Parameterization of translation-invariant two-dimensional two-state quantum walks

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Abstract

This study investigates the unitary equivalence classes of translation-invariant two-dimensional two-state quantum walks. We show that unitary equivalence classes of such quantum walks are essentially parameterized by two real parameters.

1 Introduction

Quantum walks are analogous to classical random walks. They have been studied in various fields, such as quantum information theory and quantum probability theory. A quantum walk is defined by a pair $(U, \{\mathcal{H}_v\}_{v\in V})$, in which V is a countable set, $\{\mathcal{H}_v\}_{v\in V}$ is a family of separable Hilbert spaces, and U is a unitary operator on $\mathcal{H} = \bigoplus_{v\in V} \mathcal{H}_v$ [6]. In this paper, we discuss two-dimensional two-state quantum walks, in which $V = \mathbb{Z}^2$ and $\mathcal{H}_v = \mathbb{C}^2$. These have been the subject of some previous studies [1, 2, 7].

It is important to clarify when two quantum walks are considered equal. We consider unitary equivalence of quantum walks in the sense of [4, 6]. If two quantum walks are unitarily equivalent, then many properties of their quantum walks are the same. For example, digraphs, dimensions of Hilbert spaces, spectrums of unitary operators, probability distributions of quantum walks, etc. would be the same for each quantum walk. Therefore, we can think of unitarily equivalent quantum walks as being the same.

The aim of this paper is to determine the unitary equivalence classes of translation-invariant two-dimensional two-state quantum walks. This will enable us to better understand the entirety of such quantum walks.

In the previous papers [4, 5], we considered unitary equivalence classes of one-dimensional quantum walks, and parameterized several types of one-dimensional quantum walks. Unitary equivalence classes of translation-invariant one-dimensional quantum walks were also investigated in [3], and they are parameterized by one real parameter. In this study, we extend these results to the two-dimensional case.

In Sect. 2, we consider unitary equivalence classes of translation-invariant two-dimensional quantum walks. We show that such unitary equivalence classes are essentially parameterized by two real parameters.

2 Unitary equivalence classes of translation-invariant two-dimensional two-state quantum walks

Before we investigate the unitary equivalence classes of translation-invariant two-dimensional two-state quantum walks, we must define such quantum walks.

Definition 1 *Let $\mathcal{H}_{m,n} = \mathbb{C}^2$ for $(m,n) \in \mathbb{Z}^2$. A unitary operator U on $\mathcal{H} = \bigoplus_{(m,n)\in\mathbb{Z}^2} \mathcal{H}_{m,n}$ is called a two-dimensional two-state quantum walk if*

$$U\mathcal{H}_{m,n} \subset \mathcal{H}_{m+1,n} \oplus \mathcal{H}_{m-1,n} \oplus \mathcal{H}_{m,n+1} \oplus \mathcal{H}_{m,n-1} \tag{1}$$

for all $m, n \in \mathbb{Z}$. Moreover, U is said to be translation-invariant if, for all $k, \ell, m, n \in \mathbb{Z}$,

$$P_{m,n}UP_{k,\ell} = P_{m+1,n}UP_{k+1,\ell} = P_{m,n+1}UP_{k,\ell+1}$$

as operators on $\mathbb{C}^2$, where $P_{m,n}$ is the projection onto $\mathcal{H}_{m,n}$.

A pure quantum state is represented by a unit vector in a Hilbert space. For $\lambda \in \mathbb{R}$, quantum states ξ and $e^{\mathrm{i}\lambda}\xi$ in $\mathcal{H}$ are identified. Hence, the quantum walks U and $e^{\mathrm{i}\lambda}U$ are also identified.

We recall the definition of unitary equivalence of two-dimensional two-state quantum walks.

Definition 2 *Two-dimensional two-state quantum walks U_1 and U_2 are unitarily equivalent if there exists a unitary $W = \bigoplus_{(m,n)\in\mathbb{Z}^2} W_{m,n}$ on $\mathcal{H} = \bigoplus_{(m,n)\in\mathbb{Z}^2} \mathcal{H}_{m,n}$ such that*

$$WU_1W^* = U_2.$$

Let U be a translation-invariant two-dimensional two-state quantum walk. When we consider $P_{m+1,n}UP_{m,n}$ as an operator on $\mathbb{C}^2$, we write $U_{+1,0} = P_{m+1,n}UP_{m,n}$. Note that $U_{+1,0}$ does not depend on m and n. Operators $U_{-1,0}, U_{0,+1}$ and $U_{0,-1}$ are similarly defined.

