Statistical manifolds with almost complex structures

Proceedings of the 45-th Symposium on Finsler Geometry (held joint with The 11th International Conference of Tensor Society)

Sep. 5 - 10, 2010

Tokyo, Japan

Society of Finsler Geometry

Japan

President: Hideo SHIMADA

CONTENTS

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1 Introduction.

Statistical models in the information geometry have a Fisher metric as the Riemannian metric, and admit a torsion-free affine connection which is constructed from the expectation with respect to the probability distribution ([1]). This affine connection is called α–connection, denoted by $\nabla^{(\alpha)}$, where α is a real number, and conjugate relative to the Fisher metric is $(-\alpha)$–connection. The 0–connection is the Levi-Civita connection with respect to the Fisher metric. Especially, $\nabla^{(1)}$ (resp. $\nabla^{(-1)}$) is said to be an exponential connection (resp. mixture connection) or e–connection (resp. m–connection) simply. The e and m–connections include impotant consepts in the information geometry. In [4], we studied exponential families admitting almost complex structures, and proved that there exists almost complex structures which are parallel with respect to the e and m–connections. Especially, if its space is of constant curvature, then these connections are only e and m–connections. For example, spaces of the multinomial or negative multinomial distributions are of constant curvature. Also, we showed the Poincaré upper half-space admits almost complex structures which are parallel relative to the affine connection and its dual ([3]).

In this paper, we shall express structures of the Kähler-like statistical manifold in terms of complex local coordinate systems.

2 Statistical manifolds with almost complex structure.

Let M and ∇ be a Riemannian manifold and an affine connection, respectively. We define another affine connection ∇^* by

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) \tag{2.1}$$

for vector fields X, Y and Z on M. The affine connection ∇^* is called conjugate (or dual) of ∇ with respect to g. The triple (M, g, ∇) is called a statistical manifold if both ∇ and ∇^* are torsion-free. Clearly $(\nabla^*)^* = \nabla$ holds. We denote by R and R^* the curvature tensors on M with respect to the affine connection ∇ and its conjugate ∇^*, respectively. Then we find $g(R(X,Y)Z, W) = -g(Z, R^*(X,Y)W)$ for vector fields X, Y, Z and W, where $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$. Therefore R vanishes identically if and only if so is R^*. If the curvature tensor R with respect to the affine connection ∇ satisfies $R(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\}$, then the statistical manifold (M, g, ∇) is called a space of constant curvature k.

An almost complex structure on a manifold M is a tensor field J of type (1,1) such that $J^2 = -I$, where I stands for the identity transformation. An almost complex manifold is such a manifold with a fixed almost complex structure. An almost complex manifold is necessarily orientable and must have an even dimension. We consider the Riemannian manifold on the almost complex manifold M. If J preserves the metric g, that is, $g(JX, JY) = g(X,Y)$ for vector fields X and Y on M, then (M, g, J) is an almost Hermitian manifold. Now, we consider the Riemannian manifold (M, g) with the almost complex structure J which has another tensor field J^* of type (1,1) satisfying

$$g(JX, Y) + g(X, J^*Y) = 0. \tag{2.2}$$

Then (M, g, J) is called an almost Hermite-like manifold. We see that $(J^*)^* = J$, $(J^*)^2 = -I$ and $g(JX, J^*Y) = g(X,Y)$. If J is parallel with respect to the affine connection ∇, then (M, g, ∇, J)

is called a Kähler-like statistical manifold. By virtue of (2.2), we get

$$g((\nabla_Z J)X, Y) + g(X, (\nabla^*_Z J^*)Y) = 0 \tag{2.3}$$

for vector fields X, Y and Z on M. Hence we have ([2])

LEMMA A. *(M, g, J) is an almost Hermite-like manifold if and only if so is (M, g, J^*). Moreover, (M, g, ∇, J) is a Kähler-like statistical manifold if and only if so is (M, g, ∇^*, J^*).*

Also, we get ([4])

THEOREM B. *Let (M^n, g, ∇, J) be a Kähler-like statistical manifold. If $M\,(n \geq 4)$ is of constant curvature, then M is flat.*

