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On matrix Toda brackets in the Baues-Wirsching cohomology

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Abstract. Hardie, Kamps and Marcum have considered the matrix Toda brackets introduced by Barratt in the category of topological spaces from a 2-categorical point of view. Baues and Dreckmann have shown that a class in the third Baues-Wirsching cohomology of a small category $\mathcal{C}$ governs every classical Toda bracket if the bracket is defined with a Toda category in $\mathcal{C}$. Our aim is to generalize such a relationship to that between the class in the cohomology and matrix Toda brackets in a 2-category. Moreover, the non-triviality of the third cohomology is discussed via computation of a matrix Toda bracket in the category of cochain complexes on an additive category.

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Contents

1. Introduction

The secondary operations due to Toda [26], the Toda brackets, play a crucial role in the computation of the homotopy groups of spheres and more general suspension spaces. More precisely, for a diagram $H \xrightarrow{h} G \xrightarrow{g} F \xrightarrow{f} E$ in the category of pointed topological spaces with $gh \simeq 0$ and $fg \simeq 0$, we define the classical Toda bracket $\{f, g, h\}$ in the set of homotopy classes $[\Sigma H, E]$ with an indeterminacy, where ΣH denotes the reduced suspension of H. Therefore, for three elements in appropriate homotopy sets, the Toda bracket may give a new element of a homotopy group of a space.

The construction of the Toda bracket is applicable in a triangulated category [11, 17]. Thus one might expect that such a secondary operation gives us new insights into consideration of an appropriate Abelian category, a triangulated category and a 2-category; see [3, 5, 8, 13, 15, 16, 25].

Baues and Wirsching have introduced cohomology of a small category, the Baues-Wirsching cohomology, with interpretation of the first and second cohomology. The first cohomology group can be described in terms of derivations, the second one classifies linear extensions of categories; see [9]. Baues and Dreckmann [8] have shown that the third Baues-Wirsching cohomology of a small category $\mathcal{C}$ classifies linear track extensions over $\mathcal{C}$; see [8, Theorem 4.6]. Moreover, a particular class, the so-called universal Toda bracket is defined in the third one. If the category $\mathcal{C}$ is a subcategory of the category of topological spaces, then it is shown that the universal Toda bracket governs every Toda bracket if $\mathcal{C}$ contains the Toda category, which defines the bracket.

Generalizing the classical Toda brackets, matrix Toda brackets have been introduced by Barratt [2]. Roughly speaking, the bracket can be defined for a commutative diagram of the form

$$\begin{tikzcd}
W \arrow[r, "w"] \arrow[rr, bend left, "0"] \arrow[dr, bend right, "0"'] & C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "b"] \\
 & A \arrow[r, "a"] & X
\end{tikzcd}$$

in the homotopy category of pointed topological spaces by using homotopies which give the comutativity. In particular, the matrix Toda bracket $\left\{ \begin{matrix} b \\ a \end{matrix}, \begin{matrix} g \\ f \end{matrix}, w \right\}$ is nothing but the classical one $\{b, g, w\}$ if A is a space consisting of a point. Subsequently, Hardie, Kamps and Marcum [13] have developed a categorical approach to such brackets in a 2-category. It is important to mention that the matrix Toda bracket has applications in explicit calculations of homotopy groups and more generalizations; see [3, 14, 15, 20, 21, 22, 23]. The results together with those mentioned above motivate us to investigate the generalized brackets with the cohomology of a small category.

In this manuscript, we show that the *same* class as the universal Toda bracket in the third Baues-Wirsching cohomology of a small category also governs matrix Toda brackets in the sense of Hardie, Kamps and Marcum if the bracket is decomposed into two Toda brackets; see Theorem 2.6. Moreover, using the description of the classical Toda bracket in a triangulated category due to Heller, we examine the non-triviality of a matrix Toda bracket defined in a 2-category of cochain complexes. Indeed, it is possible to represent a matrix Toda bracket with a classical one in an algebraic triangulated category; see Theorem 2.14 for more details. The formula is reminiscent of the original definition of the matrix Toda bracket due to Barratt; see also [24].

The the rest this paper is organized as follows. In Section 2, we describe Theorems 2.6 and 2.14, which are our main theorems. Section 3 gives a brief review of the matrix Toda bracket introduced by Hardie, Kamps and Marcum. After recalling a linear track extension over a small category, the universal Toda bracket is defined. In Section 4, we recall the definition of the Baues-Wirsching cohomology of a small category with coefficients in a natural system. Then we prove Theorems 2.6. Section 5 proves Theorem 2.14 and gives a computational example of a matrix Toda bracket. In order to describe the main theorems in Section 2, we need definitions and terminology although most of the explicit explanations are deferred to the latter sections. For the reader, we here summarize the places in which such key terms are mentioned. We indeed describe

- the *matrix Toda bracket* in the beginning of Section 3,
- a *track category* in Definition 3.2,
- a *linear track extension* in Definition 3.4,
- '*Hypothesis I*' in the assertion of Theorem 2.6 before Definition 3.6,
- the *universal Toda bracket* in Definition 3.6,
- the *Baues-Wirsching cohomolog* and its variants in Definition 4.1 and
- the *algebraic Toda bracket* in the sense of Heller, which is needed to state Theorem 2.14, in the beginning of Section 5.

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2. THE MAIN THEOREMS

In this section, our main theorems are described. For a small category $\mathcal{C}$, we define the category $\mathcal{F}(\mathcal{C})$ of factorizations in $\mathcal{C}$ as follows. The objects are the morphisms in $\mathcal{C}$ and a morphism from α to β is a pair (u, v) of morphisms in $\mathcal{C}$ such that $\beta = u \circ \alpha \circ v$:

$$\begin{tikzcd} \cdot \arrow[r, "u"] & \cdot \\ \cdot \arrow[u, "\alpha"] & \cdot \arrow[l, "v"] \arrow[u, "\beta"'] \end{tikzcd}$$

The composite of maps$(u, v) : \alpha \to \beta$ and $(u', v') : \beta \to \gamma$ is defined by $(u', v') \circ (u, v) = (u' \circ u, v \circ v')$. Observe that, for an object $\varphi : y \to x$ of $\mathcal{F}(\mathcal{C})$, a morphism $id_\varphi = (id_x, id_y)$ is the identity on φ. By definition, a *natural system* D on a small category $\mathcal{C}$ is a covariant functor from $\mathcal{F}(\mathcal{C})$ to the category of Abelian groups. We may write D_α for $D(\alpha)$, where $\alpha \in \mathrm{ob}(\mathcal{F}(\mathcal{C}))$.

We recall the definition of a 2-category. A category $\mathcal{G}$ is a *2-category* if the following conditions are satisfied:

(i) For objects $X, Y \in \mathrm{ob}(\mathcal{G})$, the hom-set $\mathrm{Hom}_{\mathcal{G}}(X, Y)$ constitutes a small category $\mathcal{G}(X, Y)$ with $\mathrm{ob}\mathcal{G}(X, Y) = \mathrm{Hom}_{\mathcal{G}}(X, Y)$.

(ii) For $X, Y, Z \in \mathrm{ob}(\mathcal{G})$, the composite of morphisms in $\mathcal{G}$ defines a functor $\circ : \mathcal{G}(Y, Z) \times \mathcal{G}(X, Y) \to \mathcal{G}(X, Z)$ with $1_{1_Y} \circ - = 1_{\mathcal{G}(X,Y)}$ and $- \circ 1_{1_Y} = 1_{\mathcal{G}(Y,Z)}$.

A morphism from $f : X \to Y$ to $g : X \to Y$ denoted $F : f \Rightarrow g$ in the category $\mathcal{G}(X, Y)$ is called a *2-morphism.*

An object 0 in a 2-category is a *0-object* if $\mathcal{G}(0, X)$ and $\mathcal{G}(X, 0)$ are trivial categories for any objects $X \in \mathrm{ob}(\mathcal{G})$.

Two 1-morphisms $f, g : A \to B$ of $\mathcal{G}$ are *homotopic* if there exists an invertible 2-morphism $F : f \Rightarrow g$. We write $f \simeq g$ if f and g are homotopic.

As usual, the *homotopy category* $H\mathcal{G}$ of a 2-category $\mathcal{G}$ is the category with the same class of objects as that in $\mathcal{G}$ and the hom-set $\mathrm{Hom}_{H\mathcal{G}}(A, B)$ for objects A and B in $H\mathcal{G}$ is defined to be the quotient $\mathrm{Hom}_{\mathcal{G}}(A, B)/_{\simeq}$ by the homotopy relation $\simeq$. In what follows, $\mathcal{G}$ denotes a 2-category with a 0-object unless otherwise specified.

In order to describe our main theorem (Theorem 2.6 below), we further need terminology on 2-morphisms.

Definition 2.1. Let $S : u \Rightarrow v$ and $T : r \Rightarrow s$ be 2-morphisms, where u, v, r, and s are morphisms with same source and target. We say that S and T are *conjugate* and write $S\langle conj\rangle T$ if there exists invertible morphisms $H : v \Rightarrow s$ and $K : u \Rightarrow r$ such that $H + S = T + K$.

Definition 2.2. Let $f, g : A \to B$ be morphisms and $F : f \Rightarrow g$ a 2-morphism in $\mathcal{G}$. A set $N(F)$ is defined to be the set $\{G : 0_B^A \Rightarrow 0_B^A \mid G\langle conj\rangle F\}$.

In what follows, we write $\mathcal{A}_{XY}$ for $N(1_0 : 0 \Rightarrow 0 : X \to Y)$. We need more structure on the set $\mathcal{A}_{XY}$ in order to relate matrix Toda brackets with the cohomology mentioned in the Introduction. The set $\mathcal{A}_{XY}$ is automatically a group; see [13, Proposition 2.4]. Moreover, we assume that the group $\mathcal{A}_{XY}$ is Abelian for any X and Y in $H\mathcal{G}$. For any morphisms $b : B \to C$ in $\mathcal{G}$, the maps $b_* : \mathcal{A}_{AB} \to \mathcal{A}_{AC}$ and $b^* : \mathcal{A}_{CA} \to \mathcal{A}_{BA}$ which are induced by the horizontal composition in $\mathcal{G}$ are well-defined homomorphisms by the interchange law. Observe that we have a functor $\mathcal{A}_{-,-} : H\mathcal{G} \times H\mathcal{G} \to Ab$ to the category of Abelian groups.

Consider a commutative diagram

$$(2.1): \quad \begin{tikzcd} W \arrow[r, "w"] \arrow[rr, bend left, "0"] \arrow[dr, "0"'] & C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "b"] \\ & A \arrow[r, "a"] & X \end{tikzcd}$$

in $H\mathcal{G}$. Then we can define a *matrix Toda bracket*

$$\begin{Bmatrix} b & g & \\ a, & f, & w \end{Bmatrix}$$

in the sense of Hardie, Kamps and Marcum; see Section 3 for more details. Let h and k be 1-morphisms of a 2-category $\mathcal{G}$. For a subset $\mathcal{C}$ of $\mathcal{G}(h, h)$ and an invertible 2-morphism $T : k \Rightarrow h$, let $\mathcal{C}^T$ denote the subset $\{-T + \xi + T \mid \xi \in \mathcal{C}\}$ of $\mathcal{G}(k, k)$; see [13, (6.5) Notation]. Then, we see that the matrix Toda bracket $\begin{Bmatrix} b & g & \\ a, & f, & w \end{Bmatrix}$ is in $\mathcal{A}_{WX}/(b_*\mathcal{A}_{WB} + \mathcal{D} + a_*\mathcal{A}_{WA})$, where $\mathcal{D}$ is the submodule $(\mathcal{A}(af) \circ w)^{aH}$ of $\mathcal{A}_{WX}$, where $\mathcal{A}(af)$ $=\{F : af \Rightarrow af | F : invertible\}$. Moreover, if $af \simeq 0 \simeq bg$, then $\begin{Bmatrix} b & g & \\ a, & f, & w \end{Bmatrix}$ is in $\mathcal{A}_{WX}/(b_*\mathcal{A}_{WB} + w^*\mathcal{A}_{CX} + a_*\mathcal{A}_{WA})$; see [13, Corollary 7.3]. In fact, the result [13, Proposition 6.9 (c)] yields that $\mathcal{D} = w^*\mathcal{A}_{CX}$.

Definition 2.3. Let Ab be the category of Abelian groups. A functor $D^{\mathcal{A}} : \mathcal{F}(H\mathcal{G}) \to Ab$ is defined by $D^{\mathcal{A}}_{[f]}{:=}\mathcal{A}_{s(f)t(f)} = N(1_0 : 0 \Rightarrow 0 : s(f) \to t(f))$ and $D^{\mathcal{A}}(u, v)(G) := 1_u \circ G \circ 1_v = v^*u_*G$ for $G \in D^{\mathcal{A}}_{[f]}$ and $(u, v) : [f] \to [g]$, where $\circ$ denotes the composition in $H\mathcal{G}$.

Let $\mathcal{G}$ be a track category; see Definition 3.2 below. We consider a linear track extension of the form $D^{\mathcal{A}} \xrightarrow{+} \mathcal{G}_2 \rightrightarrows \mathcal{G}_1 \xrightarrow{p} H\mathcal{G}$ in the sense of Baues and

Dreckmann; see Definition 3.4. Then, we have the *universal Toda bracket* $\langle H\mathcal{G}\rangle$ in the Baues-Wirsching cohomology $H^3(H\mathcal{G}, D^{\mathcal{A}})$ with the coefficients in the natural system $D^{\mathcal{A}}$. Here, $H\mathcal{G}$ is small; see Definition 3.6. We observe that one of important data which defines a linear track extension is a set of isomorphisms $\sigma_f : D^{\mathcal{A}}_{[f]} \to \mathcal{G}(f, f)$ determined by 1-morphisms f of $\mathcal{G}$. Such a set of isomorphisms is called the *action* of $D^{\mathcal{A}}$ to $\mathcal{G}$.

Track categories appear naturally in homotopy theory. Indeed, applying the fundamental groupoid functor to each mapping space in a topologically enriched category yields a track category; see [18]. The discussion in [10, Introduction] as well as [12, Section 8.I,8, III.1] gives a simplicial version of the construction above. Moreover, a cofibration category in the sense of Baues gives rise to a track category; see [6, Proposition II.5.6 and Corollary]. From a category equipped with a suitable cylinder functor, one can form a track category as described in [18, Theorem IV.1.11].

We here mention that the Baues-Wirsching cohomology $H^*(\mathcal{C}, D)$ of a small category $\mathcal{C}$ can be normalized with an ideal S of the category $\mathcal{C}$. We denote it by $H^*_S(\mathcal{C}, D)$; see Section 4.

Definition 2.4. Let MT be the category generated by the directed graph in displayed diagram, modulo the relation $\tilde{a} \circ \tilde{f} = \tilde{b} \circ \tilde{g}$.

$$\begin{tikzcd}
\widetilde{W} \arrow[r, "\tilde{w}"] & \widetilde{C} \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{f}"'] & \widetilde{B}. \arrow[d, "\tilde{b}"] \\
 & \widetilde{A} \arrow[r, "\tilde{a}"] & \widetilde{X}
\end{tikzcd}$$

This category MT is called the *matrix Toda category* .

Definition 2.5. Let $\mathcal{C}$ be a category with zero morphisms. For a functor $F : MT \to \mathcal{C}$ which satisfies $F(fw) = 0, F(gw) = 0$, the functor F is called a *matrix Toda diagram* in $\mathcal{C}$. We also call $F(MT)$ a *matrix Toda diagram* in $\mathcal{C}$.

