Submanifolds of statisticalmanifolds admitting almost complex structures

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1 Introduction.

Statistical models in information geometry have a Fisher metric as the Riemannian metric, and admit a torsion-free affine connection which is constructed from the expectation of the probability distribution ([1]). We have studied the exponential family admitting almost complex structures which is parallel relative to the e and m–connections. Especially, the multinomial distribution or negative multinomial distribution which are discrete distributions, the multivariate normal distribution or Dirichlet distribution which are continuous distributions, these distributions are important examples of the exponential family, and we proved that these spaces admit almost complex structures which is parallel relative to the e and m–connections. In this time, we report submanifolds of statistical manifolds admitting almost complex structures.

2 Statistical manifolds with almost complex structures.

Let (M, g) and ∇ be a semi-Riemannian manifold and affine connection, respectively. We define another affine connection ∇^* by

$$Eg(F, G) = g(\nabla_E F, G) + g(F, \nabla^*_E G) \tag{2.1}$$

for vector fields E, F and G on M. The affine connection ∇^* is called conjugate (or dual) of ∇ with respect to g. The triple (M, g, ∇) is called a statistical manifold if both ∇ and ∇^* are torsion-free. Clearly $(\nabla^*)^* = \nabla$ holds. It is easy to see that $\frac{1}{2}(\nabla + \nabla^*)$ is a metric connection. We denote by R and R^* the curvature tensors with respect to the affine connection ∇ and its conjugate ∇^*, respectively. Then we find $g(R(E, F)G, H) = -g(G, R^*(E, F)H)$ for vector fields E, F, G, H on M, where $R(E, F)G = [\nabla_E, \nabla_F]G - \nabla_{[E,F]}G$. Thus R vanishes identically if and only if so is R^*.

An almost complex structure on a manifold M is a tensor field J of type (1,1) such that $J^2 = -I$, where I stands for the identity transformation. An almost complex manifold is such a manifold with a fixed almost complex structure. An almost complex manifold is necessarily orientable and must have an even dimension. If J preserves the metric g, that is,

$$g(JE, JF) = g(E, F) \tag{2.2}$$

for vector fields E and F on M, then (M, g, J) is an almost Hermitian manifold. Now, we consider the semi-Riemannian manifold (M, g) with the almost complex structure J which has another tensor field J^* of type (1,1) satisfying

$$g(JE, F) + g(E, J^*F) = 0. \tag{2.3}$$

Then (M, g, J) is called an almost Hermite-like manifold. We see that $(J^*)^* = J$, $(J^*)^2 = -I$ and

$$g(JE, J^*F) = g(E, F). \tag{2.4}$$

If J is parallel with respect to the affine connection ∇, then (M, g, ∇, J) is called a Kähler-like statistical manifold. By virtue of (2.3), we get

$$g((\nabla_G J)E, F) + g(E, (\nabla^*_G J^*)F) = 0 \tag{2.5}$$

for vector fields E, F and G on M.

LEMMA A ([2]). *We have*

(1) (M, g, J) *is an almost Hermite-like manifold if and only if so is* (M, g, J^*),

(2) (M, g, ∇, J) *is a Kähler-like statistical manifold if and only if so is* (M, g, ∇^*, J^*).

3 Submanifolds of statistical manifolds

Let $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$ be an m-dimensional statistical manifold and M be a connected n-dimensional submanifold of $\widetilde{M}$ with the induced metric g. The letters X, Y, Z, W will always denote tangential vector fields and U, V normal vector fields on M. We denote the Gauss formulae relative to the affine connection $\widetilde{\nabla}$ and its conjugate $\widetilde{\nabla}^*$ by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y) \qquad \text{and} \qquad \widetilde{\nabla}^*_X Y = \nabla^*_X Y + \sigma^*(X, Y), \tag{3.1}$$

respectively. It is easy to see that ∇ and ∇^* are affine connections and σ, σ^* are bilinear and symmetric. Since $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$ are torsion-free, we find ∇ and ∇^* are torsion-free. Because of $X\widetilde{g}(Y, Z) = \widetilde{g}(\widetilde{\nabla}_X Y, Z) + \widetilde{g}(Y, \widetilde{\nabla}^*_X Z)$, affine connections ∇ and ∇^* are conjugate each other. Hence we have