3 Kähler-like statistical manifolds in local coordinate systems.

We shall express the Kähler-like statistical manifold in terms of complex local coordinate systems. Let M^{2n} be a Riemannian manifold with a local coordinate system $(x^1, y^1, \dots, x^n, y^n)$ on a neighborhood of a point $p \in M$. Throughout this paper, Greek indices α, $\beta, \dots$ run from 1 to n and Latin capitals A, $B, \dots$ run through $1, \dots, n, \bar{1}, \dots, \bar{n}$. We take a complex local coordinate system $(z^1, \dots, z^n)$ in M, where $z^\alpha = x^\alpha + \sqrt{-1}\, y^\alpha$. We define an endomorphism J of T_pM by

$$J\left(\frac{\partial}{\partial x^\alpha}\right) = \frac{\partial}{\partial y^\alpha}, \qquad J\left(\frac{\partial}{\partial y^\alpha}\right) = -\frac{\partial}{\partial x^\alpha}.$$

We extend J to the complexification $T_p^{\mathbb{C}}M$ of T_pM and we have

$$JZ_\alpha = \sqrt{-1}\, Z_\alpha, \qquad JZ_{\bar{\alpha}} = -\sqrt{-1}\, Z_{\bar{\alpha}}, \tag{3.1}$$

where $Z_\alpha := \frac{\partial}{\partial z^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} - \sqrt{-1}\,\frac{\partial}{\partial y^\alpha}\right)$ and $Z_{\bar{\alpha}} := \frac{\partial}{\partial \bar{z}^\alpha} = \overline{Z_\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} + \sqrt{-1}\,\frac{\partial}{\partial y^\alpha}\right)$. We set

$$g(u + \sqrt{-1}\, v,\ u' + \sqrt{-1}\, v') = \{g(u, u') - g(v, v')\} + \sqrt{-1}\,\{g(u, v') + g(v, u')\}$$

for any $u + \sqrt{-1}\, v$, $u' + \sqrt{-1}\, v' \in T_p^{\mathbb{C}}M$. Then the metric g is a symmetric bilinear on $T_p^{\mathbb{C}}M$. Moreover we denote the components of g by

$$g_{\alpha\beta} = g(Z_\alpha,\ Z_\beta), \qquad g_{\alpha\bar{\beta}} = g(Z_\alpha,\ Z_{\bar{\beta}}), \qquad g_{\bar{\alpha}\beta} = g(Z_{\bar{\alpha}},\ Z_\beta), \qquad g_{\bar{\alpha}\bar{\beta}} = g(Z_{\bar{\alpha}},\ Z_{\bar{\beta}}).$$

Then we get

$$g_{\alpha\beta} = g_{\beta\alpha}, \qquad g_{\bar{\alpha}\bar{\beta}} = g_{\bar{\beta}\bar{\alpha}}, \qquad g_{\alpha\bar{\beta}} = g_{\bar{\beta}\alpha}, \qquad \bar{g}_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}}, \qquad \bar{g}_{\alpha\bar{\beta}} = g_{\bar{\alpha}\beta}. \tag{3.2}$$

We put $Z^A = g^{AB} Z_B$. Then we find on the almost Hermite-like manifold

$$J^* Z_A = -\sqrt{-1}\,(g_{A\omega} Z^\omega - g_{A\bar{\omega}} Z^{\bar{\omega}}), \tag{3.3}$$

where g^{AB} are components of the inverse matrix of g. Thus we have

LEMMA 3.1. *$J = J^*$ if and only if $g_{\alpha\beta} = 0$ $(g_{\bar{\alpha}\bar{\beta}} = 0)$.*

Next, we put $\nabla_{Z_A} Z_B = \Gamma_{AB}^{\ E} Z_E$ and $\nabla^*_{Z_A} Z_B = \Gamma^{*\,E}_{AB} Z_E$. By virtue of (2.1), we find

$$(\nabla_{Z_A} J) Z_\beta = 2\sqrt{-1}\,\Gamma_{A\beta}^{\ \bar{\omega}}\, Z_{\bar{\omega}}, \qquad (\nabla_{Z_A} J) Z_{\bar{\beta}} = -2\sqrt{-1}\,\Gamma_{A\bar{\beta}}^{\ \omega}\, Z_\omega,$$

$$(\nabla^*_{Z_A} J) Z_\beta = 2\sqrt{-1}\,\Gamma^{*\,\bar{\omega}}_{A\beta}\, Z_{\bar{\omega}}, \qquad (\nabla^*_{Z_A} J) Z_{\bar{\beta}} = -2\sqrt{-1}\,\Gamma^{*\,\omega}_{A\bar{\beta}}\, Z_\omega.$$