For the category MT, we have a functor $\varphi : MT \to H\mathcal{G}$ with $\varphi(\tilde{\eta}) = \eta$ for $\eta \in \{w, a, b, f, g\}$; that is, the diagram (2.1) is regarded as a matrix Toda diagram . Then, the functor φ induces a homomorphism $\varphi^* : H^3_{O(H\mathcal{G})}(H\mathcal{G}, D^{\mathcal{A}}) \to H^3_S(MT, \varphi^* D^{\mathcal{A}})$. Here, $S = \{\tilde{f}\tilde{w}, \tilde{g}\tilde{w}, \tilde{b}\tilde{g}\tilde{w}\}$. Moreover, we have an isomorphism $\tilde{h} : H^3(H\mathcal{G}, D^{\mathcal{A}}) \to H^3_{O(H\mathcal{G})}(H\mathcal{G}, D^{\mathcal{A}})$; see Section 4.

We are ready to describe our main theorem.

Theorem 2.6. *Let $\mathcal{G}$ be a track category with $H\mathcal{G}$ small, which satisfies Hypothesis I ;see Section 3. If $af \simeq 0 \simeq bg$, then there exists an isomorphism*

$$\alpha : H^3_S(MT, \varphi^* D^{\mathcal{A}}) \xrightarrow{\cong} \mathcal{A}_{WX}/(b_*\mathcal{A}_{WB} + w^*\mathcal{A}_{CX} + a_*\mathcal{A}_{WA})$$

such that the composite defined by the diagram

$$\begin{tikzcd}
H^3_{O(H\mathcal{G})}(H\mathcal{G}, D^{\mathcal{A}}) \arrow[r, "\varphi^*"] & H^3_S(MT, \varphi^* D^{\mathcal{A}}) \arrow[d, "\cong"', "\alpha"] \\
H^3(H\mathcal{G}, D^{\mathcal{A}}) \arrow[u, "\tilde{h}", "\cong"'] \arrow[r, "\left\{\begin{smallmatrix} b & g \\ a, & f, \end{smallmatrix} w\right\}^*"] & \mathcal{A}_{WX}/(b_*\mathcal{A}_{WB} + w^*\mathcal{A}_{CX} + a_*\mathcal{A}_{WA})
\end{tikzcd}$$

sends the universal Toda bracket $\langle H\mathcal{G}\rangle$ to the matrix Toda bracket $\left\{\begin{matrix} b & g \\ a, & f, \end{matrix}\ w\right\}$ defined by the diagram (2.1).

We recall a linear track extension; see Definition 3.4. Then we have the following corollary.

Corollary 2.7. *Let $\mathcal{C}$ be a small subcategory of $H\mathcal{G}$ which has a matrix Toda diagram with a non-zero bracket. Then C admits a non-trivial linear track extension by the natural system $D^{\mathcal{A}}$ mentioned in Theorem 2.6.*

Proof. Let $F : MT \to \mathcal{C}$ be a functor which satisfies the condition described in Definition 2.5. For the inclusion functor $\iota : \mathcal{C} \to H\mathcal{G}$, we define the functor $\varphi : MT \to H\mathcal{G}$ by composing ι and F. Then the commutative diagram

$$\begin{tikzcd}
H\mathcal{G} & \mathcal{C} \arrow[l, "\iota"'] & MT \arrow[l, "F"'] \arrow[ll, bend right, "\varphi"']
\end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd}
H^3_{O(H\mathcal{G})}(H\mathcal{G}, D^{\mathcal{A}}) \arrow[r, "\iota^*"] \arrow[d, shift left, "i^*"] \arrow[rr, bend left, "\varphi^*"] & H^3_{O(\mathcal{C})}(\mathcal{C}, \iota^* D^{\mathcal{A}}) \arrow[r, "F^*"] \arrow[d, shift left, "i^*"] & H^3_S(MT, \varphi^* D^{\mathcal{A}}) \arrow[d, "\alpha", "\cong"'] \\
H^3(H\mathcal{G}, D^{\mathcal{A}}) \arrow[r, "\iota^*"] \arrow[u, shift left, "\tilde{h}", "\cong"'] \arrow[rr, bend right, "\left\{\begin{matrix} b & g & \\ a, & f, & w \end{matrix}\right\}^*"'] & H^3(\mathcal{C}, \iota^* D^{\mathcal{A}}) \arrow[u, shift left, "\tilde{h}", "\cong"'] & \mathcal{A}_{WX}/\mathcal{A}^\sharp,
\end{tikzcd}$$

where $\mathcal{A}^\sharp := b_*\mathcal{A}_{WB} + w^*\mathcal{A}_{CX} + a_*\mathcal{A}_{WA}$. In fact, $F(S)$ and $\iota(O(\mathcal{C}))$ are included in $O(\mathcal{C})$ and $O(H\mathcal{G})$, respectively. Then we have $F^* \circ \iota^* = \varphi^*$. The functor $\iota : \mathcal{C} \to H\mathcal{G}$ induces the commutative diagram

$$\begin{tikzcd}
\check{F}^*(O(H\mathcal{G})) \arrow[r, "\iota^\sharp"] \arrow[d, "i"] & \check{F}^*(O(\mathcal{C})) \arrow[d, "i"] \\
F^*(H\mathcal{G}, D^{\mathcal{A}}) \arrow[r, "\iota^\sharp"] & F^*(\mathcal{C}, \iota^* D^{\mathcal{A}}),
\end{tikzcd}$$

where column maps are inclusions. The diagram gives rise to the commutative diagram

$$\begin{tikzcd}
H^3_{O(H\mathcal{G})}(H\mathcal{G}, D^{\mathcal{A}}) \arrow[r, "\iota^*"] \arrow[d, "i^*"] & H^3_{O(\mathcal{C})}(\mathcal{C}, \iota^* D^{\mathcal{A}}) \arrow[d, "i^*"] \\
H^3(H\mathcal{G}, D^{\mathcal{A}}) \arrow[r, "\iota^*"] & H^3(\mathcal{C}, \iota^* D^{\mathcal{A}}).
\end{tikzcd}$$

Thus we have $\iota^* \circ i^* = i^* \circ \iota^*$. In consequence, we see that

$$\begin{aligned}
\alpha \circ F^* \circ \tilde{h}(\iota^* \langle H\mathcal{G} \rangle) &= \alpha \circ F^*(\tilde{h} \circ \iota^*(\langle H\mathcal{G} \rangle)) \\
&= \alpha \circ F^*(\iota^* \circ \tilde{h}(\langle H\mathcal{G} \rangle)) \\
&= \alpha \circ \varphi^* \circ \tilde{h}(\langle H\mathcal{G} \rangle)) \\
&= \left\{\begin{matrix} b & g & \\ a & f & w \end{matrix}\right\}^* (\langle H\mathcal{G} \rangle) \\
&= \left\{\begin{matrix} b & g & \\ a & f & w \end{matrix}\right\} \ (\textit{by theorem } 2.6.) \\
&\neq 0.
\end{aligned}$$

This implies that the element $\iota^*\langle H\mathcal{G}\rangle$ is nontrivial in $H^3(\mathcal{C}, \iota^* D^{\mathcal{A}})$. $\square$

If A in the diagram (2.1) is the 0-object , then we have a diagram

$$\begin{tikzcd}
W \arrow[r, "w"] \arrow[rr, bend left, "0", "\Downarrow K"'] & C \arrow[r, "g"] \arrow[rr, bend right, "0"', "\Uparrow L"] & B \arrow[r, "b"] & X.
\end{tikzcd}$$

Therefore, the matrix Toda bracket $\left\{\begin{matrix} b & g & \\ 0, & 0, & w \end{matrix}\right\}$ is nothing but the classical Toda bracket $\{b, g, w\}$ defined by $\{-bK + Lw \mid K : 0 \Rightarrow gw, L : 0 \Rightarrow bg, K, L : \text{invertible}\}$. We observe that $\{b\ , g\ , w\}$ is in the coset $\mathcal{A}_{WX}/(b_*\mathcal{A}_{WB} + w^*\mathcal{A}_{CX})$; see [13, Proposition 8.2].

Corollary 2.8. *One has* $\left\{\begin{matrix} b & g & \\ 0, & 0, & w \end{matrix}\right\}^*(\langle H\mathcal{G}\rangle) = \{b\ g\ w\}$.

In what follows, we may write $\{b\ g\ w\}^*$ for $\left\{\begin{matrix} b & g & \\ 0, & 0, & w \end{matrix}\right\}^*$.

Remark 2.9. We have $\left\{\begin{matrix} b & g & \\ a, & f, & w \end{matrix}\right\}^*(\langle H\mathcal{G}\rangle) = \left\{\begin{matrix} b & g & \\ a, & f, & w \end{matrix}\right\} = \{b\ g\ w\} - \{a\ f\ w\} = (\pi \circ (\{b\ g\ w\}^* - \{a\ f\ w\}^*))(\langle H\mathcal{G}\rangle)$ in $\mathcal{A}_{WX}/(b_*\mathcal{A}_{WB} + w^*\mathcal{A}_{CX} + a_*\mathcal{A}_{WA})$, where $\pi : \mathcal{A}_{WX}/(b_*\mathcal{A}_{WB} + w^*\mathcal{A}_{CX}) \oplus \mathcal{A}_{WX}/(w^*\mathcal{A}_{CX} + a_*\mathcal{A}_{WA}) \to \mathcal{A}_{WX}/(b_*\mathcal{A}_{WB} + w^*\mathcal{A}_{CX} + a_*\mathcal{A}_{WA})$ is the projection; see Lemma 4.7 below.

We here recall a result in [8] on the classical Toda bracket. Let Top be the category of based topological spaces with the based homotopy relation $\simeq_*$. Assume that the 2-category $\mathcal{G}$ in our setting is a subcategory of based co$H$spaces whose 1-morphism are continuous maps and whose 2-hom-set of $Top(X, Y)$ is defined by $\mathrm{Hom}_{Top(X,Y)}(f, g) = \{H : X \times I \to Y\}/_{\simeq_*}$ for 1-morphisms $f, g : X \to Y$. Then we see that

$$\mathcal{A}_{AB} = N(1_0 : 0^A_B \Rightarrow 0^A_B) = \{G : 0^A_B \Rightarrow 0^A_B \mid G\langle conj\rangle 1_0\} \cong [\Sigma A, B]$$

as a set and that Hypothesis I is satisfied for the 2-category $\mathcal{G}$. Observe that the based homotopy set $[\Sigma A, B]$ is an Abelian group whose addition is defined with the suspension structure of the domain as usual. Therefore, in view of Corollary 2.8, Theorem 2.6 is regarded as a generalization of the result [8, Theorem 3.3]. It is important to remark that the ideal S' used in [8, Thorem 3.3] does *not* coincide with S in Theorem 2.6. In fact, $S = \{fw, gw, bgw\}$ and $S' = \{bg, gw, bgw\}$. Hence, Corollary 2.8 gives another description of classical Toda brackets, which is stated in terms of matrix Toda brackets.

Under the same assumption as in Theorem 2.6, we have propositions. These propositions follow from [13, Theorem 5.4 and 5.5] and Theorem 2.6.

Proposition 2.10. *Let* $x : X \to Y$ *be a map. Given a diagram of the form (2.1) in* $H\mathcal{G}$, *let* $x_* : \mathcal{A}_{WX}/(b_*\mathcal{A}_{WB} + \mathcal{D} + a_*\mathcal{A}_{WA}) \to \mathcal{A}_{WY}/((xb)_*\mathcal{A}_{WB} + \mathcal{A}_{CY} \circ w + (xa)_*\mathcal{A}_{WA})$ *be the homomorphism induced by the map* x, *where* $\mathcal{D} = (\mathcal{A}(af) \circ w)^{aH}$. *Suppose further that* $xaf = 0 = xbg$ *in a commutative diagram*

$$\begin{tikzcd}
W \arrow[r, "w"] & C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "b"] & \\
& A \arrow[r, "a"] & X \arrow[r, "x"] & Y
\end{tikzcd}$$

in $H\mathcal{G}$. Then, there holds the equality

$$x_*\left\{\begin{matrix} b & g \\ a, & f, \end{matrix}\; w\right\} = \left\{\begin{matrix} xb & g \\ xa, & f, \end{matrix}\; w\right\}^* (\langle H\mathcal{G}\rangle)$$

in $\mathcal{A}_{WX}/(xb)_*\mathcal{A}_{WB} + \mathcal{A}_{CY} \circ w + (xa)_*\mathcal{A}_{WA}$.

Proof. By Theorem 2.6, we have $\left\{\begin{matrix} xb & g \\ xa, & f, \end{matrix}\; w\right\}^* (\langle H\mathcal{G}\rangle) = \left\{\begin{matrix} xb & g \\ xa, & f, \end{matrix}\; w\right\}$. By the definition of the matrix Toda bracket, there exists an element θ in $\mathcal{A}_{WX}$ such that $\left\{\begin{matrix} b & g \\ a, & f, \end{matrix}\; w\right\} = [\theta]$. It follows that $x_*\left\{\begin{matrix} b & g \\ a, & f, \end{matrix}\; w\right\} = [x \circ \theta]$. The result [13, Theorem 5.4] implies that $x \circ \left\{\begin{matrix} b & g \\ a, & f, \end{matrix}\; w\right\} \subset \left\{\begin{matrix} xb & g \\ xa, & f, \end{matrix}\; w\right\}$. Then we see that $x \circ \theta \in x \circ \left\{\begin{matrix} b & g \\ a, & f, \end{matrix}\; w\right\}$ and hence $[x \circ \theta] = \left\{\begin{matrix} xb & g \\ xa, & f, \end{matrix}\; w\right\}$ in the quotient mentioned in the assertion. This completes the proof. □

Proposition 2.11. *Given a diagram of the form (2.1) in $H\mathcal{G}$, let $\pi : \mathcal{A}_{ZX}/(b_*\mathcal{A}_{ZB} + \mathcal{D} + a_*\mathcal{A}_{ZA}) \to \mathcal{A}_{ZX}/(b_*\mathcal{A}_{ZB} + \mathcal{A}_{WX} \circ \delta + a_*\mathcal{A}_{ZA})$ be the projection, where $\delta : Z \to W$ is a map and $\mathcal{D} = (\mathcal{A}(af) \circ w\delta)^{aH}$. Suppose that $afw = 0 = bgw$ in a commutative diagram*

$$\begin{tikzcd}
Z \arrow[r, "\delta"] & W \arrow[r, "w"] & C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "b"] \\
 & & A \arrow[r, "a"] & X
\end{tikzcd}$$

in $H\mathcal{G}$. Then, one has

$$\pi\left\{\begin{matrix} b & g \\ a, & f, \end{matrix}\; w\delta\right\} = \left\{\begin{matrix} b & gw \\ a, & fw, \end{matrix}\; \delta\right\}^* (\langle H\mathcal{G}\rangle)$$

in $\mathcal{A}_{ZX}/(b_*\mathcal{A}_{ZB} + \mathcal{A}_{WX} \circ \delta + a_*\mathcal{A}_{ZA})$.

