博　士　論　文　の　内　容　の　要　旨
Abstract of Doctoral Dissertation

氏名 Full Name	矢澤　明喜子 Yazawa, Akiko
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（博士論文の内容の要旨　Abstract of Doctoral Dissertation）

We consider the generating polynomials for matroids and graded Artinian Gorenstein algebras defined by the polynomials. We study the strong Lefschetz property of the algebras and strictly log-concavity of the polynomials. We show their properties by using the Hessian matrices of the polynomials. This thesis is based on the papers [7, 8, 10, 11].

First, we recall the strong Lefschetz property. The strong Lefschetz property is a ring abstraction of the hard Lefschetz theorem. Let $A = \bigoplus_k A_k$ be a graded Artinian $\mathbb{K}$-algebra over a field $A_0 = \mathbb{K}$ of characteristic zero. We say that A has the *strong Lefschetz property* if there exists an element $L \in A_1$ such that the multiplication map $\times L^{s-2k} \colon A_k \to A_{s-k}$ is bijective for $k \in \{0, 1, \dots, \lfloor \frac{s}{2} \rfloor\}$.

We usually consider the strong Lefschetz property of a graded Artinian Gorenstein algebra. Let $A = \bigoplus_{k=0}^{s} A_k$ be a graded Artinian ring over $\mathbb{K}$. It is known that the algebra A is a standard grading Gorenstein algebra with $\mathbb{K}$ of characteristic zero if and only if there exists a homogeneous polynomial F such that $A \simeq \mathbb{K}[x_1, \dots, x_n]/\operatorname{Ann}(F)$, where $\operatorname{Ann}(F)$ is the annihilator ideal generated by the polynomials annihilating F by partial derivative operator. The Hessian criterion of the strong Lefschetz property of a graded Artinian Gorenstein algebra is known: Let $F \in \mathbb{K}[x_1, \dots, x_n]$ be a hompgeneous polynomial of degree s. We define A_F to be $\mathbb{K}[x_1, \dots, x_n]/\operatorname{Ann}(F)$. Let A_k be the homogeneous spaces of A_F, and Λ_k a basis for A_k. We define the matrix $H_F^{(k)}$ by

$$H_F^{(k)} = \left(e_i \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) e_j \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) F \right)_{e_i, e_j \in \Lambda_k}.$$

We call $H_F^{(k)}$ the *kth Hessian matrix* of F with respect to the basis Λ_k. Note that if we can take a basis $\{ x_1, \dots, x_n \}$ for A_1, then the first Hessian matrix $H_F^{(1)}$ with respect to the basis is a usual Hessian matrix H_F.

Proposition 1 ([9, 6])**.** *Consider a graded Artinian Gorenstein algebra A_F. The algebra A_F has the strong Lefschetz property at degree k if and only if $\det H_F^{(k)}(\boldsymbol{a}) \neq 0$.*

Next, we recall the strictly log-concavity of a homogeneous polynomial. We say that a homogeneous polynomial F is *strictly log-concave* if for $v, v' \in \mathbb{R}^n_{\geq 0}$ and $\lambda \in (0,1)$, we have $F(\lambda v + (1-\lambda)v') > F(v)^\lambda F(v')^{(1-\lambda)}$. We can restate the definition by using matrices: We say that a homogeneous polynomial F is strictly log-concave at $\boldsymbol{a} \in \mathbb{R}^n$ if

$$-FH_F + (\nabla F)(\nabla F)^\top \big|_{x=\boldsymbol{a}}$$

is positive definite, where ∇F is the gradient vector of F. If the log-concavity of F is already known, the degeneracy of the Hessian matrix H_F of F implies the strictly log-concavity.

Proposition 2. *If F is log-concave and $\det H_F\big|_{x=\boldsymbol{a}} \neq 0$, then F is strictly log-concave at $\boldsymbol{a}$.*

Finally, we recall matroids. A matroid is a generalization of the concept of the independency of a vector space. A matroid defined by some vectors is called a vector matroid. A vector matroid is one of important classes of matroids. We also have another important class of matroids, defined from graphs. If we think cycles in a graph as dependent sets, then a graph has the structure of a matroid. A matroid defined from a graph Γ is called the graphic matroid $M(\Gamma)$ of the graph. An independent set of the graphic matroid $M(\Gamma)$ of a graph Γ corresponds to a tree, i.e., a subgraph without cycles, in the graph Γ. For a connected graph Γ, a basis for the graphic matroid $M(\Gamma)$ corresponds to a spanning tree.

Let us consider the basis generating polynomial F_M, independent set generating polynomial P_M, and reduced independent set generating polynomial $\overline{P_M}$ of a matroid M. Let us consider the algebras defined by the polynomials. The log-concavity of the generating polynomials for a matroid is shown in [, , 1, ,]. oreover, the authors of [,] show more stronger property called Lorenztian property, which relates to the strong Lefschetz property. ur main theorem is the following:

Theorem 3 [7] **.** *Let M be a simple matroid on $[n]$ of rank $r \geq$. Then, we have*

1 *The Hessian matrices of F_M evaluated $\boldsymbol{a} \in \mathbb{R}^n_{>0}$ have exactly one positive eigenvalue. Moreover, the Hessian does not vanish.*
The Hessian of P_M evaluated $0, \boldsymbol{a}$ $\in \{ 0 \} \times \mathbb{R}^n_{>0}$ is zero.
If M is not a uniform matroid, then the Hessian matrix of $\overline{P}_M$ evaluated $\boldsymbol{a} \in \mathbb{R}^{n+1}_{>0}$ has exactly one positive eigenvalue. Moreover, the Hessian does not vanish.

y ropositions 1 and , we have the following as a corollary to Theorem .

Theorem 4 [7] **.** *Let M be a simple matroid on $[n]$ of rank $r \geq$. Then, we have*

1 *The polynomial F_M is strictly log-concave on the positive orthant.*
If M is not a uniform matroid, then $\overline{P}_M$ is strictly log-concave on the positive orthant.
The algebras A_{F_M} and $A_{\overline{P}_M}$ have the strong Lefschetz property at degree one.

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