$$\begin{aligned} A_{F_M} &= \mathbb{R}[x_1, x_2, \dots, x_n]/\operatorname{Ann}(F_M), \\ A_{P_M} &= \mathbb{R}[x_0, x_1, x_2, \dots, x_n]/\operatorname{Ann}(P_M), \\ A_{\overline{P}_M} &= \mathbb{R}[x_0, x_1, x_2, \dots, x_n]/\operatorname{Ann}(\overline{P}_M). \end{aligned}$$

By Proposition 1.14 and Theorem 6.1, we have the following.

Theorem 6.5 (Murai–Nagaoka–Yazawa [**16**]). *Let $L = a_1x_1 + a_2x_2 + \cdots a_nx_n$ and $L' = a_0x_0 + L$, where $a_i > 0$ for all i. For a simple matroid M on $[n]$ with rank $r \geq 2$, we have the following.*

(1) *A_{F_M} has the strong Lefschetz property at degree one with Lefschetz element L, and A_{F_M} satisfies the Hodge–Riemann relation at degree one with respect to L.*
(2) *A_{P_M} does not satisfy the Hodge–Riemann relation at degree one with respect to L.*
(3) *If M is not uniform matroid, then $A_{\overline{P}_M}$ has the strong Lefschetz property at degree one with Lefschetz element L', and $A_{\overline{P}_M}$ satisfies the Hodge–Riemann relation at degree one with respect to L'.*

Let A_F be a graded Artinian Gorenstein algebra, where the top degree is at most 4. If the first Hessian does not vanish, then A_F has the strong Lefschetz property. Therefore, we have the following.

Corollary 6.6. *We have the following:*

- *Let M be a matroid with rank ≤ 4. The algebra A_{F_M} has the strong Lefschetz property.*
- *Let M be a matroid with rank ≤ 3. The algebra $A_{\overline{P}_M}$ has the strong Lefschetz property.*

Remark 6.7. Let $f_{r,n}$ denote the number of labeled matroid on $[n]$ of rank r. Then we have the following:

- $f_{0,n} = 1$.
- $f_{n,n} = 1$.
- $f_{1,n} = 2^n - 1$.
- $f_{r,n} = f_{n-r,n}$, for $0 \leq r \leq n$.

Table 1 is a table of $f_{r,n}$ with $n \leq 8$. See [**9**] for more details.

TABLE 1. The number of labeled matroids

$r \setminus n$	0	1	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	1	1
1		1	3	7	15	31	63	127	255
2			1	7	36	171	813	4012	20891
3				1	15	171	2053	33442	1022217
4					1	31	813	33442	8520812
5						1	63	4012	1022217
6							1	127	20891
7								1	255
8									1
	1	2	5	16	68	406	3807	75164	10607540

TABLE 2. The number of isomorphism classes of matroids

$r \setminus n$	0	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1	1
1		1	2	3	4	5	6	7	8	9
2			1	3	7	13	23	37	58	87
3				1	4	13	38	108	325	1275
4					1	5	23	108	940	190214
5						1	6	37	325	190214
6							1	7	58	1275
7								1	8	87
8									1	9
9										1
	1	2	4	8	17	38	98	306	1724	383172

Remark 6.8. Let $f'_{r,n}$ denote the number of isomorphism classes of matroids on $[n]$ of rank r. Then we have the following:

- $f'_{0,n} = 1$.
- $f'_{n,n} = 1$.
- $f'_{1,n} = n$.
- $f'_{r,n} = f_{n-r,n}$, for $0 \leq r \leq n$.

Table 2 is a table of $f'_{r,n}$ with $n \leq 8$. See [**19**] for more details.

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