Theorem 1 *Let U be a translation-invariant two-dimensional two-state quantum walk. Then,* $\mathrm{ran}U_{+1,0}$ *and* $\mathrm{ran}U_{-1,0}$ *are orthogonal. Similarly,* $\mathrm{ran}U_{0,+1}$ *and* $\mathrm{ran}U_{0,-1}$ *are also orthogonal.*

Proof. Since U is unitary, $U\mathcal{H}_{m,n}$ and $U\mathcal{H}_{m+2,n}$ are orthogonal. In considering (1), we obtain that $P_{m+1,n}U\mathcal{H}_{m,n}$ and $P_{m+1,n}U\mathcal{H}_{m+2,n}$ are orthogonal. This means $\mathrm{ran}U_{+1,0} \perp \mathrm{ran}U_{-1,0}$.

The proof of $\mathrm{ran}U_{0,+1} \perp \mathrm{ran}U_{0,-1}$ is similar. $\square$

First, we concentrate on the rank of $U_{+1,0}$. Since $\dim \mathcal{H}_{m+1,n} = 2$, we have $0 \leq \mathrm{rank}U_{+1,0} \leq 2$.

Case 1: $\mathrm{rank}U_{+1,0} = 2$.

By Lemma 1 and the assumption, $U_{-1,0} = 0$ holds. Then,

$$U\mathcal{H}_{m,n} \subset \mathcal{H}_{m+1,n} \oplus \mathcal{H}_{m,n+1} \oplus \mathcal{H}_{m,n-1}. \tag{2}$$

Since $U\mathcal{H}_{m,n}$ and $U\mathcal{H}_{m+1,n-1}$ are orthogonal, we obtain $P_{m+1,n}U\mathcal{H}_{m,n} \perp P_{m+1,n}U\mathcal{H}_{m+1,n-1}$ by (2). The assumption $\mathrm{rank}U_{+1,0} = 2$ implies $P_{m+1,n}U\mathcal{H}_{m,n} = \mathcal{H}_{m+1,n}$, and hence, $U_{0,+1} = 0$. Similarly, we can obtain $U_{0,-1} = 0$.

Consequently, when $\mathrm{rank}U_{+1,0} = 2$, $U\mathcal{H}_{m,n} = \mathcal{H}_{m+1,n}$. This means that U is represented as a direct sum of unitary operators on $\mathbb{C}^2$.

Case 2: $\mathrm{rank}U_{+1,0} = 0$.

In this case,

$$U\mathcal{H}_{m,n} \subset \mathcal{H}_{m-1,n} \oplus \mathcal{H}_{m,n+1} \oplus \mathcal{H}_{m,n-1}.$$

Since $U\mathcal{H}_{m,n}$ and $U\mathcal{H}_{m+1,n+1}$ are orthogonal, we obtain $P_{m,n+1}U\mathcal{H}_{m,n} \perp P_{m,n+1}U\mathcal{H}_{m+1,n+1}$, and hence, $\mathrm{ran}U_{0,+1}$ and $\mathrm{ran}U_{-1,0}$ are orthogonal. Similarly, we can obtain $\mathrm{ran}U_{0,-1} \perp \mathrm{ran}U_{-1,0}$. Then, by Lemma 1, $\mathrm{ran}U_{-1,0}$, $\mathrm{ran}U_{0,+1}$ and $\mathrm{ran}U_{0,-1}$ are mutually orthogonal. This implies that one of the above three ranges is the zero vector space.