THEOREM 3.1. *(M, g, ∇) is a statistical submanifold of $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$.*

Next, we denote the Weingarten formulae relative to the affine connections $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$ by

$$\widetilde{\nabla}_X V = -A_V X + D_X V \qquad \text{and} \qquad \widetilde{\nabla}^*_X V = -A^*_V X + D^*_X V, \tag{3.2}$$

respectively. Owing to $\widetilde{g}(Y, V) = 0$, we obtain

LEMMA 3.2. *We have $\widetilde{g}(\sigma(X, Y), V) = g(Y, A^*_V X)$ and $\widetilde{g}(\sigma^*(X, Y), V) = g(Y, A_V X)$.*

LEMMA 3.3. *We get $g(A^*_V X, Y) = g(X, A^*_V Y)$ and $g(A_V X, Y) = g(X, A_V Y)$.*

Let $\overline{g}$ be an induced metric on the normal bundle $T^\perp M$. We get $\overline{g}(D_X V, U) + \overline{g}(V, D^*_X U) = X\overline{g}(V, U)$. Hence we have

LEMMA 3.4. *D and D^* are affine connections in $T^\perp M$ of M in $\widetilde{M}$ with respect to $\overline{g}$ on $T^\perp M$. Moreover D and D^* are conjugate each other.*

From $(\widetilde{\nabla}^*)^* = \widetilde{\nabla}$, we have

LEMMA 3.5. *$(\nabla^*)^* = \nabla$, $(\sigma^*)^* = \sigma$, $(A^*)^* = A$ and $(D^*)^* = D$ hold.*

For the second fundamental forms σ and σ^*, we define the covariant differentiations $\overline{\nabla}$ and $\overline{\nabla}^*$ by

$$(\overline{\nabla}_X \sigma)(Y, Z) = D_X(\sigma(Y, Z)) - \sigma(\nabla^*_X Y, Z) - \sigma(Y, \nabla_X Z), \tag{3.3}$$

$$(\overline{\nabla}^*_X \sigma^*)(Y, Z) = D^*_X(\sigma^*(Y, Z)) - \sigma^*(\nabla_X Y, Z) - \sigma^*(Y, \nabla^*_X Z). \tag{3.4}$$

Then we have

LEMMA 3.6.

$$\widetilde{g}(\widetilde{R}(X, Y)Z, W) = g(R(X, Y)Z, W) - \widetilde{g}(\sigma(Y, Z), \sigma^*(X, W)) + \widetilde{g}(\sigma(X, Z), \sigma^*(Y, W)) \tag{3.5}$$

(Equation of Gauss relative to $\widetilde{\nabla}$)

$$(\widetilde{R}(X, Y)Z)^\perp = (\overline{\nabla}_X \sigma)(Y, Z) - (\overline{\nabla}_Y \sigma)(X, Z) \tag{3.6}$$

(Equation of Codazzi relative to $\widetilde{\nabla}$)

LEMMA 3.7.

$$\widetilde{g}(\widetilde{R}^*(X, Y)Z, W) = g(R^*(X, Y)Z, W) - \widetilde{g}(\sigma^*(Y, Z), \sigma(X, W)) + \widetilde{g}(\sigma^*(X, Z), \sigma(Y, W)) \tag{3.7}$$

(Equation of Gauss relative to $\widetilde{\nabla}^*$)

$$(\widetilde{R}^*(X, Y)Z)^\perp = (\overline{\nabla}^*_X \sigma^*)(Y, Z) - (\overline{\nabla}^*_Y \sigma^*)(X, Z) \tag{3.8}$$