Proof. Theorem 2.6 enables us to deduce that $\left\{\begin{matrix} b & gw \\ a, & fw, \end{matrix}\; \delta\right\}^* (\langle H\mathcal{G}\rangle) = \left\{\begin{matrix} b & gw \\ a, & fw, \end{matrix}\; \delta\right\}$. By the definition of the matrix Toda bracket, there exists an element θ in $\mathcal{A}_{ZX}$ such that $\left\{\begin{matrix} b & g \\ a, & f, \end{matrix}\; w\delta\right\} = [\theta]$. By [13, Theorem 5.5], we see that $\left\{\begin{matrix} b & g \\ a, & f, \end{matrix}\; w\delta\right\} \subset \left\{\begin{matrix} b & gw \\ a, & fw, \end{matrix}\; \delta\right\}$. Since θ is in $\left\{\begin{matrix} b & g \\ a, & f, \end{matrix}\; w\delta\right\}$, it follows that $[\theta] = \left\{\begin{matrix} b & gw \\ a, & fw, \end{matrix}\; \delta\right\}$ in the quotient. We have the result. □

In describing Theorem 2.6, we use a matrix Toda category MT but not a Toda diagram in [8]. Therefore, the maps π and x_* in Propositions 2.10 and 2.11 are defined naturally. An advantage of the propositions above is that the non-triviality of the matrix Toda brackets follows from that of the image by the homomorphisms of the universal Toda bracket $\langle H\mathcal{G}\rangle$.

Another main theorem asserts that a matrix Toda bracket is represented by the classical one in an appropriate category. In order to describe such a result, we consider the category of cochain complexes $Ch(\mathcal{B})$ on an additive category $\mathcal{B}$.

For any objects X and Y in $Ch(\mathcal{B})$, the hom-set $Ch(\mathcal{B})(X, Y)$ admits a category structure. In fact, its objects are cochain maps from X to Y and the hom-set is a set of linear maps of degree -1 defined by

$$\mathrm{Hom}_{Ch(\mathcal{B})(X,Y)}(f, g) := \{h : X \to Y \mid f - g = d_Y h + hd_X\}/\sim .$$

Moreover, for linear maps $h, k : X \to Y$ of degree -1, by definition $h \sim k$ if and only if there exists a linear map $u : X \to Y$ of degree -2 such that $h - k = d_Y u - u d_X$.

A vertical composite

$$+ : \mathrm{Hom}_{Ch(\mathcal{B})(X,Y)}(g, h) \times \mathrm{Hom}_{Ch(\mathcal{B})(X,Y)}(f, g) \to \mathrm{Hom}_{Ch(\mathcal{B})(X,Y)}(f, h)$$

is defined by $+(h_1, h_2) = h_1 + h_2$. Moreover, the composite

$$\circ : Ch(\mathcal{B})(Y, Z) \times Ch(\mathcal{B})(X, Y) \to Ch(\mathcal{B})(X, Z)$$

in the category $Ch(\mathcal{B})$ gives rise to a functor whose behavior in the hom-sets

$$\circ : \mathrm{Hom}_{Ch(\mathcal{B})(Y,Z)}(f', g') \times \mathrm{Hom}_{Ch(\mathcal{B})(X,Y)}(f, g) \to \mathrm{Hom}_{Ch(\mathcal{B})(X,Z)}(f' \circ f, g' \circ g)$$

is defined by $k \circ h := f'k + hg$ for $k \in \mathrm{Hom}_{Ch(\mathcal{B})(Y,Z)}(f', g')$ and $h \in \mathrm{Hom}_{Ch(\mathcal{B})(X,Y)}(f, g)$.

The homotopy category $H(Ch(\mathcal{B}))$ admits the triangulated category structure whose distinguished triangles are constructed by the mapping cone and suspension functors; see [1, Theorem 2.3.1].

More generally, we recall the definition of an algebraic triangulated category.

Definition 2.12. [19, 3.2] An *exact functor* $T \to \mathcal{U}$ between triangulated categories is a pair (υ, η) consisting of a functor $\upsilon : \mathcal{T} \to \mathcal{U}$ and natural isomorphism $\eta : \upsilon \circ \Sigma_{\mathcal{T}} \to \Sigma_{\mathcal{U}} \circ \upsilon$ such that for every exact triangle $X \overset{\alpha}{\to} Y \overset{\beta}{\to} Z \overset{\gamma}{\to} \Sigma X$ in $\mathcal{T}$ the triangle

$$\upsilon X \xrightarrow{\upsilon\alpha} \upsilon Y \xrightarrow{\upsilon\beta} \upsilon Z \xrightarrow{\eta X \circ \upsilon\gamma} \Sigma(\upsilon X)$$

is exact in $\mathcal{U}$.

Definition 2.13. [19, 7.5] A triangulated category $\mathcal{T}$ is called *algebraic* if there exists a fully faithful exact functor $\vartheta : \mathcal{T} \to H(Ch(\mathcal{B}))$ with a natural isomorphism $\eta : \vartheta \circ \Sigma \to \Sigma \circ \vartheta$ for some additive category $\mathcal{B}$.

A matrix Toda diagram (2.1) in a triangulated category $\mathcal{T}$ gives rise to a Toda diagram of the form

$$W \xrightarrow{w} C \xrightarrow{-f \vee g} A \oplus B \xrightarrow{\nabla(a,b)} X.$$

Then we have $(-f \vee g)w = 0$ and $\nabla(a, b)(-f \vee g) = 0$ in $\mathcal{T}$. In fact, the direct sum of two objects A and B is an object $A \oplus B$ together with morphisms

$$i : A \to A \oplus B \ , \ j : B \to A \oplus B$$

making $(A \oplus B, i, j)$ into the coproduct. Moreover, we have the product $(A \oplus B, p, q)$ with morphisms

$$p : A \oplus B \to A \ , \ q : A \oplus B \to B.$$

These maps are related by equations [18, page 444]

$$pi = id_A \ , \ qj = id_B \ , \ pj = 0 \ , \ qi = 0 \ \ ip + jq = id_{A \oplus B}.$$

We shall prove $(-f \vee g)w = 0$ and $\nabla(a, b)(-f \vee g) = 0$. Consider the diagram

$$\begin{tikzcd}
 & & B \arrow[d, shift left, "j"] \arrow[rd, "b"] & \\
W \arrow[r, "w"] & C \arrow[ru, "g"] \arrow[r, "-f\vee g"] \arrow[rd, "-f"'] & A \oplus B \arrow[u, shift left, "q"] \arrow[d, shift right, "p"'] \arrow[r, "\nabla(a,b)"] & X. \\
 & & A \arrow[u, shift right, "i"'] \arrow[ru, "a"'] &
\end{tikzcd}$$

For $(-f \vee g)w : W \to A \oplus B$, we see that $p \circ ((-f \vee g)w) = (p \circ (-f \vee g) \circ w = -f \circ w = 0 = p \circ 0$ and $q \circ ((-f \vee g)w) = g \circ w = 0 = q \circ 0$. Since $(-f \vee g)w$ is unique, it follows that $(-f \vee g)w = 0$.

$$\begin{tikzcd}
& B \\
W \arrow[ur, "0"] \arrow[r, "(-f\vee g)w"] \arrow[dr, "0"'] & A \oplus B \arrow[u, "q"'] \arrow[d, "p"] \\
& A
\end{tikzcd}$$

The second equality follows from relations between morphism mentioned above. In fact, we have

$$\begin{aligned}
\nabla(a,b) \circ (-f \vee g) &= \nabla(a,b) \circ 1_{A\oplus B} \circ (-f \vee g) \\
&= \nabla(a,b) \circ (i \circ p + j \circ q) \circ (-f \vee g) \\
&= \nabla(a,b) \circ i \circ p \circ (-f \vee g) + \nabla(a,b) \circ j \circ q \circ (-f \vee g) \\
&= (\nabla(a,b) \circ i) \circ (p \circ (-f \vee g)) + (\nabla(a,b) \circ j) \circ (q \circ (-f \vee g)) \\
&= a \circ (-f) + b \circ g \\
&= -a \circ f + b \circ g \\
&= 0.
\end{aligned}$$

Let $\{\varphi, \psi, \eta\}$ denote the classical Toda bracket for $X \xrightarrow{\eta} Y \xrightarrow{\psi} Z \xrightarrow{\varphi} W$ in $\mathcal{T}$ defined by Heller [17, Section 13]. Then we have the following theorem.

Theorem 2.14. *For a matrix Toda diagram* (2.1) *in an algebraic triangulated category $\mathcal{T}$ with a fully faithful exact functor (ϑ, η), via ϑ, one has*

$$\left\{\begin{matrix} b, & g & \\ & & w \\ a, & f & \end{matrix}\right\} = \{\nabla(a,b), -f \vee g, w\},$$

that is

$$\left\{\begin{matrix} \vartheta b, & \vartheta g & \\ & & \vartheta w \\ \vartheta a, & \vartheta f & \end{matrix}\right\} = \vartheta\{\nabla(a,b), -f \vee g, w\} \circ \eta_W^{-1}.$$

3. Brief recollection on matrix Toda brackets and the universal Toda bracket

We begin by considering a diagram

$$\begin{tikzcd}
W \arrow[r, "w"] \arrow[rr, bend left, "0"] \arrow[dr, bend right, "0"'] & C \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{S}{\Longrightarrow}"] & B \arrow[d, "b"] \\
& A \arrow[r, "a"] & X
\end{tikzcd}$$

in a 2-category $\mathcal{G}$, where $S : af \Rightarrow bg$ is a 2-morphism. Then, we define a set $\sigma(S, w)$ by

$\sigma(S, w) := \{-bK + Sw + aH \mid H : 0 \Rightarrow fw, K : 0 \Rightarrow gw, H$ and K are invertible$\}$.

We define the matrix Toda bracket $\left\{\begin{matrix} b & g & \\ a, & f, & w \end{matrix}\right\}$ to be the union $\bigcup_{S\in Z} \sigma(S, w)$, where $Z = \{S \mid S : af \Rightarrow bg$ is invertible$\}$.

Remark 3.1. For a 2-morphism $S : af \Rightarrow bg$ and 1-morphism $w : W \to$ C, we see that $\sigma(S, w) \subset N(Sw)$.

Let $\theta = -bK+Sw+aH$ in $N(1_{afw}) = \mathcal{A}_{WX}$; see [13, Proposition (2.4)]. Suppose that $af \simeq 0 \simeq bg$ and $\mathcal{A}_{WX}$ is an Abelian group. Then, the matrix Toda bracket $\left\{\begin{matrix} b & g & \\ a, & f, & w \end{matrix}\right\}$ coincides with the coset $(\theta + b \circ N(1_{gw}) + N(S) \circ w + a \circ N(1_{fw}))$. We observe that $a \simeq a',\ b \simeq b',\ f \simeq f',\ g \simeq g'$ and $w \simeq w'$ then $\left\{\begin{matrix} b & g & \\ a, & f, & w \end{matrix}\right\} = \left\{\begin{matrix} b' & g' & \\ a', & f', & w' \end{matrix}\right\}$ as a coset; see [13, Proposition (5.3)].

In general, $k \simeq l$ then $N(1_k) = N(1_l)$; see [13, Proposition (2.2)]. Then, we see that $\left\{\begin{matrix} b & g & \\ a, & f, & w \end{matrix}\right\} = [\theta]$ is in $\mathcal{A}_{WX}/(b_*\mathcal{A}_{WB} + w^*\mathcal{A}_{CX} + a_*\mathcal{A}_{WA})$.

Let $\mathcal{C}$ be a small category and D a natural system. For α, $u \in \mathrm{mor}(\mathcal{C})$ with $s(u) = t(\alpha)$, we write u_* for the homomorphism $D(u, \mathrm{id}_{s(\alpha)}) : D_\alpha \to D_{u\circ\alpha}$. Similarly, the homomorphism $D(\mathrm{id}_{t(\alpha)}, v) : D_\alpha \to D_{\alpha\circ v}$ is written as v^*, where $v \in \mathrm{mor}(\mathcal{C})$ with $t(v) = s(\alpha)$.

We here recall the definition of the nth *Baues–Wirsching cohomology* of $\mathcal{C}$ with coefficients in D. For $n \geq 1$, let $N_n(\mathcal{C})$ be the set of n-simplices of the nerve of $\mathcal{C}$; that is, $N_n(\mathcal{C}) = \{\lambda = (\lambda_1, \lambda_2,\dots, \lambda_n) \mid X_0 \xleftarrow{\lambda_1} X_1 \xleftarrow{\lambda_2} X_2 \leftarrow \cdots \xleftarrow{\lambda_n} X_n\ \}$. Let F^n be an Abelian group defined by

$$F^n = F^n(\mathcal{C}, D) := \{c : N_n(\mathcal{C}) \to \cup_{g\in mor(\mathcal{C})} D_g \mid c(\lambda_1, \dots, \lambda_n) \in D_{\lambda_1\circ\cdots\circ\lambda_n}\}.$$

The boundary operator $\delta^n : F^{n-1} \to F^n$ is defined by

$$\begin{aligned}(\delta^n(c))(\lambda) \quad &= \quad \lambda_{1*}c(\lambda_2, \dots, \lambda_n) \\ &\quad + \sum_{i=1}^{n-1}(-1)^i c(\lambda_1, \dots, \lambda_i \circ \lambda_{i+1}, \dots, \lambda_n) + (-1)^n \lambda_n^* c(\lambda_1, \dots, \lambda_{n-1}),\end{aligned}$$

for $\lambda = (\lambda_1, \dots, \lambda_n) \in N_n(\mathcal{C})$ and $c \in F^{n-1}$. The cohomology $H^n(F^*(\mathcal{C}, D))$ of the complex $F^*(\mathcal{C}, D) := \{F^n(\mathcal{C}, D), \delta^n\}_{n\in\mathbb{Z}}$ is called the *nth Baues-Wirsching cohomology*, denoted $H^n(\mathcal{C}, D)$.

Definition 3.2. A *track category* $\mathcal{T}$ is a 2-category enriched in groupoids. More precisely, $\mathcal{T}$ is category consisting of following data:

(i) For objects A and B of $\mathcal{T}$, the set $\mathcal{T}(A, B)$ of 1-morphisms is a groupoid.
(ii) The composite $* : \mathcal{T}(A, B)\times\mathcal{T}(A', A) \to \mathcal{T}(A', B)$ is a functor for $A', A, B \in ob(\mathcal{T})$.

We may write $\mathcal{T}_1$ and $\mathcal{T}_2$ for the underlying 1-category of a track category and the sets of 2-morphisms, respectively; allowed Line 11of Page 4. (be used Line 11of Page 4.)

Remark 3.3. (i) For $f, f' \in \mathrm{Hom}_{\mathcal{T}_1}(A, B) = ob\mathcal{T}(A, B), H \in \mathrm{Hom}_{\mathcal{T}(A,B)}(f, f')$ and $G \in \mathrm{Hom}_{\mathcal{T}(A',A)}(g, g')$, the function $*$ carries the pair of tracks (H, G) to a track $H{*}G$ in $\mathrm{Hom}_{\mathcal{T}(A',B)}(f{\circ}g, f'{\circ}g')$. For the vertical composite $+ : \mathrm{Hom}_{\mathcal{T}(A,B)}(f', f'')\times \mathrm{Hom}_{\mathcal{T}(A,B)}(f, f') \to \mathrm{Hom}_{\mathcal{T}(A,B)}(f, f'')$, we write $+\ (H', H) = H' + H$. Observe that the functoriality of $*$ gives rise to equalities $H * G = g'^* H + f_* G = f'_* G + g^* H$, where $f_*G := 1_f * G$ and $g^*H := H * 1_g$.
(ii) $(1_A)_*G = G,\ (1_A)^*H = H$.
(iii) The operation $*$ in (i) is associative.