Then we get

LEMMA 3.2. *An almost complex structure J is parallel with respect to ∇ (resp. ∇^*) if and only if $\Gamma_{A\bar{\beta}}{}^{\bar{\omega}} = 0$ and $\Gamma_{A\bar{\beta}}{}^{\omega} = 0$ (resp. $\Gamma^*_{A\beta}{}^{\bar{\omega}} = 0$ and $\Gamma^*_{A\bar{\beta}}{}^{\omega} = 0$) hold.*

We assume that $J = J^*$ on the Kähler-like statistical manifold. From (2.1), Lemmas 3.1 and 3.2, we have

THEOREM 3.1. *On the Kähler-like statistical manifold (M, g, ∇, J), if $J = J^*$, then $\nabla = \nabla^*$, that is, (M, g, ∇, J) is a Kählerian manifold.*

4 Curvatures on the Kähler-like statistical manifold.

We denote the curvature tensor with respect to the affine connection ∇ by

$$R_{ABC}{}^{D} = Z_A\Gamma_{BC}{}^{D} - Z_B\Gamma_{AC}{}^{D} + \Gamma_{BC}{}^{E}\Gamma_{AE}{}^{D} - \Gamma_{AC}{}^{E}\Gamma_{BE}{}^{D}.$$

Then we get

LEMMA 4.1. *On the Kähler-like statistical manifold, we obtain*

$$R_{AB\gamma}{}^{\bar{\delta}} = 0, \qquad R_{AB\bar{\gamma}}{}^{\delta} = 0,$$

$$R_{\alpha\beta\gamma}{}^{\delta} = Z_\alpha\Gamma_{\beta\gamma}{}^{\delta} - Z_\beta\Gamma_{\alpha\gamma}{}^{\delta} + \Gamma_{\beta\gamma}{}^{\varepsilon}\Gamma_{\alpha\varepsilon}{}^{\delta} - \Gamma_{\alpha\gamma}{}^{\varepsilon}\Gamma_{\beta\varepsilon}{}^{\delta},$$

$$R_{\bar{\alpha}\beta\gamma}{}^{\delta} = Z_{\bar{\alpha}}\Gamma_{\beta\gamma}{}^{\delta}, \qquad R_{\bar{\alpha}\bar{\beta}\gamma}{}^{\delta} = 0, \qquad R_{\alpha\beta\bar{\gamma}}{}^{\bar{\delta}} = 0, \qquad R_{\alpha\bar{\beta}\bar{\gamma}}{}^{\bar{\delta}} = Z_\alpha\Gamma_{\bar{\beta}\bar{\gamma}}{}^{\bar{\delta}},$$

$$R_{\bar{\alpha}\bar{\beta}\bar{\gamma}}{}^{\bar{\delta}} = Z_{\bar{\alpha}}\Gamma_{\bar{\beta}\bar{\gamma}}{}^{\bar{\delta}} - Z_{\bar{\beta}}\Gamma_{\bar{\alpha}\bar{\gamma}}{}^{\bar{\delta}} + \Gamma_{\bar{\beta}\bar{\gamma}}{}^{\bar{\varepsilon}}\Gamma_{\bar{\alpha}\bar{\varepsilon}}{}^{\bar{\delta}} - \Gamma_{\bar{\alpha}\bar{\gamma}}{}^{\bar{\varepsilon}}\Gamma_{\bar{\beta}\bar{\varepsilon}}{}^{\bar{\delta}}.$$

We define the Ricci tensor Ric with respect to ∇ by $\mathrm{Ric}_{AB} = R_{DAB}{}^{D} = R_{DABE}g^{ED}$. Thus we have