When $\mathrm{ran}U_{-1,0} = \{0\}$,

$$U\mathcal{H}_{m,n} \subset \mathcal{H}_{m,n+1} \oplus \mathcal{H}_{m,n-1}.$$

This means that U can be represented as a direct sum of one-dimensional quantum walks. When $\mathrm{ran}U_{0,\pm1} = \{0\}$,

$$U\mathcal{H}_{m,n} \subset \mathcal{H}_{m-1,n} \oplus \mathcal{H}_{m,n\mp1}.$$

Then, U on $\mathcal{H} = \bigoplus_{k\in\mathbb{Z}} \left(\bigoplus_{n\in\mathbb{Z}} \mathcal{H}_{n,k\mp n}\right)$ can also be considered as a direct sum of one-dimensional quantum walks.

Consequently, when $\mathrm{rank}U_{+1,0} = 0, 2$, U is a direct sum of unitary operators on $\mathbb{C}^2$ or of one-dimensional quantum walks. Similarly, when $\mathrm{rank}U_{-1,0} = 0, 2$ or $\mathrm{rank}U_{0,\pm1} = 0, 2$, we have the same result.

Hence, in the following, we assume that $\mathrm{rank}U_{\pm1,0} = \mathrm{rank}U_{0,\pm1} = 1$. Let ξ_1, ξ_2, ζ_1 and ζ_2 be unit vectors in $\mathbb{C}^2$ with $\mathrm{ran}U_{+1,0} = \mathbb{C}\xi_1$, $\mathrm{ran}U_{-1,0} = \mathbb{C}\xi_2$, $\mathrm{ran}U_{0,+1} = \mathbb{C}\zeta_1$ and $\mathrm{ran}U_{0,-1} = \mathbb{C}\zeta_2$. By Lemma 1, $\{\xi_1, \xi_2\}$ and $\{\zeta_1, \zeta_2\}$ are orthonormal bases in $\mathbb{C}^2$. When we consider the vectors as in $\mathcal{H}_{m,n}$, we write $\xi_1^{m,n}$, $\xi_2^{m,n}$ and so on.

Next, we clarify the structure of U. To do this, we prepare three lemmas.

Theorem 2 *Let η be a vector in $\mathbb{C}^2$, and let*

$$U\eta^{m,n} = a\xi_1^{m+1,n} + b\zeta_1^{m,n+1} + c\zeta_2^{m,n-1} + d\xi_2^{m-1,n}$$

for some $a, b, c, d \in \mathbb{C}$. If one of a, b, c and d is zero, then two of them are zero.

Proof. We need only show the proof for the case when $a = 0$; the other cases are proven similarly.

Since $U\eta^{m,n}$, $U\eta^{m-1,n-1}$ and $U\eta^{m-1,n+1}$ are mutually orthogonal, $d\xi_2^{m-1,n}$, $b\zeta_1^{m-1,n}$ and $c\zeta_2^{m-1,n}$ are mutually orthogonal. Since $\dim \mathcal{H}_{m-1,n} = 2$, one of b, c and d is zero. $\square$

Theorem 3 *There exists an orthonormal basis $\{\eta_1^{m,n}, \eta_2^{m,n}\}$ in $\mathcal{H}_{m,n} = \mathbb{C}^2$ such that*

$$U\eta_1^{m,n} = \alpha\xi_1^{m+1,n} + \beta\zeta_1^{m,n+1}, \quad U\eta_2^{m,n} = \gamma\zeta_2^{m,n-1} + \delta\xi_2^{m-1,n} \tag{3}$$

or

$$U\eta_1^{m,n} = \alpha\xi_1^{m+1,n} + \beta\zeta_2^{m,n-1}, \quad U\eta_2^{m,n} = \gamma\zeta_1^{m,n+1} + \delta\xi_2^{m-1,n} \tag{4}$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{C}\backslash\{0\}$.

Proof. Since $\mathrm{rank}U_{-1,0} = 1$, $\dim\ker U_{-1,0} = 1$. Let η_1 be a unit vector in $\ker U_{-1,0}$, and let η_2 be a unit vector in $\mathbb{C}^2$ which is orthogonal to η_1. Then,

$$U\eta_1^{m,n} = a\xi_1^{m+1,n} + b\zeta_1^{m,n+1} + c\zeta_2^{m,n-1} \tag{5}$$

for some $a, b, c \in \mathbb{C}$. By Lemma 2, one of a, b and c is zero.