Definition 3.4. [8, Definition (4.3)] Let $\mathcal{C}$ be a category and D a natural system on $\mathcal{C}$, namely a functor $D : \mathcal{FC} \to Ab$. A *linear track extension* $\mathcal{E}$ of $\mathcal{C}$ by D consists of a track category $\mathcal{T}$, a functor $p : \mathcal{T}_1 \to \mathcal{C}$ and an *action* σ of D on $\mathcal{T}$, which is

a set $\sigma = \{\sigma_f : D_{p(f)} \to \mathcal{T}(f,f) \mid \sigma_f$ is an isomorphism of groups for $f \in$mor$(\mathcal{T}_1)\}$. Moreover, the following conditions are required:
(i) p is identity on $ob(\mathcal{C})$ and full, moreover, p satisfies $p(f) = p(g) \Leftrightarrow f \simeq g$.
(ii) For $H \in \mathcal{T}(f,h)$ and $\alpha \in D_{p(f)}, \sigma_h(\alpha) + H = H + \sigma_f(\alpha)$.
(iii) For $\alpha, \beta \in D_{p(f)}$ and $f, g \in mor(\mathcal{T}_1)$, one has $g^*\sigma_f(\alpha) = \sigma_{fg}(g^*\alpha)$ and $f_*\sigma_g(\beta) = \sigma_{fg}(f_*\beta)$.

Following [8], we may write $D \xrightarrow{+} \mathcal{T}_2 \rightrightarrows \mathcal{T}_1 \xrightarrow{p} \mathcal{C}$ for the linear track extension $\mathcal{E}$ mentioned above, see also [4, Section 1] for the notation.

Let $\mathcal{E}$ be a linear track extension of a small category $\mathcal{C}$ by D and let τ : mor $\mathcal{C} \to$ mor $\mathcal{T}_1$ and $H : N_2\mathcal{C} \to \bigcup_{f,g \in mor(\mathcal{T}_1)} \mathcal{T}(f,g)$ be functions with $p \circ \tau$=1 and $H(f,g) \in \mathcal{T}(\tau f \circ \tau g, \tau(fg))$. We define a cochain $C_{\mathcal{E}}(\tau, H) : N_3(\mathcal{C}) \to \bigcup_{f \in mor(\mathcal{C})} D_f$ by

$$C_{\mathcal{E}}(\tau, H)(f,g,h) := \sigma^{-1}_{\tau(fgh)}(\Delta),$$

with $\Delta = -H(f,gh) - (\tau f)_*H(g,h) + (\tau h)^*H(f,g) + H(fg,h)$. See the diagram below. Observe that Δ belongs to $T(\tau(fgh), \tau(fgh))$ and $\sigma^{-1}_{\tau(fgh)}$ is an isomorphism from $\mathcal{T}(\tau(fgh), \tau(fgh))$ to $D(p\tau(fgh)) = D(fgh)$.

$$\begin{tikzcd}[column sep=large, row sep=large]
C & F \arrow[l, "\tau(f)"'] \arrow[l, bend right=50, "\tau(f)"'] \arrow[l, phantom, bend right=25, "\Downarrow 1\tau(f)"] & G \arrow[l, "\tau(g)"'] \arrow[ll, bend left=40, "\tau(fg)"] \arrow[l, phantom, bend left=30, "\Downarrow H(f,g)"] & H \arrow[l, "\tau(h)"] \arrow[l, bend left=50, "\tau(h)"] \arrow[l, phantom, bend left=25, "\Downarrow 1\tau(h)"] \arrow[ll, bend right=40, "\tau(gh)"'] \arrow[l, phantom, bend right=35, "\Downarrow -H(g,h)"] \arrow[lll, bend right=60, "\tau(fgh)"'] \arrow[ll, phantom, bend right=55, "\Downarrow -H(f,gh)"] \arrow[lll, bend left=60, "\tau(fgh)"] \arrow[ll, phantom, bend left=50, "\Downarrow H(fg,h)"]
\end{tikzcd}$$

Let F^n denote the Abelian group $F^n(\mathcal{C}, D)$. For the boundary operator δ^4 : $F^3 \to F^4$, we see that $\delta^4(C(\tau, H)) = 0$. Observe that the cocycle $C(\tau, H)$ does depend on the choice of τ and H; its cohomology class$[C(\tau, H)]$ dose not ; see [8, (A.1) Lemma (c)].

Remark 3.5. Suppose that $\mathcal{T}$ has a 0-object, then we can choose τ and H so that $\tau(0) = 0$ and $H(f, 0) = H(0, f) = 1_0$. Indeed, $p \circ \tau = 1$ and p is full, then there exist $\tau(f) \in \mathrm{Hom}_{\mathcal{T}_1}(A, B)$ such that $p(\tau(f)) = f$ for $f \in mor\mathcal{C}$. Observe that $p(\tau(fg)) = fg = p(\tau(f) \circ \tau(g))$, then there exist a 2-morphism H between $\tau f \circ \tau g$ and $\tau(fg)$; see Definition 3.4(i).

We here define the universal Toda bracket.
Hypothesis I. Let $\mathcal{G}$ be a track category which satisfies the following conditions:

(i) The group $\mathcal{A}_{XY}$ is Abelian for any X and Y in $H\mathcal{G}$.
(ii) For the functor $D^{\mathcal{A}} : \mathcal{F}(H\mathcal{G}) \to Ab$ defined in Definition 2.3, there exists a linear track extension

$$D^{\mathcal{A}} \xrightarrow{+} \mathcal{G}_2 \rightrightarrows \mathcal{G}_1 \xrightarrow{p} H\mathcal{G}$$

such that $\sigma_0 = id$, where the functor $p : \mathcal{G}_1 \to H\mathcal{G}$ is the natural projection.

We observe that σ_0 denotes the element determined by the zero map 0 in the action $\sigma = \{\sigma_f : D^{\mathcal{A}}_{[f]} \to \mathcal{G}(f,f) \mid \sigma_f$ is an isomorphism of groups for $f \in \mathrm{mor}(\mathcal{G}_1)\}$ of $D^{\mathcal{A}}$ which defines the linear track extension above; see [7, proof of Theorem 3.1], [8, Example 4.7] and Example 5.2 below for examples which satisfy Hypothesis I.

In particular, Example 5.2 gives a linear track extension whose action consists of identities; that is $\sigma_f = id$ for each morphism f in $\mathcal{G}$.

Definition 3.6. Let $\mathcal{G}$ be a track category which satisfies Hypothesis I and $H\mathcal{G}$ small category. Let $\mathcal{E}(H\mathcal{G})$ be the linear track extension in Hypothesis I. Then the class $\langle H\mathcal{G}\rangle := [C_{\mathcal{E}(H\mathcal{G})}(\tau, H)]$ which belongs to $H^3(H\mathcal{G}, D^{\mathcal{A}})$ is called the *universal Toda bracket*.

4. Proof of Theorem 2.6

We begin by recalling a normalized version of the Baues-Wirsching cohomology. Let $S \subset \mathrm{mor}(\mathcal{C})$ be a subclass of morphism in $\mathcal{C}$. We say that S is an *ideal* in $\mathcal{C}$ if $f \circ g \in S,\ g \circ h \in S$ for any $g \in S$ and $(f, g, h) \in N_3(\mathcal{C})$. A natural system D on $\mathcal{C}$ is *S-trivial* if $f^* = 0,\ f_* = 0$ (zero map) for any $f \in S$. For a 0-object $*$, a set $O(\mathcal{C})$ is defined by $O(\mathcal{C}) := \{0 : A \to * \to B | A, B \in \mathrm{ob}(\mathcal{C})\}$. It is readily seen that $O(\mathcal{C})$ is an ideal.

Definition 4.1. Let S be an ideal of $mor(\mathcal{C})$. Abelian subgroups $F^n(S)$ and $\check{F}^n(S)$ of $F^n(\mathcal{C}, D) = F^n$ are denoted by

$$F^n(S) := \begin{cases} \{c \in F^n | c(\lambda_1, \dots, \lambda_n) = 0 \text{ if } \lambda_i \in S \text{ for every } \ i \in \{1, \dots, n\}\} & (n \geq 1) \\ \{c \in F^0 |\ c(A) = 0 \text{ if } 1_A \in S\} & (n = 0) \end{cases}$$

and

$$\check{F}^n(S) := \begin{cases} \{c \in F^n | c(\lambda_1, \dots, \lambda_n) = 0 \text{ if } \lambda_i \in S \text{ for some } \ i \in \{1, \dots, n\}\} & (n \geq 1) \\ \{c \in F^0 \mid c(A) = 0 \text{ if } 1_A \in S\} & (n = 0) \end{cases}$$

respectively. We say that $c \in F^n(S)$ is a cochain *relative* to S. We say that $c \in \check{F}^n(S)$ is a *normalized* cochain.
Observe that we have a sequence $\check{F}^n(S) \subset F^n(S) \subset F^n(\mathcal{C}, D)$ of inclusions.

Let K be a subcategory of a small category $\mathcal{C}$ and S an ideal of $mor(\mathcal{C})$. We define $H^n_S(\mathcal{C}, K; D)$ by $H^n(F^*(morK) \cap \check{F}^*(S), \delta)$, which is called the S-normalized cohomology group of the pair $(\mathcal{C}, K)$.

In order to define the map α in Theorem 2.6, important results in [8] concerning the normalized cohomology are described below.

Theorem 4.2. [8, Theorem (1.9)] *Let S be an ideal in $\mathcal{C}$ and D an S-trivial natural system on $\mathcal{C}$. Then the inclusion $j : S \subset S \cup Ob(\mathcal{C})$ induces an isomorphism $j^* : H^n_{S \cup Ob(\mathcal{C})}(\mathcal{C}, K; D) \xrightarrow{\cong} H^n_S(\mathcal{C}, K; D)$ for $n \geq 0$.*

Theorem 4.3. [8, Theorem (1.10)] *Let $\mathcal{C}$ be a small category which has a zero object and O($\mathcal{C}$) the ideal of zero morphisms. Let K be a subcategory of $\mathcal{C}$ which contains the zero morphism $0 : A \to A$ for every object A in Ob(K). Moreover, let S be an ideal in $\mathcal{C}$ and D a natural system on $\mathcal{C}$ which is $S \cup O(\mathcal{C})$-trivial. Then the inclusion $i : S \subset S \cup O(\mathcal{C})$ induces an isomorphism $i^* : H^n_{S \cup O(\mathcal{C})}(\mathcal{C}, K; D) \xrightarrow{\cong} H^n_S(\mathcal{C}, K; D)$.*

In what follows, we drop the tildes for objects and morphisms of MT and S denotes the ideal $\{fw, gw, bgw\}$ in MT. We shall have a commutative diagram

$$\begin{tikzcd}
(4.1) : & \check{F}^2 \arrow[r, "\check{\delta}^3"] \arrow[d, shift left, "m'"] & \check{F}^3 \arrow[r, "\check{\delta}^4"] \arrow[d, shift left, "m"] & \check{F}^4 \arrow[d, equal] \\
& D^\sharp \arrow[u, shift left, "n'", "\cong"'] \arrow[r, "m \circ \check{\delta}^3 \circ n'"] & D_{bgw} \times D_{afw} \arrow[u, shift left, "n", "\cong"'] \arrow[r, "0"] & 0,
\end{tikzcd}$$

where $D^\sharp := D_{gw} \times D_{bg} \times D_{fw} \times D_{af} \times D_{bgw}$ and $\check{F}^i$denotes $\check{F}^i(S \cup ob(MT))$; see Appendix A for the commutativity of the diagram and the maps. This allows us to deduce the following lemma.

Lemma 4.4. *The map n induces an isomorphism*

$$\tilde{n} : D_{bgw} \times D_{bgw}/I = D_{bgw} \times D_{afw}/I \cong H^3_{S \cup ob(MT)}(MT, D),$$

where $I = \operatorname{Im}(m \circ \check{\delta}^3 \circ n')$

The following lemma is proved in Appendix A.

Lemma 4.5. *The map* $k : D_{bgw}/(b_* D_{gw} + w^* D_{bg} + a_* D_{fw}) \to D_{bgw} \times D_{bgw}/\operatorname{Im}(m \circ \check{\delta}^3 \circ n')$ *defined by* $k([y]) = [(y, 0)]$ *is a well-defined isomorphism.*

Thus, we have the following result.

Theorem 4.6. *Let D be a natural system on MT which is S-trivial. Then $\tilde{n} \circ k$ is an isomorphism from $D_{bgw}/(b_* D_{gw} + w^* D_{bg} + a_* D_{fw})$ to $H^3_{S \cup ob(MT)}(MT, D)$.*

Let MT be the matrix Toda category, $D^{\mathcal{A}}$ the natural system of automorphism on $H\mathcal{C}$ and S an ideal of MT and of the form $\{fw, gw, bgw\}$. Suppose that there exists a functor $\varphi : MT \to H\mathcal{G}$ which satisfies the condition that $\varphi(fw) = 0$ and $\varphi(gw) = 0$. We define $(\varphi^\sharp)^3 : \check{F}^3(\mathrm{O}(H\mathcal{G})) \to \check{F}^3(S)$ by

$$(\varphi^\sharp)^3(c)(\lambda_1, \lambda_2, \lambda_3) = c(\varphi(\lambda_1), \varphi(\lambda_2), \varphi(\lambda_3)).$$

Moreover,we define $\varphi^* : H^3_{O(H\mathcal{G})}(H\mathcal{G}; D^{\mathcal{A}}) \to H^3_S(MT; \varphi^* D^{\mathcal{A}})$ by $\varphi^*([c]) := [(\varphi^\sharp)^3(c)]$ for $[c] \in H^3_{O(H\mathcal{G})}(H\mathcal{G}; D^{\mathcal{A}})$.

We define $\left\{\begin{matrix} b & g & \\ a' & f' & w \end{matrix}\right\}^*$ by the composite which fits into the commutative diagram

$$\begin{tikzcd}
H^3_{O(H\mathcal{G})}(H\mathcal{G}; D^{\mathcal{A}}) \arrow[r, "\varphi^*"] & H^3_S(MT; \varphi^* D^{\mathcal{A}}) \arrow[r, shift left, "\cong"] & H^3_{S \cup ob(MT)}(MT; \varphi^* D^{\mathcal{A}}) \arrow[l, shift left, "j^*"] \arrow[d, shift left, "\cong"] \\
H^3(H\mathcal{G}; D^{\mathcal{A}}) \arrow[u, "\tilde{h}", "\cong"'] \arrow[rr, "\left\{\begin{matrix} b & g & \\ a' & f' & w \end{matrix}\right\}^*"] & & \mathcal{A}_{WX}/b * \mathcal{A}_{WB} + w^* \mathcal{A}_{CX} + a_* \mathcal{A}_{WA}, \arrow[u, shift left, "\tilde{n} \circ k"]
\end{tikzcd}$$

where $\tilde{h}$ is the inverse of the isomorphism described in Theorem 4.3; see also Theorem 4.6.

In order to prove Theorem 2.6, we recall the definition of $\tilde{h}$ in [8, Appendix B]. Under the same assumption as in theorem 4.3, let F^* denote the cochain complex $\{F^n(\mathrm{mor}K) \cap \check{F}^n(S), \delta^n\}$. We write F^n for the module $F^n(\mathrm{mor}K) \cap \check{F}^n(S)$.