LEMMA 4.2. *We get*

$$\mathrm{Ric}_{\alpha\beta} = R_{\varepsilon\alpha\beta}{}^{\varepsilon}, \qquad \mathrm{Ric}_{\alpha\bar{\beta}} = R_{\bar{\varepsilon}\alpha\bar{\beta}}{}^{\bar{\varepsilon}} = -Z_\alpha\Gamma_{\bar{\varepsilon}\bar{\beta}}{}^{\bar{\varepsilon}},$$

$$\mathrm{Ric}_{\bar{\alpha}\beta} = R_{\varepsilon\bar{\alpha}\beta}{}^{\varepsilon} = -Z_{\bar{\alpha}}\Gamma_{\varepsilon\beta}{}^{\varepsilon}, \qquad \mathrm{Ric}_{\bar{\alpha}\bar{\beta}} = R_{\bar{\varepsilon}\bar{\alpha}\bar{\beta}}{}^{\bar{\varepsilon}}.$$

LEMMA 4.3. *We have*

$$R^*_{\alpha\beta C}{}^{D} = -R_{\alpha\beta\varepsilon}{}^{\omega}g_{\omega C}g^{\varepsilon D},$$

$$R^*_{\alpha\bar{\beta}C}{}^{D} = Z_{\bar{\beta}}\Gamma_{\alpha\varepsilon}{}^{\omega}\cdot g_{\omega C}g^{\varepsilon D} - Z_\alpha\Gamma_{\bar{\beta}\bar{\varepsilon}}{}^{\bar{\omega}}\cdot g_{\bar{\omega}C}g^{\bar{\varepsilon}D},$$

$$R^*_{\bar{\alpha}\bar{\beta}C}{}^{D} = -R_{\bar{\alpha}\bar{\beta}\bar{\varepsilon}}{}^{\bar{\omega}}g_{\bar{\omega}C}g^{\bar{\varepsilon}D}.$$

LEMMA 4.4. *We obtain*

$$\mathrm{Ric}^*_{\alpha\beta} = -R_{\varepsilon\alpha\sigma}{}^{\omega}g_{\omega\beta}g^{\sigma\varepsilon} - Z_{\bar{\varepsilon}}\Gamma_{\alpha\sigma}{}^{\omega}\cdot g_{\omega\beta}g^{\sigma\bar{\varepsilon}} + Z_\alpha\Gamma_{\bar{\varepsilon}\bar{\sigma}}{}^{\bar{\omega}}\cdot g_{\bar{\omega}\beta}g^{\bar{\sigma}\bar{\varepsilon}},$$

$$\mathrm{Ric}^*_{\alpha\bar{\beta}} = -R_{\varepsilon\alpha\sigma}{}^{\omega}g_{\omega\bar{\beta}}g^{\sigma\varepsilon} - Z_{\bar{\varepsilon}}\Gamma_{\alpha\sigma}{}^{\omega}\cdot g_{\omega\bar{\beta}}g^{\sigma\bar{\varepsilon}} + Z_\alpha\Gamma_{\bar{\varepsilon}\bar{\sigma}}{}^{\bar{\omega}}\cdot g_{\bar{\omega}\bar{\beta}}g^{\bar{\sigma}\bar{\varepsilon}},$$

$$\mathrm{Ric}^*_{\bar{\alpha}\beta} = Z_{\bar{\alpha}}\Gamma_{\varepsilon\sigma}{}^{\omega}\cdot g_{\omega\beta}g^{\sigma\varepsilon} - Z_\varepsilon\Gamma_{\bar{\alpha}\bar{\sigma}}{}^{\bar{\omega}}\cdot g_{\bar{\omega}\beta}g^{\bar{\sigma}\varepsilon} - R_{\bar{\varepsilon}\bar{\alpha}\bar{\sigma}}{}^{\bar{\omega}}g_{\bar{\omega}\beta}g^{\bar{\sigma}\bar{\varepsilon}},$$

$$\mathrm{Ric}^*_{\bar{\alpha}\bar{\beta}} = Z_{\bar{\alpha}}\Gamma_{\varepsilon\sigma}{}^{\omega}\cdot g_{\omega\bar{\beta}}g^{\sigma\varepsilon} - Z_\varepsilon\Gamma_{\bar{\alpha}\bar{\sigma}}{}^{\bar{\omega}}\cdot g_{\bar{\omega}\bar{\beta}}g^{\bar{\sigma}\varepsilon} - R_{\bar{\varepsilon}\bar{\alpha}\bar{\sigma}}{}^{\bar{\omega}}g_{\bar{\omega}\bar{\beta}}g^{\bar{\sigma}\bar{\varepsilon}}.$$

Also, we define the scalar cuvatures r and r^* by $r = \mathrm{Ric}_{AB}g^{AB}$ and $r^* = \mathrm{Ric}^*_{AB}g^{AB}$, respectively. We find

LEMMA 4.5. $r = r^*$.