Assume that $a = 0$. Let

$$U\eta_2^{m,n} = p\xi_1^{m+1,n} + q\zeta_1^{m,n+1} + r\zeta_2^{m,n-1} + s\xi_2^{m-1,n}.$$

By the assumption $\mathrm{rank}U_{\pm 1,0} = 1$, p and s are not zero. Moreover, since $U\eta_1^{m,n}$, $U\eta_2^{m-1,n+1}$ and $U\eta_2^{m+1,n+1}$ are mutually orthogonal, $b\zeta_1^{m,n+1}$, $p\xi_1^{m,n+1}$ and $s\xi_2^{m,n+1}$ are mutually orthogonal, and hence $b = 0$. Similarly, we obtain $c = 0$, and therefore $\eta_1 = 0$. This is a contradiction.

Hence, assume that $a \neq 0$ and $c = 0$ in (5). If, in addition, $b = 0$, then $U\eta_1^{m,n} = a\xi_1^{m+1,n}$ and

$$U\eta_2^{m,n} = q\zeta_1^{m,n+1} + r\zeta_2^{m,n-1} + s\xi_2^{m-1,n}.$$

Again, by Lemma 2, one of q, r and s is zero, and this contradicts the assumption that $\mathrm{rank}U_{\pm 1,0} = \mathrm{rank}U_{0,\pm 1} = 1$. Therefore, $b \neq 0$.

Let

$$U\eta_1^{m,n} = a\xi_1^{m+1,n} + b\zeta_1^{m,n+1}, \quad U\eta_2^{m,n} = p\xi_1^{m+1,n} + q\zeta_1^{m,n+1} + r\zeta_2^{m,n-1} + s\xi_2^{m-1,n}$$

for some $a, b, p, q, r, s \in \mathbb{C}$ with $a, b, r, s \neq 0$. Since $U\eta_1^{m,n} \perp U\eta_2^{m,n}$, $a\bar{p} + b\bar{q} = 0$. On the other hand,

$$U(p\eta_1^{m,n} - a\eta_2^{m,n}) = (bp - aq)\zeta_1^{m,n+1} - ar\zeta_2^{m,n-1} - as\xi_2^{m-1,n}.$$

By Lemma 2 and the assumption $a, r, s \neq 0$, we have $bp - aq = 0$. These equations imply $b(|p|^2 + |q|^2) = 0$, and therefore, $p = q = 0$. Hence, we conclude

$$U\eta_1^{m,n} = \alpha\xi_1^{m+1,n} + \beta\zeta_1^{m,n+1}, \quad U\eta_2^{m,n} = \gamma\zeta_2^{m,n-1} + \delta\xi_2^{m-1,n}$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{C}\backslash\{0\}$.

Similarly, if we assume $a \neq 0$ and $b = 0$ in (5), we obtain

$$U\eta_1^{m,n} = \alpha\xi_1^{m+1,n} + \beta\zeta_2^{m,n-1}, \quad U\eta_2^{m,n} = \gamma\zeta_1^{m,n+1} + \delta\xi_2^{m-1,n}$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{C}\backslash\{0\}$. $\square$

Theorem 4 *If U satisfies* (3)*, there exist $\theta_1, \theta_2 \in \mathbb{R}$ such that*

$$\zeta_1 = e^{\mathrm{i}\theta_1}\xi_2, \quad \zeta_2 = e^{\mathrm{i}\theta_2}\xi_1,$$

where $\mathrm{i} = \sqrt{-1}$*. If U satisfies* (4)*, there exist $\theta_1, \theta_2 \in \mathbb{R}$ such that*

$$\zeta_1 = e^{\mathrm{i}\theta_1}\xi_1, \quad \zeta_2 = e^{\mathrm{i}\theta_2}\xi_2.$$

Proof. We need only show the proof for the case when U satisfies (3); the other case is proven similarly

The condition $U\eta_1^{m,n} \perp U\eta_1^{m+1,n-1}$ implies $\xi_1^{m+1,n} \perp \zeta_1^{m+1,n}$. Since $\{\xi_1, \xi_2\}$ and $\{\zeta_1, \zeta_2\}$ are orthonormal bases in $\mathbb{C}^2$, $\zeta_1 = e^{\mathrm{i}\theta_1}\xi_2$ and $\zeta_2 = e^{\mathrm{i}\theta_2}\xi_1$ for some $\theta_1, \theta_2 \in \mathbb{R}$. $\square$

As a consequence of these lemmas, we have the following theorem.