(Equation of Codazzi relative to $\widetilde{\nabla}^*$)

Let $e_{n+1}, \ldots, e_m$ be an othonormal basis in $T^\perp_x M$ for each $x \in M$, that is, $\widetilde{g}(e_a, e_b) = \varepsilon_a \delta_{ab}$ $(a, b, \ldots = n+1, \ldots, m; \varepsilon_a = +1$ or $-1)$, and we set $A_{e_a} = A_a$, $A^*_{e_a} = A^*_a$. Then we have

LEMMA 3.8. *The equations of Gauss relative to $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$ are rewritten as follows:*

$$\widetilde{g}(\widetilde{R}(X, Y)Z, W) = g(R(X, Y)Z, W) - \sum_{a=n+1}^{m} \varepsilon_a \{g(Y, A^*_a Z) g(X, A_a W) - g(X, A^*_a Z) g(Y, A_a W)\},$$

$$\widetilde{g}(\widetilde{R}^*(X, Y)Z, W) = g(R^*(X, Y)Z, W) - \sum_{a=n+1}^{m} \varepsilon_a \{g(Y, A_a Z) g(X, A^*_a W) - g(X, A_a Z) g(Y, A^*_a W)\}.$$

We now define the curvature tensors $R^{\perp}$ and $(R^{\perp})^*$ of the normal bundle of M with respect to the affine connections $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$ by

$$R^{\perp}(X,Y)V = D_X(D_Y V) - D_Y(D_X V) - D_{[X,Y]}V, \tag{3.9}$$

$$(R^*)^{\perp}(X,Y)V = D^*_X(D^*_Y V) - D^*_Y(D^*_X V) - D^*_{[X,Y]}V, \tag{3.10}$$

respectively. Therefore we have

$$\widetilde{R}(X,Y)V = -(\overline{\nabla}_X A)_V Y + (\overline{\nabla}_Y A)_V X + R^{\perp}(X,Y)V - \sigma(X, A_V Y) + \sigma(Y, A_V X),$$

$$\widetilde{R}^*(X,Y)V = -(\overline{\nabla}^*_X A^*)_V Y + (\overline{\nabla}^*_Y A^*)_V X + (R^{\perp})^*(X,Y)V - \sigma^*(X, A^*_V Y) + \sigma^*(Y, A^*_V X).$$

Hence we find

LEMMA 3.9.

$$\widetilde{g}(\widetilde{R}(X,Y)V,U) = \widetilde{g}(R^{\perp}(X,Y)V,U) + g([A^*_U, A_V]X,Y) \qquad \textit{(Equation of Ricci relative to } \widetilde{\nabla}) \tag{3.11}$$

$$\widetilde{g}(\widetilde{R}^*(X,Y)V,U) = \widetilde{g}((R^{\perp})^*(X,Y)V,U) + g([A_U, A^*_V]X,Y) \qquad \textit{(Equation of Ricci relative to } \widetilde{\nabla}^*) \tag{3.12}$$

Let $e_1, \ldots, e_n$ denote an othonormal basis of T_xM for each $x \in M$, that is, $\widetilde{g}(e_i, e_j) = \varepsilon_i \delta_{ij}$ ($i, j, \ldots = 1, \ldots, n; \varepsilon_i = +1$ or -1). The mean curvature vectors μ and μ^* of M are defined to be $\mu = \frac{1}{n}\sum \varepsilon_i \sigma(e_i, e_i)$ and $\mu^* = \frac{1}{n}\sum \varepsilon_i \sigma^*(e_i, e_i)$, respectively. If $\sigma(X,Y) = g(X,Y)\mu$ (resp. $\sigma(X,Y) = 0$), then M is said to be totally umbilical (resp. totally geodesic) relative to $\widetilde{\nabla}$. Let $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$ be a space of constant curvature $\widetilde{c}$ relative to $\widetilde{\nabla}$, namely, $\widetilde{R}(X,Y)Z = \widetilde{c}\{\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y\}$. Then $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$ is of constant curvature $\widetilde{c}$ relative to $\widetilde{\nabla}$ if and only if so is $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla}^*)$ relative to $\widetilde{\nabla}^*$. The equations of Gauss and Codazzi are given by