5 A Kähler-like statistical manifold satisfying certain condition.

We assume that the curvature tensor with respect to the affine connection ∇ of the Kähler-like statistical manifold (M, g, ∇, J) is the following equation:

$$
\begin{aligned}
(5.1) \qquad R(Z_A, Z_B)Z_C = \frac{c}{4}[\, & g(Z_B, Z_C)\, Z_A - g(Z_A, Z_C)\, Z_B - g(Z_B, JZ_C)JZ_A \\
& + g(Z_A, JZ_C)JZ_B + \{g(Z_A, JZ_B) - g(JZ_A, Z_B)\}JZ_C\,],
\end{aligned}
$$

where c is a constant. Then we get

$$
\begin{aligned}
R_{\alpha\beta\gamma}{}^{\omega} &= \frac{c}{2}(g_{\beta\gamma}\delta_{\alpha}^{\omega} - g_{\alpha\gamma}\delta_{\beta}^{\omega}), &\qquad R_{\alpha\bar{\beta}\gamma}{}^{\omega} &= \frac{c}{2}(g_{\bar{\beta}\gamma}\delta_{\alpha}^{\omega} + g_{\alpha\bar{\beta}}\delta_{\gamma}^{\omega}), \\
R_{\alpha\bar{\beta}\bar{\gamma}}{}^{\bar{\omega}} &= -\frac{c}{2}(g_{\alpha\bar{\gamma}}\delta_{\bar{\beta}}^{\bar{\omega}} + g_{\alpha\bar{\beta}}\delta_{\bar{\gamma}}^{\bar{\omega}}), &\qquad R_{\bar{\alpha}\bar{\beta}\bar{\gamma}}{}^{\bar{\omega}} &= \frac{c}{2}(g_{\bar{\beta}\bar{\gamma}}\delta_{\bar{\alpha}}^{\bar{\omega}} - g_{\bar{\alpha}\bar{\gamma}}\delta_{\bar{\beta}}^{\bar{\omega}}).
\end{aligned}
$$

Therefore we obtain

$$
\mathrm{Ric}_{\beta\gamma} = \frac{c}{2}(n-1)g_{\beta\gamma}, \quad \mathrm{Ric}_{\alpha\bar{\beta}} = \mathrm{Ric}_{\bar{\beta}\alpha} = \frac{c}{2}(n+1)g_{\alpha\bar{\beta}}, \quad \mathrm{Ric}_{\bar{\beta}\bar{\gamma}} = \frac{c}{2}(n-1)g_{\bar{\beta}\bar{\gamma}}.
$$

Thus we find

LEMMA 5.1. *The Ricci tensor is symmetric on the Kähler-like statistical manifold satisfying* (5.1).

Moreover, we have $r = c\{n(n+1) - 2g_{\varepsilon\omega}g^{\varepsilon\omega}\}$. Similarly, we find $r^* = c\{n(n+1) - 6g_{\varepsilon\omega}g^{\varepsilon\omega} + 4g_{\alpha\varepsilon}g_{\beta\omega}g^{\alpha\beta}g^{\varepsilon\omega}\}$. Hence we have from Lemma 4.5

LEMMA 5.2. *If the Kählerian statistical manifold satisfies* (5.1), *then* $c = 0$ *or* $\mathrm{tr}\,(AB) = \mathrm{tr}\,(AB)^2$, *where* $A = (g_{\alpha\beta})$ *and* $B = (g^{\alpha\beta})$.

REMARK. When M is a Kählerian manifold, the space satisfying (5.1) is of constant holomorphic sectional curvature c.

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