Theorem 1 *For a translation-invariant two-dimensional two-state quantum walk U with* $\mathrm{rank}U_{\pm1,0} = \mathrm{rank}U_{0,\pm1} = 1$*, there exist orthonormal bases $\{\xi_1, \xi_2\}$ and $\{\eta_1, \eta_2\}$ in $\mathbb{C}^2$, $r \in (0,1)$ and $a, b, c, d \in \mathbb{R}$ such that*

$$\begin{aligned} U &= \sum_{(m,n)\in\mathbb{Z}^2} |e^{\mathrm{i}a} r\xi_1^{m+1,n} + e^{\mathrm{i}b}\sqrt{1-r^2}\xi_2^{m,n\pm1}\rangle\langle\eta_1^{m,n}| \\ &\quad + |e^{\mathrm{i}c}\sqrt{1-r^2}\xi_1^{m,n\mp1} + e^{\mathrm{i}d} r\xi_2^{m-1,n}\rangle\langle\eta_2^{m,n}| \end{aligned}$$

and $a - c = b - d + \pi \pmod{2\pi}$.

Proof. By the lemmas above, we can assume that

$$U = \sum_{(m,n)\in\mathbb{Z}^2} |\alpha\xi_1^{m+1,n} + \beta\xi_2^{m,n\pm1}\rangle\langle\eta_1^{m,n}| + |\gamma\xi_1^{m,n\mp1} + \delta\xi_2^{m-1,n}\rangle\langle\eta_2^{m,n}|$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{C}\backslash\{0\}$.

Since $U\eta_1^{m,n}$ and $U\eta_2^{m+1,n\pm1}$ are orthogonal, $\alpha\xi_1^{m+1,n} + \beta\xi_2^{m,n\pm1}$ and $\gamma\xi_1^{m+1,n} + \delta\xi_2^{m,n\pm1}$ are orthogonal. Hence, there exist $r \in (0,1)$ and $a, b, c, d \in \mathbb{R}$ such that

$$\alpha = e^{\mathrm{i}a} r, \quad \beta = e^{\mathrm{i}b}\sqrt{1-r^2}, \quad \gamma = e^{\mathrm{i}c}\sqrt{1-r^2}, \quad \delta = e^{\mathrm{i}d} r$$

and $a - c = b - d + \pi \pmod{2\pi}$. $\square$

In the following, we assume that there exist orthonormal bases $\{\xi_1, \xi_2\}$ and $\{\eta_1, \eta_2\}$ in $\mathbb{C}^2$, $r \in (0,1)$ and $a, b, c, d \in \mathbb{R}$ such that

$$\begin{aligned} U &= \sum_{(m,n)\in\mathbb{Z}^2} |e^{\mathrm{i}a} r \xi_1^{m+1,n} + e^{\mathrm{i}b}\sqrt{1-r^2}\xi_2^{m,n+1}\rangle\langle \eta_1^{m,n}| \\ &\quad + |e^{\mathrm{i}c}\sqrt{1-r^2}\xi_1^{m,n-1} + e^{\mathrm{i}d} r \xi_2^{m-1,n}\rangle\langle \eta_2^{m,n}| \end{aligned}$$

and $a - c = b - d + \pi \pmod{2\pi}$. We can analyze the other case in the same way.

Now, we consider unitary equivalence of translation-invariant two-dimensional two-state quantum walks.