$$\begin{aligned} g(R(X,Y)Z,W) = \widetilde{c}\{&g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ &+\widetilde{g}(\sigma(Y,Z), \sigma^*(X,W)) - \widetilde{g}(\sigma(X,Z), \sigma^*(Y,W)), \end{aligned} \tag{3.13}$$

$$(\overline{\nabla}_X \sigma)(Y,Z) = (\overline{\nabla}_Y \sigma)(X,Z). \tag{3.14}$$

Hence we have

THEOREM 3.10. *Let $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$ be of constant curvature $\widetilde{c}$ relative to $\widetilde{\nabla}$. If M^n $(n \geq 2)$ is totally umbilical of $\widetilde{M}$ relative to both $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$, then we have*

(1) *$\widetilde{g}(\mu, \mu^*)$ is a constant on M,*

(2) *(M, g, ∇) is of constant curvature $\widetilde{c} + \widetilde{g}(\mu, \mu^*)$ relative to ∇,*

(3) *(M, g, ∇^*) is of constant curvature $\widetilde{c} + \widetilde{g}(\mu, \mu^*)$ relative to ∇^*.*

4 Submanifolds of the Kähler-like statistical manifold

Let $(\widetilde{M}, \widetilde{g}, \widetilde{J})$ be an almost Hermite-like manifold. We put

$$\widetilde{J}X = PX + FX, \quad \widetilde{J}V = tV + fV, \quad \widetilde{J}^*X = P^*X + F^*X, \quad \widetilde{J}^*V = t^*V + f^*V, \tag{4.1}$$

where PX, tV, P^*X, t^*V are tangential components and FX, fV, F^*X, f^*V are normal components. By virtue of (2.3), we have

$$\begin{aligned} &g(PX,Y) + g(X, P^*Y) = 0, \qquad &&\widetilde{g}(FX,U) + g(X, t^*U) = 0, \\ &g(tU,X) + \widetilde{g}(U, F^*X) = 0, \qquad &&\widetilde{g}(fU,V) + \widetilde{g}(U, f^*V) = 0. \end{aligned} \tag{4.2}$$

LEMMA 4.1. *Let $(\widetilde{M}, \widetilde{g}, \widetilde{J})$ be an almost Hermite-like manifold. Then $\widetilde{J}(T_xM) \subset T_xM$ (resp. $\widetilde{J}^*(T_xM) \subset T_xM$) is equivalent to $\widetilde{J}^*(T_xM)^{\perp} \subset (T_xM)^{\perp}$ (resp. $\widetilde{J}(T_xM)^{\perp} \subset (T_xM)^{\perp}$).*

From $\widetilde{J}$ and $\widetilde{J}^*$ are almost complex structures, we have

$$\begin{aligned} &P^2 = -I - tF, \quad &&FP + fF = 0, \quad &&Pt + tf = 0, \quad &&f^2 = -I - Ft, \\ &(P^*)^2 = -I - t^*F^*, \quad &&F^*P^* + f^*F^* = 0, \quad &&P^*t^* + t^*f^* = 0, \quad &&(f^*)^2 = -I - F^*t^*. \end{aligned} \tag{4.3}$$

We define the covariant derivatives $\nabla_X P$ of P and $\nabla_X F$ of F by $(\nabla_X P)Y = \nabla_X(PY) - P(\nabla_X Y)$ and $(\nabla_X F)Y = D_X(FY) - F(\nabla_X Y)$, respectively. Also, we define the covariant derivative $\nabla_X t$ (resp. $\nabla_X f$) of t (resp. f) by $(\nabla_X t)V = \nabla_X(tV) - t(D_X V)$ (resp. $(\nabla_X f)V = D_X(fV) - f(D_X V)$). Similarly, we can define the covariant derivative with respect to ∇^*. Hence we get