For $n \geq 1$ and $0 \leq i \leq n$, let $t^i : F^{n+1} \to F^n$ be a homomorphism defined by

$$t^i c(\lambda_1, \ldots, \lambda_n) := \begin{cases} c(\lambda_1, \ldots, \lambda_i, 0, \lambda_{i+1}, \ldots, \lambda_n) & \text{if } \lambda_1 \circ \ldots \circ \lambda_n = 0 \\ 0 \quad \text{otherwise.} \end{cases}$$

By definition, we see that for $n = 0$, $t^0 : F^1 \to F^0$ is the trivial map. For each non-negative integer k, define a submodule F^n_k of F^n by

$$F^n_k = \{c \in F^n : c(\lambda_1, \ldots, \lambda_n) = 0 \text{ if } \lambda_i = 0 \text{ for some } i \leq k\}.$$

Then F^n_k defines a decreasing sequence of subcomplexes of (F^*, δ) such that $F^n_0 = F^n$ and $F^n_k = F^n(\mathrm{mor}K) \cap \check{F}^n(S \cup O(\mathcal{C}))$ for $k \geq n$. We define a map $h^k : F^*_k \to F^*_k$ by $h^k := 1 - \tilde{t^k}\delta^{n+1} - \delta^n \tilde{t^k}$, where

$$\tilde{t^k}(c) := \begin{cases} (-1)^{k+1} t^k & \text{if } k \leq n-1 \\ 0 \quad \text{otherwise.} \end{cases} \quad \text{for } c \in F^n$$

A direct calculation shows that h^k is a chain map; see [8, page 337] . Thus we see that h^k is homotopic to the identity. The homomorphism h^k satisfies the condition that (i) $h^k c = c$ for $c \in F^*_{k+1}$ and (ii) $h^k(F^*_k) \subset F^*_{k+1}$.

It follows from the condition (ii) that h^k is a homomorphism from F^n_k to F^n_{k+1}. Thus we have a commutative diagram

$$\begin{tikzcd}
{} \arrow[r] & F^{n-1}_{n+1} \arrow[r, "\delta^n"] & F^n_{n+1} \arrow[r, "\delta^n"] & F^{n+1}_{n+1} \arrow[r, "\delta^n"] & {} \\
{} \arrow[r] & F^{n-1}_{n} \arrow[u, "id"] \arrow[r, "\delta^n"] & F^n_{n} \arrow[u, "id"] \arrow[r, "\delta^n"] & F^{n+1}_{n} \arrow[u, "h^n"] \arrow[r, "\delta^n"] & {} \\
{} \arrow[r] & F^{n-1}_{n-1} \arrow[u, "id"] \arrow[r, "\delta^n"] & F^n_{n-1} \arrow[u, "h^{n-1}"] \arrow[r, "\delta^n"] & F^{n+1}_{n-1} \arrow[u, "h^{n-1}"] \arrow[r, "\delta^n"] & {} \\
 & \vdots \arrow[u, "h^{n-2}"] & \vdots \arrow[u, "h^{n-2}"] & \vdots \arrow[u, "h^{n-2}"] & \\
{} \arrow[r] & F^{n-1}_{0} \arrow[u, "h^0"] \arrow[r, "\delta^n"] & F^n_{0} \arrow[u, "h^0"] \arrow[r, "\delta^{n+1}"] & F^{n+1}_{0} \arrow[u, "h^0"] \arrow[r] & {}
\end{tikzcd}$$

This diagram enables us to define a chain map $h : F^n_0 = F^n \to F^n_n = F^n(\mathrm{mor}K) \cap \check{F}^n(S \cup O(\mathcal{C}))$ by $h = h^{n-1} \circ h^{n-2} \circ \cdots \circ h^0$. The homomorphism $\tilde{h}$ in Theorem 2.6 is defined by $\tilde{h}([c]) := [hc]$.

We describe Lemma 4.7 which is necessary for Lemma 4.8.

Lemma 4.7. *Suppose that $S : af \Rightarrow bg$ is an invertible 2-morphism in $\mathcal{G}$. If $N(S)$ is a non-empty set, then there exist two invertible 2-morphisms $U : af \Rightarrow 0$ and $V : bg \Rightarrow 0$ such that $S = -V + U$.*

Proof. Since $N(S) \neq 0$, there exists 2-morphism $G : 0 \Rightarrow 0$ such that $G \langle \mathrm{conj} \rangle S$; that is, there exist invertible 2-morphisms F_1 and F_2 such that $F_1 + G = S + F_2$. Then we have an equality $S = F_1 + G - F_2$. We define invertible 2-morphisms $U : af \Rightarrow 0$ and $V : bg \Rightarrow 0$ by $V = -G - F_1$ and $U = -F_2$, respectively. Then we see that $S = -V + U : af \Rightarrow bg$. □

With two important lemmas below, we prove Theorem 2.6; see Appendix B for the proofs of the lemmas 4.8, 4.9.

Lemma 4.8. *For the map k in Lemma 4.5, one has $k([\theta]) = [(-bK - Vw, -aH - Uw)]$, where V and U are the 2-morphisms defined by the 2-morphism $S : af \Rightarrow bg$ in Lemma 4.7.*

Lemma 4.9. *For the composite $h = h^2 \circ h^1 \circ h^0$, one has*

$$(\varphi^\sharp)^3(h\mathcal{C}_{\mathcal{E}(C)}(\tau, H))(b, g, w) = \mathcal{C}_{\mathcal{E}(C)}(\tau, H)(b, g, w) = -bK - Vw.$$

Proof of Theorem 2.6. We define a homomorphism

$$\alpha : H^3_S(MT, \varphi^* D^{\mathcal{A}}) \xrightarrow{\cong} \mathcal{A}_{WX}/(b_*\mathcal{A}_{WB} + w^*\mathcal{A}_{CX} + a_*\mathcal{A}_{WA})$$

by $\alpha := (j^* \circ \tilde{n} \circ k)^{-1}$.

It follows from Lemma 4.8, 4.4, Theorem 4.2 and Lemma A.1 that

$$\begin{aligned}
\alpha^{-1}([\theta]) &= j^* \circ \tilde{n} \circ k([\theta]) \\
&= (j^* \circ \tilde{n})[(-bK - Vw, -aH - Uw)] \\
&= j^*[n(-bK - Vw, -aH - Uw)] = [n(-bK - Vw, -aH - Uw)] \\
&= [c'_{(-bK - Vw, -aH - Uw)}]
\end{aligned}$$

in $H^3_S(MT; \varphi^* D^{\mathcal{A}})$. For $[c'_{(-bK-Vw,-aH-Uw)}] \in H^3_S(MT; \varphi^* D^{\mathcal{A}})$, we have

$$c'_{(-bK-Vw,-aH-Uw)}(b, g, w) = -bK - Vw.$$

The same argument enables us to deduce that $c'_{(-bK-Vw,-aH-Uw)}(a,f,w) = -aH-Uw$. The result above and Lemma 4.9 yield that $\varphi^* \circ \tilde{h}(\langle \mathcal{C} \rangle) = \alpha^{-1}([\theta])$. This completes the proof. □

5. PROOF OF THEOREM 2.14

In this section, we describe a matrix Toda bracket in a triangulated category in terms of a classical Toda bracket defined in [17, Chapter II].

Let $\mathcal{T}$ be a triangulated category with suspension functor Σ. We recall here the Toda bracket in $\mathcal{T}$ defined by Heller [17]. Given a diagram $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ with $hg = 0$ and $gf = 0$ in $\mathcal{T}$. Then we have a distinguished triangle of the form $W \xrightarrow{f} X \xrightarrow{i} C_f \xrightarrow{j} \Sigma W$ and a commutative diagram

$$\begin{tikzcd}
W \arrow[r, "f"] \arrow[d, equal] & X \arrow[r, "i"] \arrow[d, equal] & C_f \arrow[r, "j"] \arrow[d, "s"] & \Sigma W \arrow[d, "t"] \\
W \arrow[r, "f"'] & X \arrow[r, "g"'] & Y \arrow[r, "h"'] & Z
\end{tikzcd}$$

in $\mathcal{T}$. The Toda bracket $\{h, g, f\} \in \mathcal{T}(\Sigma W, Z)/(h\mathcal{T}(\Sigma W, Y)) + \mathcal{T}(\Sigma X, Z)\Sigma f)$ is defined by the class $[t]$ in the quotient.

Let $\mathcal{B}$ be an additive category and $Ch(\mathcal{B})$ the category of cochain complexes on $\mathcal{B}$ mentioned in Section 2. The homotopy category $H(Ch(\mathcal{B}))$, which admits the triangulated category structure whose distinguished triangles are constructed by the mapping cone and suspension functors. Observe that the suspension functor $\Sigma : H(Ch(\mathcal{B})) \to H(Ch(\mathcal{B}))$ is defined by $(\Sigma X)^i = X^{i+1}$ and $d_{\Sigma X} = -d_X$ for a cochain complex (X, d_X).

Let $\mathcal{T}$ be an algebraic triangulated category with a fully faithful exact functor $\vartheta : \mathcal{T} \to H(Ch(\mathcal{B}))$. In what follows, we identify morphisms in $\mathcal{T}$ with their images by ϑ in $H(Ch(\mathcal{B}))$.

Proposition 5.1. *$\mathcal{A}_{WX} = \mathcal{A}(0 : W \to X) = \mathcal{T}(\Sigma W, X)$.*

Proof. By definition, we see that

$$\begin{aligned}
\mathcal{A}(0 : W \to X) &= N(1_0 : 0_X^W \Rightarrow 0_X^W) = \mathrm{Hom}_{\mathcal{T}(W,X)}(0_X^W, 0_X^W) \\
&= \mathrm{Hom}_{ch(\mathcal{B})(W,X)}(0_X^W, 0_X^W) \\
&= \{h : W \to X \mid 0 - 0 = d_X h + h d_W\}/\sim \\
&= \{h : \Sigma W \to X \mid h : \text{a cochain map}\}/\text{chain homotopy relation} \\
&= H(Ch(\mathcal{B}))(\Sigma W, X) \\
&= \mathcal{T}(\Sigma W, X).
\end{aligned}$$

This completes the proof. □

Here we verify that the matrix Toda bracket and the Toda bracket in Theorem 2.14 are in the same abelian group in $H(Ch(\mathcal{B}))(\Sigma(\vartheta W), \vartheta X)$.

The matrix Toda bracket $\left\{\begin{matrix} \vartheta b, & \vartheta g & \\ & & \vartheta w \\ \vartheta a, & \vartheta f & \end{matrix}\right\}$ belongs to $\mathcal{A}_{\vartheta W \vartheta X}/(\vartheta b_* \mathcal{A}_{\vartheta W \vartheta B} + \vartheta w^* \mathcal{A}_{\vartheta C \vartheta X} + \vartheta a_* \mathcal{A}_{\vartheta W \vartheta A})$ and $\mathcal{A}_{\vartheta W \vartheta X}$ is nothing but the group $H(Ch(\mathcal{B}))(\Sigma(\vartheta W)), \vartheta X)$ by Proposition 5.1. The classical Toda bracket $\vartheta\{\nabla(a,b), -f \vee g, w\}$ in $H(Ch(\mathcal{B})$ belong to $\vartheta\mathcal{T}(\Sigma W, X)/\vartheta(\nabla(a,b)\mathcal{T}(\Sigma W, A \oplus B) + \mathcal{T}(\Sigma C, X)\Sigma w)$ and $\vartheta\mathcal{T}(\Sigma W, X)$ is equal to $H(Ch(\mathcal{B}))(\vartheta(\Sigma W), \vartheta X)$. Thus we have $H(Ch(\mathcal{B}))(\vartheta(\Sigma W), \vartheta X) \circ \eta_W^{-1} = H(Ch(\mathcal{B}))(\Sigma(\vartheta W), \vartheta X)$.

We denote by MD the diagram (2.1) in $\mathcal{T}$. Then MD gives rise to a sequence $T(MD) : W \overset{w}{\to} C \overset{-f\vee g}{\to} A\oplus B \overset{\nabla(a,b)}{\to} X$ with $(-f\vee g)w = 0$ and $\nabla(a,b)(-f\vee g) = 0$. Thus $T(MD)$ is regarded as a Toda category by definition.

Proof of Theorem 2.14. We see that the both sides are in the same Abelian group. In fact, the natural maps $i : A \to A\oplus B$ and $j : B \to A\oplus B$ induce an isomorphism $(i_*, j_*) : \mathcal{T}(\Sigma W, A) \oplus \mathcal{T}(\Sigma W, B) \overset{\cong}{\to} \mathcal{T}(\Sigma W, A \oplus B)$. It follows from the universality of the direct sum that the composite $\nabla(a, b)_* \circ (i_*, j_*)$ is nothing but $\nabla(a_*, b_*) : \mathcal{T}(\Sigma W, A) \oplus \mathcal{T}(\Sigma W, B) \to \mathcal{T}(\Sigma W, X)$. Thus we have

$$\nabla(a, b)_*(\Sigma W, A \oplus B) = a_*\mathcal{T}(\Sigma W, A) + b_*\mathcal{T}(\Sigma W, B).$$

Indeed, we have

$$\begin{aligned}\nabla(a, b)_*\mathcal{T}(\Sigma W, A \oplus B) &= \nabla(a, b)_*(i_*, j_*)(\mathcal{T}(\Sigma W, A) \oplus \mathcal{T}(\Sigma w, B))\\ &= \nabla(a_*, b_*)(\mathcal{T}(\Sigma W, A) \oplus \mathcal{T}(\Sigma w, B))\\ &= a_*\mathcal{T}(\Sigma W, A) + b_*\mathcal{T}(\Sigma W, B).\end{aligned}$$

$$\begin{tikzcd}
 & \mathcal{T}(\Sigma W, B) \arrow[d, "j_*"] \arrow[dr, "b_*"] & \\
\mathcal{T}(\Sigma W, A) \oplus \mathcal{T}(\Sigma W, B) \arrow[ur, "(0,j_*)"] \arrow[r, "(i_*,j_*)"] \arrow[dr, "(i_*,0)"'] & \mathcal{T}(\Sigma W, A \oplus B) \arrow[r, "\nabla(a,b)_*"] & \mathcal{T}(\Sigma W, X) \\
 & \mathcal{T}(\Sigma W, A) \arrow[u, "i_*"'] \arrow[ur, "a_*"'] &
\end{tikzcd}$$

The result and Proposition 5.1 enable us to conclude that the matrix Toda bracket and the classical Toda bracket are in the same quotient group.