Step 1. Define a unitary W_1 on $\mathcal{H}$ by

$$W_1 \xi_1^{m,n} = e_1^{m,n}, \quad W_1 \xi_2^{m,n} = e_2^{m,n},$$

where $\{e_1^{m,n}, e_2^{m,n}\}$ is the canonical basis in $\mathcal{H}_{m,n}$. W_1 can also be represented as

$$W_1 = \bigoplus_{(m,n)\in\mathbb{Z}^2} |e_1^{m,n}\rangle\langle \xi_1^{m,n}| + |e_2^{m,n}\rangle\langle \xi_2^{m,n}|.$$

Since U is written as

$$\begin{aligned} U &= \sum_{(m,n)\in\mathbb{Z}^2} |e^{\mathrm{i}a} r \xi_1^{m+1,n} + e^{\mathrm{i}b} s \xi_2^{m,n+1}\rangle\langle \eta_1^{m,n}| \\ &\quad + |e^{\mathrm{i}c} s \xi_1^{m,n-1} + e^{\mathrm{i}d} r \xi_2^{m-1,n}\rangle\langle \eta_2^{m,n}| \end{aligned}$$

for some $r \in (0,1)$ and $a, b, c, d \in \mathbb{R}$ with $s = \sqrt{1-r^2}$, $W_1 U W_1^*$ is calculated as

$$\begin{aligned} & W_1 U W_1^* \\ = & \sum_{(m,n)\in\mathbb{Z}^2} |e^{\mathrm{i}a} r W_1 \xi_1^{m+1,n} + e^{\mathrm{i}b} s W_1 \xi_2^{m,n+1}\rangle\langle W_1 \eta_1^{m,n}| \\ & + |e^{\mathrm{i}c} s W_1 \xi_1^{m,n-1} + e^{\mathrm{i}d} r W_1 \xi_2^{m-1,n}\rangle\langle W_1 \eta_2^{m,n}| \\ = & \sum_{(m,n)\in\mathbb{Z}^2} |e^{\mathrm{i}a} r e_1^{m+1,n} + e^{\mathrm{i}b} s e_2^{m,n+1}\rangle\langle W_1 \eta_1^{m,n}| \\ & + |e^{\mathrm{i}c} s e_1^{m,n-1} + e^{\mathrm{i}d} r e_2^{m-1,n}\rangle\langle W_1 \eta_2^{m,n}|. \end{aligned}$$

Here, $\{W_1\eta_1, W_1\eta_2\}$ is an orthonormal basis in $\mathbb{C}^2$. Therefore, there exist $p \in [0,1]$ and $x, y, z, w \in \mathbb{R}$ such that

$$W_1\eta_1 = e^{\mathrm{i}x} p e_1 + e^{\mathrm{i}y} q e_2, \quad W_1\eta_2 = e^{\mathrm{i}z} q e_1 + e^{\mathrm{i}w} p e_2,$$

where $q=\sqrt{1-p^2}$ and $x-z=y-w+\pi \pmod{2\pi}$. Consequently, we obtain

$$\begin{aligned}W_1UW_1^* &= \sum_{(m,n)\in\mathbb{Z}^2} |e^{\mathrm{i}a}re_1^{m+1,n}+e^{\mathrm{i}b}se_2^{m,n+1}\rangle\langle e^{\mathrm{i}x}pe_1^{m,n}+e^{\mathrm{i}y}qe_2^{m,n}| \\ &\quad +|e^{\mathrm{i}c}se_1^{m,n-1}+e^{\mathrm{i}d}re_2^{m-1,n}\rangle\langle e^{\mathrm{i}z}qe_1^{m,n}+e^{\mathrm{i}w}pe_2^{m,n}|.\end{aligned}$$

Step 2. Define a unitary W_2 on $\mathcal{H}$ by

$$W_2=\bigoplus_{(m,n)\in\mathbb{Z}^2} e^{-\mathrm{i}x}|e_1^{m,n}\rangle\langle e_1^{m,n}|+e^{-\mathrm{i}y}|e_2^{m,n}\rangle\langle e_2^{m,n}|.$$