LEMMA 4.2. *Let $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$ be a statistical manifold admitting the Hermite-like structuter $\widetilde{J}$. Then we have*

(1) $\nabla P = 0$ *is equivalent to* $\nabla^* P^* = 0$, (2) $\nabla F = 0$ *is equivalent to* $\nabla^* t^* = 0$,

(3) $\nabla t = 0$ *is equivalent to* $\nabla^* F^* = 0$, (4) $\nabla f = 0$ *is equivalent to* $\nabla^* f^* = 0$.

LEMMA 4.3. *Let $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla}, \widetilde{J})$ be a Kähler-like statistical manifold. Then $\widetilde{\nabla}\widetilde{J} = 0$ is equivalent to the following equations:*

$$(\nabla_X P)Y = A_{FY}X + t\,\sigma(X, Y), \tag{4.4}$$

$$(\nabla_X F)Y = -\sigma(X, PY) + f\,\sigma(X, Y), \tag{4.5}$$

$$(\nabla_X t)V = -P(A_V X) + A_{fV}X, \tag{4.6}$$

$$(\nabla_X f)V = -F(A_V X) - \sigma(X, tV). \tag{4.7}$$

Finally, we discuss that M is a $\widetilde{J}$-invariant submanifold of the almost Hermite-like manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$, namely, $\widetilde{J}(T_x M) \subset T_x M$ for each $x \in M$. We call such a submanifold M an almost Hermite-like submanifold. Owing to Lemma 4.1, we can put $\widetilde{J}X = JX$, $\widetilde{J}V = tV + \overline{J}V$, $\widetilde{J}^*X = J^*X + F^*X$ and $\widetilde{J}^*V = \overline{J}^*V$. Since $\widetilde{J}$ and $\widetilde{J}^*$ are almost complex structures, we obtain $J^2 = -I$, $Jt + t\overline{J} = 0$, $\overline{J}^2 = -I$, $(J^*)^2 = -I$, $F^*J^* + \overline{J}^*F^* = 0$ and $(\overline{J}^*)^2 = -I$. Also, we find $g(JX, Y) + g(X, J^*Y) = 0$, $g(tU, X) + \overline{g}(U, F^*X) = 0$ and $\overline{g}(\overline{J}U, V) + \overline{g}(U, \overline{J}^*V) = 0$ from (4.2). Thus we have

LEMMA 4.4. *Let $(\widetilde{M}, \widetilde{g}, \widetilde{J})$ be an almost Hermite-like manifold and M be a $\widetilde{J}$-invariant submanifold. Then we get*

(1) (M, g, J) *is an almost Hermite-like submanifold,*

(2) (M, g, J^*) *is an almost Hermite-like submanifold.*

Let $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla}, \widetilde{J})$ be a Kähler-like statistical manifold and M be a $\widetilde{J}$-invariant submanifold. From (4.5), we find $\sigma(X, JY) = \overline{J}\sigma(X, Y)$. Hence we have

THEOREM 4.5. *Let $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla}, \widetilde{J})$ be a Kähler-like statistical manifold and M be a $\widetilde{J}$-invariant submanifold. If M is totally umbilical relative to $\widetilde{\nabla}$, then M is totally geodesic relative to $\widetilde{\nabla}$. Moreover (M, g, ∇, J) is a Kähler-like statistical manifold.*

REFERENCES

[1] S. AMARI AND H. NAGAOKA, Methods of Information Geometry, *AMS & Oxford University Press*, 2000.

[2] K. TAKANO, Statistical manifolds with almost complex structures and its statistical submersions, *Tensor, N. S.* **65** (2004), 128–142.

[3] ____________, Statistical manifolds with almots contact structures and its statistical submersions, *J. Geom.* **85** (2006), 171–187.

[4] K. YANO AND M. KON, *Structures on Manifolds,* World Scientific, 1984.

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