In order to show the equality of Theorem 2.14, we consider a commutative diagram

$$\begin{tikzcd}
\mathcal{T}(W, A \oplus B) \arrow[d, "\nabla(a,b)_*"] & \mathcal{T}(C, A \oplus B) \arrow[l, "w^*"'] \arrow[d, "\nabla(a,b)_*"] & \mathcal{T}(C_w, A \oplus B) \arrow[l, "i^*"'] \arrow[d, "\nabla(a,b)_*"] & \mathcal{T}(\Sigma W, A \oplus B) \arrow[l, "j_*"'] \arrow[d, "\nabla(a,b)_*"] \\
\mathcal{T}(W, X) & \mathcal{T}(C, X) \arrow[l, "w^*"'] & \mathcal{T}(C_w, X) \arrow[l, "i^*"'] & \mathcal{T}(\Sigma W, X) \arrow[l, "j_*"']
\end{tikzcd}$$

in which row sequences are exact. Chasing the diagram, we construct t and s in the definition of the Toda bracket $\{\nabla(a, b), -f\vee g, w\}$. Since $w^*(-f\vee g) = (-f\vee g)w = 0$ for $-f\vee g$ in $\mathcal{T}(C, A\oplus B)$, it follows that $i^*(s) = -f\vee g$ for some morphism $s : C_w \to A\oplus B$. In fact, we can choose the morphism $\begin{pmatrix} -f & H \\ g & -K \end{pmatrix} : C_w = C\oplus \Sigma W \to A\oplus B$ in $\mathcal{T}(C_w, A \oplus B)$ as s. The equation $C_w = C \oplus \Sigma W$ is meant in the underlying category of graded objects in $\mathcal{B}$, not in $\mathcal{T}$. Moreover, we see that $j^*(-bK + Sw + aH) = (0, -bK + Sw + aH)$, $\nabla(a, b)_* \begin{pmatrix} -f & H \\ g & -K \end{pmatrix} = (-af + bg, aH - bK)$ and

$$j^*(-bK + Sw + aH) - \nabla(a, b)_* \begin{pmatrix} -f & H \\ g & -K \end{pmatrix} = (S, 0) \begin{pmatrix} d_C & w \\ 0 & -d_W \end{pmatrix} + d_X(S, 0).$$

This implies that we can choose $-bK + Sw + aH$ as t and $j_*(t) = \nabla(a, b)_*s$; see Appendix C in detail. This completes the proof. □

We describe a computational example with Theorem 2.14.

Example 5.2. Let $\mathcal{G}$ be the 2-category $Ch(\mathcal{B})$ mentioned above. One has a linear track extension of the form $D^{\mathcal{A}} \xrightarrow{+} \mathcal{G}_2 \rightrightarrows \mathcal{G}_1 \xrightarrow{p} H\mathcal{G}$. By definition, we see that $D^{\mathcal{A}}_{[f]} = \mathcal{A}(0 : s(f) \to t(f)) = \mathcal{G}(0,0) = \mathcal{G}(f,f)$. Then, we define the action σ by $\sigma_f = id$ for any f. Indeed, $\mathcal{G}(f,f) = \{h : X \to Y \mid f - f = d_Y h + h d_X\}/\sim = \{h : X \to Y \mid 0 = 0 - 0 = d_Y h + h d_X\}/\sim = \mathcal{G}(0,0)$. In this example, all conditions in Theorem 2.6 are satisfied.

Example 5.3. Let $\mathcal{T}$ be the triangulated category $H(Ch(\mathcal{B}))$, where $\mathcal{B}$ is the category of $\mathbb{Z}$-modules. Here $\mathcal{T}$ is clearly algebraic. Let $X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{j} \Sigma X$ be the mapping cone construction of a map f in $\mathcal{T}$. Suppose that there exists an integer k such that (i) $H^k(X) \neq 0$ while $\mathrm{Hom}(H^k(Y), H^k(X)) = 0$ and (ii) $H(j) : H^{k-1}(C_f) \to H^{k-1}(\Sigma X)$ is not surjective. Then we see that $1_{\Sigma(X\oplus X)}$ is a non-trivial element in the quotient Q defined by

$$\mathcal{T}(\Sigma(X\oplus X), \Sigma(X\oplus X))/(j,j)\mathcal{T}(\Sigma(X\oplus X), C_f\oplus C_f) + \mathcal{T}(\Sigma(Y\oplus Y), \Sigma(X\oplus X))\Sigma(f,f).$$

In fact, If $1_{\Sigma(X\oplus X)}$ is trivial in Q, then we have

$$1_{\Sigma(X\oplus X)} = (j,j)(\alpha) + \beta \circ \Sigma(f,f)$$

for some $\alpha \in H^0(\mathrm{Hom}(\Sigma(X\oplus X), C_f \oplus C_f))$ and $\beta \in H^0(\mathrm{Hom}(\Sigma(Y\oplus Y), \Sigma(X\oplus X))$.

Observe that in our setting, the Hom-set $\mathcal{T}(U,V)$ is nothing but the 0th cohomology $H^0(\mathrm{Hom}((U,d_U),(V,d_V)))$ of the cochain complex $\mathrm{Hom}(U,V)$ with the differential δ defined by $\delta(\varphi) = d_V\varphi - (-1)^{\deg \varphi}\varphi d_U$ for a homomorphism $\varphi : U \to V$.

Let $h : H^0(\mathrm{Hom}((U,d_U),(V,d_V))) \to \prod_{-p+q=0} \mathrm{Hom}(H^p(U), H^q(V))$ be the homomorphism defined by assigning a cochain map the map induced in the cohomology. Then the equality above enables us to deduce that

$$h(1_{\Sigma(X\oplus X)}) = (H(j),H(j))h(\alpha) + h(\beta) \circ \Sigma(H(f),H(f)) = (H(j),H(j))h(\alpha)$$

in the degree $k-1$. The second equality follows from the condition (i). The left hand side is the identity map $1_{H(\Sigma(X\oplus X))}$ while the right hand side is not surjective because of (ii), which is a contradiction.

We consider a diagram

$$\begin{tikzcd} MD_f : & X \oplus X \arrow[r, "{(f,f)}"] & Y \oplus Y \arrow[r, "{(i,0)}"] \arrow[d, "{-(0,i)}"'] & C_f \arrow[d, "j\vee 0"] \\ & & C_f \arrow[r, "0\vee j"'] & \Sigma(X \oplus X) \end{tikzcd}$$

in $\mathcal{T}$. It is readily seen that this is a matrix Toda diagram. Thus the diagram gives rise to a Toda diagram of the form

$$\begin{tikzcd} T(MD_f) : & X \oplus X \arrow[r, "{(f,f)}"] & Y \oplus Y \arrow[r, "{(i,i)}"] & C_f \oplus C_f \arrow[r, "{(j,j)}"] & \Sigma(X \oplus X), \end{tikzcd}$$

which is a distinguished triangle in $\mathcal{T}$. Theorem 2.14 implies that in Q,

$$\left\{ \begin{matrix} j \vee 0, & (i,0) \\ 0 \vee j, & -(0,i) \end{matrix} \quad (f,f) \right\} = \{(j,j),(i,i),(f,f)\} = [1_{\Sigma(X\oplus X)}] \neq 0.$$

The last equality follows from the definition of the classical Toda bracket.

We describe a toy example; see diagram below. As the example which satisfies the conditions (i) and (ii) above, we give complexes X, Y and a cochain map $f : X \to Y$, where $X^0 = \mathbb{Z}$, $X^i = 0$ if $i \neq 0$, $Y^{-1} = Y^0 = \mathbb{Z}$, $Y^k = 0$ for $k \neq -1, 0$, the differential $d_Y : Y^{-1} \to Y^0$ is defined by $d_Y(y) = 2y$ and $f^0 = 1 : X^1 = \mathbb{Z} \to Y^1 = \mathbb{Z}$. Observe that $H^0(Y) = \mathbb{Z}/2\mathbb{Z}$, $H^0(X) = \mathbb{Z}$ and $H(j) : H^{-1}(C_f) = 2\mathbb{Z} \to H^{-1}(\Sigma X) = \mathbb{Z}$ is the inclusion; see Appendix D.

The result [8, Theorem (4.6)] asserts that linear track extensions of $\mathcal{C}$ by a natural system D are classified with the Baues-Wirsching cohomology $H^3(\mathcal{C}, D)$. Then, Theorem 2.6 enables one to deduce the following result.

Example 5.4. Let $\mathcal{G}$ be a small sub 2-category of $Ch(\mathcal{B})$ which contains the matrix Toda diagram MD_f described in Example 5.3. There exists a linear track extension of $H\mathcal{G}$ by $D^{\mathcal{A}}$ with $\sigma_0 = 1$ which is non-trivial up to equivalence. This follows from Corollary 2.7.

APPENDIX A. PROOF OF LEMMA 4.5

By the same argument as in the proof of [8, Lemma (1.12)], we have Lemma 4.5. We first define isomorphism m, n, m' and n' in diagram (4.1).

Lemma A.1. *The Abelian subgroup* $\check{F}^3$ *of* F^3 *is isomorphic to* $D_{bgw} \times D_{afw}$.

Proof. We define homomorphisms $m : \check{F}^3 \to D_{bgw} \times D_{afw}$ and $n : D_{bgw} \times D_{afw} \to \check{F}^3$ by $m(c) := (c(b, g, w), c(a, f, w))$ and $n((x_1, x_2)) := c'_{(x_1,x_2)}$ respectively, where c'_{x_1,x_2} is defined by

$$c'_{(x_1,x_2)}(\alpha_1, \alpha_2, \alpha_3) := \begin{cases} x_1 & (\alpha_1, \alpha_2, \alpha_3) = (b, g, w) \\ x_2 & (\alpha_1, \alpha_2, \alpha_3) = (a, f, w) \\ 0 \text{ if there exists } i \text{ such that } \alpha_i \text{ is in } S \cup ob(MT). \end{cases}$$

The conditions (ii) and (iii) in the definition of MT yield that the map n is well defined. It is readily seen that $n \circ m = id_{\check{F}^3}$ and $m \circ n = id_{D_{bgw} \times D_{afw}}$. □

Lemma A.2. *The Abelian subgroup* $\check{F}^2$ *of* F^2*is isomorphic to* $D^\sharp = D_{gw} \times D_{bg} \times D_{fw} \times D_{af} \times D_{bgw}$.

Proof. We define homomorphisms $m' : \check{F}^2 \to D^\sharp$ and $n' : D^\sharp \to \check{F}^2$ by $m'(c) := (c(g, w), c(b, g), c(f, w), c(a, f), (bg, w))$ and $n'((x_1, x_2, x_3, x_4, x_5)) := c'_{(x_1,x_2,x_3,x_4,x_5)}$, respectively. Here,

$$c'_{(x_1,x_2,x_3,x_4,x_5)}(\alpha_1, \alpha_2) := \begin{cases} x_1 & (\alpha_1, \alpha_2) = (g, w) \\ x_2 & (\alpha_1, \alpha_2) = (b, g) \\ x_3 & (\alpha_1, \alpha_2) = (f, w) \\ x_4 & (\alpha_1, \alpha_2) = (a, f) \\ x_5 & (\alpha_1, \alpha_2) = (bg, w) \\ 0 \text{ if there exists } i \text{ such that } \alpha_i \text{ is in } S \cup ob(MT). \end{cases}$$

The well-definedness of n' follows from the condition (i) in the definition of MT and the argument of Lemma A.1. A direct calculation shows that $m' \circ n' = id_{D^\sharp}$ and $n' \circ m' = id_{\check{F}^2}$. □

We show the following Lemmas to describe cochains $\check{F}^4$ and the map $m \circ \check{\delta}^3 \circ n'$ explicitly.

Lemma A.3. *The Abelian subgroup* $\check{F}^n$ *of* F^n *is trivial for* $n \geq 4$.

Proof. For $(\lambda_1, \ldots, \lambda_n) \in N_n(MT)$, there exists i such that λ_i is the identity. Therefore, $c(\lambda_1, \ldots, \lambda_n) = 0$ for any $c \in \check{F}^n$. □

Lemma A.4. *For* $(x_1, x_2, x_3, x_4, x_5) \in D^\sharp$*, one has*

$$m \circ \check{\delta}^3 \circ n'(x_1, x_2, x_3, x_4, x_5) = (b_* x_1 - w^* x_2 - x_5, a_* x_3 - w^* x_4 - x_5).$$

Proof. By definition, it follows that

$$\begin{array}{rl} & m \circ \check{\delta}^3(n'(x_1, x_2, x_3, x_4, x_5)) \\ = & m \circ \check{\delta}^3(c'_{(x_1,x_2,x_3,x_4,x_5)}) \\ = & m(\check{\delta}^3(c'_{(x_1,x_2,x_3,x_4,x_5)})) \\ = & (\check{\delta}^3(c'_{(x_1,x_2,x_3,x_4,x_5)})(b, g, w),\ \check{\delta}^3(c'_{(x_1,x_2,x_3,x_4,x_5)})(a, f, w)) \\ = & (\delta^3(c'_{(x_1,x_2,x_3,x_4,x_5)})(b, g, w),\ \delta^3(c'_{(x_1,x_2,x_3,x_4,x_5)})(a, f, w)). \end{array}$$

For the first component , we have

$$\begin{array}{rl} & \delta^3(c'_{(x_1,x_2,x_3,x_4,x_5)})(b, g, w) \\ = & b_*c'_{(x_1,x_2,x_3,x_4,x_5)}(g, w) + (-1)^1 c'_{(x_1,x_2,x_3,x_4,x_5)}(bg, w) \\ & +(-1)^2 c'_{(x_1,x_2,x_3,x_4,x_5)}(b, gw) + (-1)^3 w^*(c'_{(x_1,x_2,x_3,x_4,x_5)}(b, g)) \\ = & b_*x_1 - x_5 + 0 - w^*x_2 = b_*x_1 - w^*x_2 - x_5. \end{array}$$

For the second one, we see that

$$\begin{array}{rl} & \delta^3(c'_{(x_1,x_2,x_3,x_4,x_5)})(a, f, w) \\ = & a_*c'_{(x_1,x_2,x_3,x_4,x_5)}(f, w) + (-1)^1 c'_{(x_1,x_2,x_3,x_4,x_5)}(af, w) \\ & +(-1)^2 c'_{(x_1,x_2,x_3,x_4,x_5)}(a, fw) + (-1)^3 w^*(c'_{(x_1,x_2,x_3,x_4,x_5)}(a, f)) \\ = & a_*x_3 - x_5 + 0 - w^*x_4 = a_*x_3 - w^*x_4 - x_5. \end{array}$$

This completes the proof. □

Proof of Lemma 4.5. We define maps

$$l : D_{bgw} \times D_{bgw}/\operatorname{Im}(m \circ \check{\delta}^3 \circ n') \to\ D_{bgw}/(b_*D_{gw} + w^*D_{bg} + a_*D_{fw}),$$

$$k : D_{bgw}/(b_*D_{gw} + w^*D_{bg} + a_*D_{fw}) \to D_{bgw} \times D_{bgw}/\operatorname{Im}(m \circ \check{\delta}^3 \circ n')$$

by

$$l([(x_1, x_2)]) := [x_1 - x_2] \text{ for } [(x_1, x_2)] \in D_{bgw} \times D_{afw}\operatorname{Im}(m \circ \check{\delta}^3 \circ n'),$$

$$k([y]) := [(y, 0)] \text{ for } [y] \in D_{bgw}/(b_*D_{gw} + w^*D_{bg} + a_*D_{fw}).$$

We show that l and k are well-defined homomorphisms. For any element (z_1, z_2) in $\operatorname{Im}(m \circ \check{\delta}^3 \circ n')$, there exists an element $(x_1, x_2, x_3, x_4, x_5) \in D^\sharp$ such that $m \circ \check{\delta}^3 \circ n'(x_1, x_2, x_3, x_4, x_5) = (z_1, z_2)$. By Lemma A.4, we see that $z_1 = b_*x_1 - w^*x_2 - x_5$, $z_2 = a_*x_3 - w^*x_4 - x_5$ and

$$\begin{array}{rcl} z_1 - z_2 & = & b_*x_1 - w^*x_2 - x_5 - (a_*x_3 - w^*x_4 - x_5) \\ & = & b_*x_1 - a_*x_3 - w^*(x_2 - x_4) \in b_*D_{gw} + a_*D_{fw} + w^*D_{bg}. \end{array}$$

Next, for any element u in $b_*D_{gw} + a_*D_{fw} + w^*D_{bg}$, we write $u = b_*x + w^*y + a_*z$. Then $m \circ \check{\delta}^3 \circ n'(x, -y, -z, 0, -a_*z) = (u, 0)$. Observe that $a_*z \in a_*D_{fw} \subset D_{afw} = D_{bgw}$. It is easily seen that $l \circ k = 1_{D_{bgw}/(b_*D_{gw}+w^*D_{bg}+a_*D_{fw})}$ and $k \circ l = 1_{D_{bgw} \times D_{bgw}/\operatorname{Im}(m\circ\check{\delta}^3\circ n')}$. Indeed , $l \circ k([y]) = l([(y, 0)]) = [y - 0] = [y]$ and

$$k \circ l([(x_1, x_2)]) = k([x_1 - x_2]) = [(x_1 - x_2, 0)] = [(x_1, x_2)].$$

In fact, $[(x_1, x_2)] - [(x_1 - x_2, 0)] = [(x_1, x_2) - (x_1 - x_2, 0)] = [(x_2, x_2)]$. Moreover, since $m \circ \check{\delta}^3 \circ n'(0, 0, 0, 0, -x_2) = (b_*0 - w^*0 - (-x_2), a_*0 - w^*0 - (-x_2)) = (x_2, x_2)$, it follows that (x_2, x_2) is in $\operatorname{Im}(m \circ \check{\delta}^3 \circ n')$. We see that $[(x_2, x_2)] = 0$ and hence $[(x_1, x_2)] = [(x_1 - x_2, 0)]$. □

APPENDIX B. PROOFS OF LEMMAS 4.8 AND 4.9

Explicit calculation gives the proof of the Lemma 4.8 and we describe a proof of Lemma 4.9 in detail.