Then, since $x-z=y-w+\pi \pmod{2\pi}$,

$$\begin{aligned}& W_2W_1UW_1^*W_2^* \\ =& \sum_{(m,n)\in\mathbb{Z}^2} |e^{\mathrm{i}a}rW_2e_1^{m+1,n}+e^{\mathrm{i}b}sW_2e_2^{m,n+1}\rangle\langle e^{\mathrm{i}x}pW_2e_1^{m,n}+e^{\mathrm{i}y}qW_2e_2^{m,n}| \\ & +|e^{\mathrm{i}c}sW_2e_1^{m,n-1}+e^{\mathrm{i}d}rW_2e_2^{m-1,n}\rangle\langle e^{\mathrm{i}z}qW_2e_1^{m,n}+e^{\mathrm{i}w}pW_2e_2^{m,n}| \\ =& \sum_{(m,n)\in\mathbb{Z}^2} |e^{\mathrm{i}(a-x)}re_1^{m+1,n}+e^{\mathrm{i}(b-y)}se_2^{m,n+1}\rangle\langle pe_1^{m,n}+qe_2^{m,n}| \\ & +|e^{\mathrm{i}(c-x)}se_1^{m,n-1}+e^{\mathrm{i}(d-y)}re_2^{m-1,n}\rangle\langle e^{\mathrm{i}(z-x)}qe_1^{m,n}+e^{\mathrm{i}(w-y)}pe_2^{m,n}| \\ =& \sum_{(m,n)\in\mathbb{Z}^2} |e^{\mathrm{i}(a-x)}re_1^{m+1,n}+e^{\mathrm{i}(b-y)}se_2^{m,n+1}\rangle\langle pe_1^{m,n}+qe_2^{m,n}| \\ & +|e^{\mathrm{i}(c-x-w+y)}se_1^{m,n-1}+e^{\mathrm{i}(d-w)}re_2^{m-1,n}\rangle\langle e^{\mathrm{i}(z-x-w+y)}qe_1^{m,n}+pe_2^{m,n}| \\ =& \sum_{(m,n)\in\mathbb{Z}^2} |e^{\mathrm{i}(a-x)}re_1^{m+1,n}+e^{\mathrm{i}(b-y)}se_2^{m,n+1}\rangle\langle pe_1^{m,n}+qe_2^{m,n}| \\ & +|-e^{\mathrm{i}(c-z)}se_1^{m,n-1}+e^{\mathrm{i}(d-w)}re_2^{m-1,n}\rangle\langle -qe_1^{m,n}+pe_2^{m,n}|.\end{aligned}$$

Step 3. Let $\ell=(b+c-y-z)/2$. Define a unitary W_3 on $\mathcal{H}$ by

$$W_3=\bigoplus_{(m,n)\in\mathbb{Z}^2} e^{\mathrm{i}(m(-a+x+\ell)+n(-b+y+\ell))}I_{m,n},$$

where $I_{m,n}$ is the identity operator on $\mathcal{H}_{m,n}$. Then, for any $\xi,\zeta\in\mathbb{C}^2$ and $h,k,m,n\in\mathbb{Z}$,

$$|W_3\xi^{m,n}\rangle\langle W_3\zeta^{h,k}|=|e^{\mathrm{i}((m-h)(-a+x+\ell)+(n-k)(-b+y+\ell))}\xi^{m,n}\rangle\langle\zeta^{h,k}|.$$

Therefore,

$$\begin{aligned}
& e^{-\mathrm{i}\ell} W_3 W_2 W_1 U W_1^* W_2^* W_3^* \\
= & e^{-\mathrm{i}\ell} \sum_{(m,n)\in\mathbb{Z}^2} |e^{\mathrm{i}(a-x)} r W_3 e_1^{m+1,n} + e^{\mathrm{i}(b-y)} s W_3 e_2^{m,n+1}\rangle\langle W_3(p e_1^{m,n} + q e_2^{m,n})| \\
& + |-e^{\mathrm{i}(c-z)} s W_3 e_1^{m,n-1} + e^{\mathrm{i}(d-w)} r W_3 e_2^{m-1,n}\rangle\langle W_3(-q e_1^{m,n} + p e_2^{m,n})| \\
= & \sum_{(m,n)\in\mathbb{Z}^2} |r e_1^{m+1,n} + s e_2^{m,n+1}\rangle\langle p e_1^{m,n} + q e_2^{m,n}| \\
& + |-e^{\mathrm{i}(b+c-y-z-2\ell)} s e_1^{m,n-1} + e^{\mathrm{i}(a+d-x-w-2\ell)} r e_2^{m-1,n}\rangle\langle -q e_1^{m,n} + p e_2^{m,n}| \\
= & \sum_{(m,n)\in\mathbb{Z}^2} |r e_1^{m+1,n} + s e_2^{m,n+1}\rangle\langle p e_1^{m,n} + q e_2^{m,n}| \\
& + |-s e_1^{m,n-1} + r e_2^{m-1,n}\rangle\langle -q e_1^{m,n} + p e_2^{m,n}|.
\end{aligned}$$

Now, we are ready to prove the next theorem.

Theorem 2 *A translation-invariant two-dimensional two-state quantum walk U with $\mathrm{rank} U_{\pm 1,0} = \mathrm{rank} U_{0,\pm 1} = 1$ is unitarily equivalent to*

$$\begin{aligned}
U_{r,p,\pm} = & \sum_{(m,n)\in\mathbb{Z}^2} |r e_1^{m+1,n} + s e_2^{m,n\pm 1}\rangle\langle p e_1^{m,n} + q e_2^{m,n}| \\
& + |-s e_1^{m,n\mp 1} + r e_2^{m-1,n}\rangle\langle -q e_1^{m,n} + p e_2^{m,n}|
\end{aligned}$$

for some $0 < r < 1$ and $0 \le p \le 1$, where $s = \sqrt{1-r^2}$ and $q = \sqrt{1-p^2}$. Moreover, $U_{r,p,\varepsilon}$ and $U_{r',p',\varepsilon'}$ are unitarily equivalent if and only if $r = r'$, $p = p'$ and $\varepsilon = \varepsilon'$.

Proof. We have already proven the first part of this theorem. Hence, we need only prove that $U_{r,p,\varepsilon}$ and $U_{r',p',\varepsilon'}$ are unitarily equivalent if and only if $r = r'$, $p = p'$ and $\varepsilon = \varepsilon'$

Assume that $U_{r,p,\varepsilon}$ and $U_{r',p',\varepsilon'}$ are unitarily equivalent. Then, there exist $\lambda \in \mathbb{R}$ and a unitary operator $W = \bigoplus_{(m,n)\in\mathbb{Z}^2} W_{m,n}$ on $\mathcal{H} = \bigoplus_{(m,n)\in\mathbb{Z}^2} \mathcal{H}_{m,n}$ such that

$$e^{\mathrm{i}\lambda} W U_{r,p,\varepsilon} W^* = U_{r',p',\varepsilon'}.$$

The equation

$$U_{r',p',\varepsilon'}(p e_1^{m,n} + q e_2^{m,n}) = e^{\mathrm{i}\lambda} W U_{r,p,\varepsilon} W^* (p e_1^{m,n} + q e_2^{m,n})$$

implies $\varepsilon = \varepsilon'$.

For all $(m,n) \in \mathbb{Z}^2$, $P_{m\pm1,n} e^{i\lambda} W U_{r,p,\varepsilon} W^* P_{m,n} = P_{m\pm1,n} U_{r',p',\varepsilon'} P_{m,n}$. Therefore, $We_1^{m,n}$ and $We_2^{m,n}$ are described as $We_1^{m,n} = e^{iu_{m,n}} e_1^{m,n}$ and $We_2^{m,n} = e^{iv_{m,n}} e_2^{m,n}$ for some $u_{m,n}, v_{m,n} \in \mathbb{R}$. Since W commutes with $P_{m,n}$ for all $(m,n) \in \mathbb{Z}^2$,

$$\begin{aligned} r &= \|P_{m+1,n} U_{r,p,\varepsilon} P_{m,n}\| = \|e^{i\lambda} W P_{m+1,n} U_{r,p,\varepsilon} P_{m,n} W^*\| \\ &= \|P_{m+1,n} U_{r',p',\varepsilon'} P_{m,n}\| = r'. \end{aligned}$$

Furthermore,

$$\begin{aligned} rp &= \|P_{m+1,n} U_{r,p,\varepsilon} e_1^{m,n}\| = \|e^{-i\lambda} P_{m+1,n} W^* U_{r',p',\varepsilon'} W e_1^{m,n}\| \\ &= \|P_{m+1,n} U_{r',p',\varepsilon'} e_1^{m,n}\| = r'p'. \end{aligned}$$

This implies $p = p'$. □

This theorem says that unitary equivalence classes of translation-invariant two-dimensional two-state quantum walks are essentially parameterized by two real parameters.

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