Proof of Lemma 4.8. Recall that ${D^{\mathcal{A}}}_{bgw} := \mathcal{A}(0 : s(bgw) \to t(bgw)) = \mathcal{A}(0 : W \to X)$. We see that $Uw : afw \Rightarrow 0w = 0$ and then $-Uw : 0 \Rightarrow afw : W \to X$ and $-aH : afw \Rightarrow 0 : W \to X$. This implies that $-aH - Uw : 0 \Rightarrow 0 : W \to X$ and that $((0, 0, 0, 0, -(Uw + aH)))$ is in $D^{\sharp} = D^{\mathcal{A}}_{gw} \times D^{\mathcal{A}}_{bg} \times D^{\mathcal{A}}_{fw} \times D^{\mathcal{A}}_{af} \times {D^{\mathcal{A}}}_{bgw}$. Then we have $[(-bK + (-Vw) + Uw + aH, 0)] = [(-bK - Vw, -aH - Uw)]$. In fact, it follows that

$$\begin{aligned} & (-bK + (-Vw) + Uw + aH, 0) - (-bK - Vw, -aH - Uw) \\ = \; & (-bK + (-Vw) + Vw + bK + Uw + aH, 0 - (-aH - Uw)) \\ = \; & (Uw + aH, Uw + aH) \\ = \; & m \circ \check{\delta}^3 \circ n'((0, 0, 0, 0, -(Uw + aH))) \in \mathrm{Im}(m \circ \check{\delta}^3 \circ n'). \end{aligned}$$

The last equality follows from Lemma A.4. Therefore, we see that

$$\begin{aligned} k([\theta]) := [(\theta, 0)] = [(-bK + Sw + aH, 0)] \; &= \; [(-bK + (-V + U)w + aH, 0)] \\ &= \; [(-bK + (-Vw) + Uw + aH, 0)] \\ &= \; [(-bK - Vw, -aH - Uw)]. \end{aligned}$$

This completes the proof. □

Proof of Lemma 4.9. We see that

$$\mathcal{C}_{\mathcal{E}(C)}(\tau, H)(b, g, w) := \sigma^{-1}_{\tau(bgw)}(\Delta) = \sigma^{-1}_{\tau(0)}(\Delta) = \sigma_0^{-1}(\Delta) = \Delta,$$

since $\sigma_0 = id$. We have

$$\begin{aligned} \Delta \; &= \; -H(b, gw) - (\tau b)_* H(g, w) + (\tau w)^* H(b, g) + H(bg, w) \\ &= \; -1_{0_*} - (\tau b)_* H(g, w) + (\tau w)^* H(b, g) + 1_{0_*} \quad (gw,\ bg \in S; \text{see Theorem 4.6}) \\ &= \; -(\tau b) \circ H(g, w) + H(b, g) \circ (\tau w) \\ &= \; -bK - Vw. \;\; (\tau b = b, H(g, w) = K, \tau w = w, H(b, g) = -V) \end{aligned}$$

This yields the second equality in Lemma 4.9.

Next, we shall show the first equality in Lemma 4.9. We denote $\mathcal{C}_{\mathcal{E}(C)}(\tau, H)$ by c. By the definitions of $(\varphi^{\sharp})^3$ and the inclusion φ, we have $(\varphi^{\sharp})^3(hc)(b, g, w) = hc(\varphi(b), \varphi(g), \varphi(w)) = hc(b, g, w)$; see the definition of $(\varphi^{\sharp})^3$. Since $h = h^2 \circ h^1 \circ h^0$, it follows that $hc(b, g, w) = (h^2 \circ h^1 \circ h^0)c(b, g, w)$, where $h^0 = (1 - \tilde{t}^0\delta^4 - \delta^3\tilde{t}^0)$, $h^1 = (1 - \tilde{t}^1\delta^4 - \delta^3\tilde{t}^1)$ and $h^2 = (1 - \tilde{t}^2\delta^4 - \delta^3\tilde{t}^2)$.

We determine $h^0c(b, g, w)$, $h^1(h^0c(b, g, w))$ and $h^2(h^1(h^0c(b, g, w)))$. As for the element $h^0c(b, g, w)$, we have

$$\begin{aligned} h^0c(b, g, w) \; &= \; (1 - \tilde{t}^0\delta^4 - \delta^3\tilde{t}^0)c(b, g, w) \\ &= \; (c - \tilde{t}^0\delta^4(c) - \delta^3\tilde{t}^0(c))(b, g, w) \\ &= \; c(b, g, w) - \delta^4(c)\tilde{t}^0(b, g, w) - \delta^3\tilde{t}^0(c)(b, g, w). \end{aligned}$$

Here, $-\delta^4(c)\tilde{t}^0(b,g,w)$ and $-\delta^3\tilde{t}^0(c)(b,g,w)$ are determined as follows. By the definitions of δ^4 and $\tilde{t}^k$, we have

$$\begin{aligned}
-\delta^4(c)\tilde{t}^0(b,g,w) &= -(\tilde{t}^0(\delta^4(c))(b,g,w)) \\
&= -\tilde{t}^0(\delta^4(c)(b,g,w)) \\
&= -(-1)^{(0+1)}t^0(\delta^4(c)(b,g,w)) \\
&= \delta^4(c)(0,b,g,w) \\
&= 0_*c(b,g,w) - c(0b,g,w) + c(0,bg,w) - c(0,b,gw) + w^*c(0,b,g) \\
&= -c(0,g,w) + c(0,bg,w) - c(0,b,gw) + w^*c(0,b,g).
\end{aligned}$$

Moreover, we see that

$$\begin{aligned}
-\delta^3\tilde{t}^0(c)(b,g,w) &= -((-1)^1t^0(c)b_*(g,w) - (-1)^1t^0(c)(bg,w) \\
&\quad +(-1)^1t^0(c)(b,gw) - w^*(-1)^1t^0(c)(b,g)) \\
&= b_*c(0,g,w) - c(0,bg,w) + c(0,b,gw) - w^*c(0,b,g).
\end{aligned}$$

The definition of h^k enables us to calculate $h^0c(b,g,w)$ as follows:

$$\begin{aligned}
h^0c(b,g,w) &= c(b,g,w) - \delta^4(c)\tilde{t}^0(b,g,w) - \delta^3\tilde{t}^0(c)(b,g,w) \\
&= c(b,g,w) - c(0,g,w) + c(0,bg,w) - c(0,b,gw) + w^*c(0,b,g) \\
&\quad +b_*c(0,g,w) - c(0,bg,w) + c(0,b,gw) - w^*c(0,b,g) \\
&= c(b,g,w) - c(0,g,w) + b_*c(0,g,w).
\end{aligned}$$

Then, we have

$$\begin{aligned}
h^1(h^0(b,g,w)) &= h^1(c(b,g,w) - c(0,g,w) + b_*c(0,g,w)) \\
&= h^1c(b,g,w) - h^1c(0,g,w) + h^1b_*c(0,g,w).
\end{aligned}$$

Here, we calculate only the first term $h^1c(b,g,w)$. The direct calculation shows that

$$\begin{aligned}
&h^1c(b,g,w) \\
= \ &(1 - \tilde{t}^1\delta^4 - \delta^3\tilde{t}^1)c(b,g,w) \\
= \ &c(b,g,w) - \tilde{t}^1\delta^4(c)(b,g,w) - \delta^3\tilde{t}^1(c)(b,g,w) \\
= \ &c(b,g,w) - (-1)^2t^1(\delta^4(c)(b,g,w)) - \delta^3((-1)^2t^1(c)(b,g,w)) \\
= \ &c(b,g,w) - (b_*c(0,g,w) - c(b0,g,w) + c(b,0g,w) - c(b,0,gw) + w^*(b,0,g)) \\
&-(b_*t^1(c)(g,w) - t^1(c)(bg,w) + t^1(c)(b,gw) - w^*t^1(c)(b,g) \\
= \ &c(b,g,w) - (b_*c(0,g,w) - c(0,g,w) + c(b,0,w) - c(b,0,gw) + w^*(b,0,g)) \\
&-(b_*c(g,0,w) - c(bg,0,w) + c(b,0,gw) - w^*c(b,0,g) \\
= \ &c(b,g,w) - b_*c(0,g,w) + c(0,g,w) - c(b,0,w) - b_*c(g,0,w) + c(bg,0,w).
\end{aligned}$$

Then, we see that

$$\begin{aligned}
&h^1c(b,g,w) - h^1c(0,g,w) + h^1b_*c(0,g,w) \\
= \ &c(b,g,w) - b_*c(0,g,w) + c(0,g,w) - c(b,0,w) - b_*c(g,0,w) + c(bg,0,w) \\
&-(c(0,g,w) - 0_*c(0,g,w) + c(0,g,w) - c(0,0,w) - 0_*c(g,0,w) + c(0,0,w)) \\
&+b_*(c(0,g,w) - 0_*c(0,g,w) + c(0,g,w) - c(0,0,w) - 0_*c(g,0,w) + c(0,0,w)) \\
= \ &c(b,g,w) - c(b,0,w) - b_*c(g,0,w) + c(bg,0,w) - c(0,g,w) + b_*c(0,g,w).
\end{aligned}$$

As for the term $h^2(h^1(h^0c(b,g,w)))$, we have

$$\begin{aligned}
& h^2(h^1(h^0c(b,g,w))) \\
= \; & h^2(c(b,g,w) - c(b,0,w) - b_*c(g,0,w) + c(bg,0,w) - c(0,g,w) + b_*c(0,g,w)) \\
= \; & h^2c(b,g,w) - h^2c(b,0,w) - h^2b * c(g,0,w) \\
& +h^2c(bg,0,w) - h^2c(0,g,w) + h^2b_*c(0,g,w).
\end{aligned}$$

As for the first term $h^2c(b,g,w)$, it follows that

$$\begin{aligned}
& h^2c(b,g,w) \\
= \; & (1 - \tilde{t}^2\delta^4 - \delta^3\tilde{t}^2)c(b,g,w) \\
= \; & c(b,g,w) - (-1)^3\delta^4t^2(c)(b,g,w) - \delta^3(-1)^3t^2(c)(b,g,w) \\
= \; & c(b,g,w) + (b_*c(g,0,w) - c(bg,0,w) + c(b,0,w) - c(b,g,0) + h^*c(b,g,0)) \\
& +(b_*t^2(c)(g,w) - t^2(c)(bg,w) + t^2(c)(b,gw) - h^*t^2(c)(b,g)) \\
= \; & c(b,g,w) + (b_*c(g,0,w) - c(bg,0,w) + c(b,0,w) - c(b,g,0) + h^*c(b,g,0)) \\
& +(b_*c(g,w,0) - c(bg,w,0) + c(b,gw,0) - h^*c(b,g,0)) \\
= \; & c(b,g,w) + b_*c(g,0,w) - c(bg,0,w) + c(b,0,w) - c(b,g,0) \\
& +b_*c(g,w,0) - c(bg,w,0) + c(b,gw,0).
\end{aligned}$$

Thus, combining the terms with other calculation, we have the following equations:

$$\begin{aligned}
& h^2(h^1(h^0c(b,g,w))) \\
= \; & h^2c(b,g,w) - h^2c(b,0,w) - h^2b_*c(g,0,w) + h^2c(bg,0,w) \\
& -h^2c(0,g,w) + h^2b_*c(0,g,w) \\
= \; & c(b,g,w) + b_*c(g,0,w) - c(bg,0,w) + c(b,0,w) - c(b,g,0) + b_*c(g,w,0) \\
& -c(bg,w,0) + c(b,gw,0) - (c(b,0,w) + b_*c(0,0,w) - c(0,0,w) + c(b,0,w) \\
& -c(b,0,0) + b_*c(0,w,0) - c(0,w,0) + c(b,0,0)) - b_*(c(g,0,w) + g_*c(0,0,w) \\
& -c(0,0,w) + c(g,0,w) - c(g,0,0) + g_*c(0,w,0) - c(0,w,0) + c(g,0,0)) \\
& +(c(bg,0,w) + (bg)_*c(0,0,w) - c(0,0,w) + c(bg,0,w) - c(bg,0,0) \\
& +(bg)_*c(0,w,0) - c(0,w,0) + c(bg,0,0)) - (c(0,g,w) + 0_*c(g,0,w) \\
& -c(0,0,w) + c(0,0,w) - c(0,g,0) + 0_*c(g,w,0) - c(0,w,0) + c(0,gw,0)) \\
& +b_*(c(0,g,w) + 0_*c(g,0,w) - c(0,0,w) + c(0,0,w) \\
& -c(0,g,0) + 0_*c(g,w,0) - c(0,w,0) + c(0,gw,0)) \\
= \; & c(b,g,w) - c(b,g,0) - c(bg,w,0) + c(b,gw,0) - c(b,0,w) + c(bg,0,w) \\
& -c(0,g,w) + c(0,g,0) + c(0,w,0) - c(0,gw,0) + b_*(c(g,w,0) - c(g,0,w) \\
& +c(0,g,w) - c(0,g,0) - c(0,w,0) + c(0,gw,0)) \\
= \; & c(b,g,w) - c(b,g,0) - c(0,w,0) + c(b,0,0) - c(b,0,w) + c(0,0,w) \\
& -c(0,g,w) + c(0,g,0) + c(0,w,0) - c(0,0,0) + b_*(c(g,w,0) - c(g,0,w) \\
& +c(0,g,w) - c(0,g,0) - c(0,w,0) + c(0,0,0)).
\end{aligned}$$

Here, considering the first terms and second one, we have

$$\begin{aligned}
& c(b,g,w) \\
= \; & \mathcal{C}_{\mathcal{E}(C)}(\tau,H)(b,g,w) := \sigma^{-1}_{\tau(bgw)}(\Delta) = \sigma^{-1}_{\tau(0)}(\Delta) = \Delta \\
= \; & -H(b,gw) - (\tau b)_*H(g,w) + (\tau w)^*H(b,g) + H(bg,w) \\
= \; & -id_{0_*} - (\tau b)_*H(g,w) + (\tau w)^*H(b,g) + id_{0_*} \\
= \; & -(\tau b)_*H(g,w) + (\tau w)^*H(b,g) \\
= \; & -(\tau b) \circ H(g,w) + H(b,g) \circ (\tau w),
\end{aligned}$$

$$\begin{aligned}
& c(b,g,0) \\
= \;& -H(b,0) - (\tau b)_* H(g,0) + (\tau 0)^* H(b,g) + H(bg,0) \\
= \;& -id - (\tau b)_* H(g,0) + (0)^* H(b,g) + id \\
= \;& -(\tau b)_* id_\alpha + 0^* H(b,g) \\
= \;& -id_{(\tau b)\circ\alpha} + 0^* H(b,g) \\
= \;& H(b,g) \circ 0_*.
\end{aligned}$$

In consequence, we see that

$$\begin{aligned}
& h^2(h^1(h^0 c(b,g,w))) \\
= \;& c(b,g,w) - H(b,g)\circ 0_* + H(b,0)\circ 0_* \\
& -(-(\tau b)\circ H(0,w) + H(b,0)\circ(\tau w)) + (-0\circ H(0,w) + H(0,0)\circ(\tau w)) \\
& -(-0\circ H(g,w) + H(0,g)\circ(\tau w)) + H(0,g)\circ 0_* - H(0,0)\circ 0_* \\
& +b_*[H(g,w)\circ 0_* - \{(-\tau g)\circ H(0,w) + H(g,0)\circ(\tau w)\} - 0\circ H(g,w) \\
& +H(0,g)\circ(\tau w) - H(0,g)\circ 0_* - H(0,w)\circ 0_* + H(0,0)\circ 0_*].
\end{aligned}$$

In the equation above, observe that

$$\begin{aligned}
& H(b,0)\circ(\tau w) = id\circ\tau w = id_\alpha \circ id_{\tau w} = id_{\alpha(\tau w)} = id_0, \\
& \tau g\circ H(0,w) - H(g,0)\circ(\tau w) + H(0,g)\circ(\tau w) \\
& = (\tau g)\circ id - id\circ(\tau w) + id\circ(\tau w) = id_{\tau g}\circ id = id_{(\tau g)\circ\alpha} = id_0, \\
& -H(0,g)\circ 0_* + H(0,0)\circ 0_* = -id\circ 0_* + id\circ 0_*, \\
& b_*(id_0) = id_{b0} = id_0, \\
& -b_*\circ(H(0,w)\circ 0_*) = -b_*\circ(id\circ 0_*) = -b_*\circ id_0 = -(\tau b)\circ H(0,w).
\end{aligned}$$

Then, we have

$$\begin{aligned}
h^2(h^1(h^0 c(b,g,w))) &= c(b,g,w) - H(b,0)\circ(\tau w) + H(0,0)\circ(\tau w) \\
&\quad -H(0,g)\circ(\tau w) + b_*\circ id_0 \\
&= c(b,g,w) - id_0 + id\circ\tau w - id\circ\tau w + id_0 \\
&= c(b,g,w) \\
&= \mathcal{C}_{\mathcal{E}(C)}(\tau,H)(b,g,w) \\
&= -bK - Vw.
\end{aligned}$$

In the same argument as above enables us to conclude that $\mathcal{C}_{\mathcal{E}(C)}(\tau,H)(a,f,w) = -aH - Uw$. This completes the proof. □

Appendix C. On the equalities $i^*(s) = -f\vee g$ and $j^*(t) = \nabla(a,b)_*(s)$ in the proof of Theorem 2.14

We recall the morphisms s and t constructed in the proof of Theorem 2.14. We show that diagram (I) is commutative; that is, $i^*(s) = -f\vee g$ and $j^*(t) = \nabla(a,b)_*(s)$.

$$(I): \begin{tikzcd}
W \arrow[r, "w"] \arrow[d, equal] & X \arrow[r, "i"] \arrow[d, equal] & C_w \arrow[r, "j"] \arrow[d, "s"] & \Sigma W \arrow[d, "t"] \\
W \arrow[r, "w"'] & X \arrow[r, "-f\vee g"'] & A\oplus B \arrow[r, "\nabla(a,b)"'] & X
\end{tikzcd}$$

First equality $i^*(s) = -f \vee g$ is showed as follows. Let s be the morphism $\begin{pmatrix} -f & H \\ g & -K \end{pmatrix} : C_w = C \oplus \Sigma W \to A \oplus B$ in $\mathcal{T}(C_w, A \oplus B)$. Then we see that $i^*(s) = \begin{pmatrix} -f & H \\ g & -K \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -f \\ g \end{pmatrix}$, $-f \vee g := \begin{pmatrix} -f \\ g \end{pmatrix}$ and $w^*(-f \vee g) = (-f \vee g)w = 0$.

Next we will verify the equality $j^*(t) = \nabla(a,b)_*(s)$.

Let t be $-bK + Sw + aH$ and s be $\begin{pmatrix} -f & H \\ g & -K \end{pmatrix}$. We write $j^*(t)$, $\nabla(a,b)_*(s)$ and $j^*(t) - \nabla(a,b)_*(s)$ with homotopy as follows.

$$\begin{aligned} & j^*(t) \\ = \; & j^*(-bK + Sw + aH) \\ = \; & (-bK + Sw + aH) \circ j \\ = \; & (-bK + Sw + aH)(0,1) \\ = \; & (0, -bK + Sw + aH). \end{aligned}$$

$$\begin{aligned} & \nabla(a,b)_*(s) \\ = \; & \nabla(a,b) \begin{pmatrix} -f & H \\ g & -K \end{pmatrix} \\ = \; & (a,b) \begin{pmatrix} -f & H \\ g & -K \end{pmatrix} \\ = \; & (-af + bg, aH - bK). \end{aligned}$$

$$\begin{aligned} j^*(t) - \nabla(a,b)_*(s) &= (0 \quad -bK + Sw + aH) - (-af + bg \quad aH - bK) \\ &= (af - bg \quad Sw). \end{aligned}$$

Moreover, we see that

$$\begin{aligned} & (S\ 0) \begin{pmatrix} d_C & w \\ 0 & -d_W \end{pmatrix} + d_X(S \quad 0) \\ = \; & (Sd_C \quad Sw) + (d_X S \quad 0) \\ = \; & (Sd_C + d_X S \quad Sw). \end{aligned}$$

Since S is a homotopy between af and bg, it follows that $af - bg = Sd_C + d_X S$. This enables us to deduce that

$$\begin{aligned} & j^*(t) - \nabla(a,b)_*(s) \\ = \; & j^*(-bK + Sw + aH) - \nabla(a,b)_* \begin{pmatrix} -f & H \\ g & -K \end{pmatrix} \\ = \; & (S\ 0) \begin{pmatrix} d_C & w \\ 0 & -d_W \end{pmatrix} + d_X(S\ 0). \end{aligned}$$

We have $j^*(t) = \nabla(a,b)_*(s)$ since $[j^*(t)] = [\nabla(a,b)_*(s)]$.

Appendix D. On the non-triviality of $1_{\Sigma(X \oplus X)}$

By definition, we have $\mathrm{Hom}(X,Y)^k = \{f : X \to Y \mid f :$ R-homomorphism, deg $f = k$, i.e, $f : X^i \to Y^{i+k}\}$ and in the complex $\cdots \to \mathrm{Hom}(X,Y)^{-1} \xrightarrow{\delta} \mathrm{Hom}(X,Y)^0 \xrightarrow{\delta} Hom(X,Y)^1 \longrightarrow$, the differential δ is defined by $\delta(f) = d_Y f = -(-)^{degf} f d_X$. Since $H^0(\mathrm{Hom}(\Sigma(Y \oplus Y), \Sigma(X \oplus X))) = \mathrm{Hom}_{ch(\mathcal{B})}(\Sigma(Y \oplus Y), \Sigma(X \oplus X))/_\sim$ $= \mathcal{T}\ (\Sigma(Y \oplus Y), \Sigma(X \oplus X))$, consider a commutative diagram

$$\begin{tikzcd}
H^0(\mathrm{Hom}\Sigma(Y\oplus Y),\Sigma(X\oplus X))) \arrow[r, "h"] \arrow[d, "\Sigma(f,f)"] & \prod_{-p+q=0}\mathrm{Hom}(H^p(\Sigma(Y\oplus Y)),H^q(\Sigma(X\oplus X))) \arrow[d, "\Sigma(H(f),H(f)^*"] \\
H^0(\mathrm{Hom}(\Sigma(X\oplus X),\Sigma(X\oplus X))) \arrow[r, "h"] & \prod_{-p+q=0}\mathrm{Hom}(H^p(\Sigma(X\oplus X)),H^q(\Sigma(X\oplus X))) \\
H^0(\mathrm{Hom}(\Sigma(X\oplus X),C_f\oplus C_f)) \arrow[r, "h"] \arrow[u, "(j,j)_*"] & \prod_{-p+q=0}\mathrm{Hom}(H^p(\Sigma(X\oplus X),H^q(C_f\oplus C_f))). \arrow[u, "(H(j),H(j))_*"]
\end{tikzcd}$$

Here h denotes the map defined by taking the homology. For $1_{\Sigma(X\oplus X)}\in H^0(\mathrm{Hom}(\Sigma(X\oplus X),\Sigma(X\oplus X))$, $\beta\in H^0(\mathrm{Hom}(\Sigma(Y\oplus Y),\Sigma(X\oplus X)),\alpha\in H^0(\mathrm{Hom}(\Sigma(X\oplus X),C_f\oplus C_f))$, we see that

$$\begin{aligned}
h(1_{\Sigma(X\oplus X)}) &= h((j,j)\circ(\alpha)+\beta\circ\Sigma(f,f)) \\
&= h((j,j)\circ(\alpha))+h(\beta\circ\Sigma(f,f)) \\
&= h((j,j)_*(\alpha))+h(\Sigma(f,f)^*(\beta)) \\
&= (H(j),H(j))_*\circ h(\alpha)+\Sigma(H(f),H(f))^*\circ h(\beta).
\end{aligned}$$

By the condition (i) there exists an integer k such that $\mathrm{Hom}(H^{k-1}(\Sigma(Y\oplus Y)),H^{k-1}(\Sigma(X\oplus X))=0$ $(h(\beta)=0)$, $h(1_{\Sigma(X\oplus X)})=h((j,j)_*(\alpha))$. We have $h(1_{\Sigma(X\oplus X)})=(H(j),H(j))h(\alpha)$. Since $h(1_{\Sigma(X\oplus X)})\in\mathrm{Hom}(H^p(\Sigma(Y\oplus Y)),H^q(\Sigma(X\oplus X)))$ and $p=q$, $1_H:H^p\to H^p$ and $1_{H(\Sigma(X\oplus X))}\in\mathrm{Hom}(H^p(\Sigma(Y\oplus Y)),H^q(\Sigma(X\oplus X)))$. Then $H(1_{\Sigma(X\oplus X)})=1_{H(\Sigma(X\oplus X))}$ is surjective. On the other hand, $(H(j),H(j)):H^q(C_f\oplus C_f))\to H^q(\Sigma(X\oplus X))$ is not surjective by the condition (ii) and then $(H(j),H(j))h(\alpha)$ is not surjective, which is contradiction. Then we see that $1_{\Sigma(X\oplus X)}$ is a nontrivial element in the quotient Q.

The toy example in Example 5.3 is pictured as follow.

$$\begin{tikzcd}
X:\cdots \arrow[r, "0"] & 0 \arrow[r, "0"] \arrow[d, "f^{-2}"] & 0 \arrow[r, "0"] \arrow[d, "f^{-1}"] & \mathbb{Z} \arrow[r, "0"] \arrow[d, "f^0=id"] & 0 \arrow[r, "0"] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots:\textit{cochain complex} \\
Y:\cdots \arrow[r, "0"] & 0 \arrow[r, "0"] & \mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z} \arrow[r, "0"] & 0 \arrow[r, "0"] & 0 \arrow[r] & \cdots:\textit{cochain complex}
\end{tikzcd}$$

with

$$H^n(X)=\begin{cases}\mathbb{Z}, & n=0\\ 0, & n\neq 0\end{cases}\quad \textit{and}\quad H^n(Y)=\begin{cases}\mathbb{Z}/2\mathbb{Z}, & n=0\\ 0, & n\neq 0.\end{cases}$$

We check there exists an integer k such that (i)$H^k(X)\neq 0$ while $\mathrm{Hom}(H^k(Y),H^k(X))=0$ and (ii) $H(j):H^{k-1}(C_f)\to H^{k-1}(\Sigma X)$ is not surjective.
(i) $Hom(\mathbb{Z}/2,\mathbb{Z})=0$.
In fact, for a homomorphism $g:\mathbb{Z}/2\to\mathbb{Z}$; $g([0])=0$. $g([0])=g([1]+[1])=0$, $g([1])+g([1])=0$, $2g([1])=0$. Then $g([1])=0$. In consequence, we have g is *zero map.*
(ii) $H(j):H^{-1}(C_f)\to H^{-1}(\Sigma X)$ is not surjective.
In fact, for mapping cone $X\xrightarrow{f}Y\xrightarrow{i}C_f\xrightarrow{j}\Sigma X$ and $C_f^i=Y^i\oplus\Sigma X^i$, we have the cochain complex

$$C_f:\cdots\longrightarrow Y^{-2}\oplus\Sigma X^{-2}\xrightarrow{d^{-1}}Y^{-1}\oplus\Sigma X^{-1}\xrightarrow{d^0}Y^0\oplus\Sigma X^0\xrightarrow{d^1}Y^1\oplus\Sigma X^1\longrightarrow.$$

Here, $Y^{-2}\oplus\Sigma X^{-2}=0\oplus X^{-1}=0\oplus 0=0$, $Y^{-1}\oplus\Sigma X^{-1}=\mathbb{Z}\oplus X^0=\mathbb{Z}\oplus\mathbb{Z}$, $Y^0\oplus\Sigma X^0=\mathbb{Z}\oplus X^1=\mathbb{Z}\oplus 0=\mathbb{Z}$.
By the definition $d^n_{C(f)}=\begin{pmatrix}d^n_Y & f^n\\ 0 & -d^n_X\end{pmatrix}$, $d^0_Y(y)=2y$, $f^0=id$, we have $\begin{pmatrix}d^0_Y & f^0\\ 0 & -d^0_X\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$

$= \begin{pmatrix} d_Y^0 x + f^0 y \\ 0 - d_X^0 y \end{pmatrix} = \begin{pmatrix} 2x + y \\ 0 + 0 \end{pmatrix} = \begin{pmatrix} 2x + y \\ 0 \end{pmatrix}$; and hence $d_{C_f}^0(x, y) = (2x + y, 0)$. Then, we see that Ker $d_{C_f}^0$ $=\{(x, y) \in Z \oplus Z \mid 2x + y = 0\} = \{(x, -2x)\}$. By $(\Sigma X)^i = X^{i+1}$, $H^{-1}(\Sigma X) = H^0(X) \cong \mathbb{Z}$. Since j is the projection to the second factor, it follows that $H(j)$ is not surjective.

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