Some geometric properties of system spaces of autoregressive process of degree 2

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■ November 16, 2018

- $14:10 \sim 14:50$ **T. Aikou** (Kagoshima Univ.)
  One-sided projective changes and invariant curvature tensors

- $15:00 \sim 15:40$ **T. Nagano** (Univ. of Nagasaki)
  On transformations of linear parallel displacement

- $16:00 \sim 16:40$ **N. Innami** (Niigata Univ.)
  Geodesics and geodesic circles in a Finsler surface with genus $\geqq 1$

■ November 17, 2018

- $10:00 \sim 10:40$ **K. Shiohama** (Fukuoka Inst. of Tech.)
  Affine Functions on Complete Finsler Manifolds

- $10:50 \sim 11:30$ **S. Sabau** (Tokai Univ., Sapporo)
  $K = 1$ Finsler metrics on S^2

- $14:00 \sim 14:40$ **H. Shimada** (Tokai Univ., Sapporo)
  About the rigidity of Busemann convex Finsler metrics

- $15:00 \sim 16:00$ **H. Tadano** (Tokyo Univ. of Science)
  Some Compactness Theorems in Riemannian and Finslerian Geometry

■ November 18, 2018

- $10:00 \sim 10:40$ **Z. Şentürk** (Istanbul Technical Univ.)
  On Locally Conformal Kaehler Manifolds

- $11:00 \sim 11:40$ **K. Takano** (Shinshu Univ.)
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Some geometric properties of system spaces of autoregressive process of degree 2

Kazuhiko Takano

1 Introduction

Let $\{\varepsilon_t\}$ and $\{x_t\}$ be time series of the input signal and the output signal. The input-output relation of the system can be expressed by

$$\sum_{i=0}^{p} a_i x_{t-i} = \sum_{i=1}^{q} b_i \varepsilon_{t-i+1} \qquad (a_0 \neq 0),$$

where $\{\varepsilon_t\}$ is a sequence of independent identically normal distributions with zero mean and variance σ^2. If $a_0 = 1$, then it is said to be an autoregressive moving average (ARMA) process of degree (p, q). This process is denoted by ARMA(p, q). When $q = 1$ and $b_1 = 1$ (resp. $p = 0$ and $a_0 = 1$), this model is called an autoregressive (AR) process of degree p (resp. moving average (MA) process of degree q). It is denoted by AR(p) (resp. MA(q)). In [2], they studied the $(p + q + 1)$–dimensional system space of ARMA(p, q) such that $a_0 = 1$ with the local coordinate system $(\rho_1, \dots, \rho_p, \delta_1, \dots, \delta_q, \sigma^2)$, where ρ_i $(i = 1, \dots, p)$ and δ_j $(j = 1, \dots, q)$ denote the roots of p-degree and q-degree polynomials $\sum_{i=0}^{p} a_i z^i = 0$ $(a_0 = 1)$ and $\sum_{i=0}^{q} b_i z^i = 0$ $(b_0 = 1)$ with respect to z, respectively. They calculated the Riemannian metric and α–connection. Also, in [4], they studied the $(p+1)$–dimensional system space of AR(p) such that $a_0 = 1$ with the local coordinate system $(\rho_1, \dots, \rho_p, \sigma^2)$ and sought the sectional curvature. In [3], we study a system space of the autoregressive process of degree 1. This space is a 2-dimensional α–flat statistical manifold, we investigate α–geodesics, almost complex structures which are parallel with respect to the α–connection.

In this time, we introduce some geometric properties of the system space of AR(2).

2 Statistical manifolds and system spaces

Let (M, g) and ∇ be a Riemannian manifold and affine connection, respectively. We define another affine connection ∇^* by

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) \tag{2.1}$$

for vector fields X, Y and Z on M. An affine connection ∇^* is called conjugate of ∇ with respect to g. The triple (M, g, ∇) is called a statistical manifold if both ∇ and ∇^* are torsion-free. Clearly $(\nabla^*)^* = \nabla$ holds. It is easy to see that $\frac{1}{2}(\nabla + \nabla^*)$ is a metric connection.

Next, we call a system for which the output x_t at time t may be expressed in terms of the previous p values $x_{t-1}, \dots, x_{t-p}$ and the current input ε_t as

$$\sum_{i=0}^{p} a_i x_{t-i} = \varepsilon_t \qquad (a_0 \neq 0) \tag{2.2}$$

an autoregressive process of degree p, where ε_t is a normal distribution with zero mean and variance σ^2. The power spectrum for this model is

$$S(\omega; a) = \sigma^2 \left| \sum_{s=0}^{p} a_s e^{-i\omega s} \right|^{-2}, \tag{2.3}$$

where $a = (a_0, a_1, \dots, a_p)$ are called AR parameters ([1]). Let M^{p+1} be a system space of AR(p) with the local coordinate system $(a_0, a_1, \dots, a_p)$. For the local coordinate system, we define components of the Fisher metric g by

$$g_{ij} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \partial_i \log S(\omega; a)\, \partial_j \log S(\omega; a)\, d\omega, \tag{2.4}$$

where $\partial_i = \partial/\partial a_i$. Moreover we put for a real number α

$$\Gamma^{(\alpha)}_{ij,k} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\{\partial_i\partial_j \log S(\omega;a) - \alpha\,\partial_i \log S(\omega;a)\,\partial_j \log S(\omega;a)\}\,\partial_k \log S(\omega;a)\,d\omega \tag{2.5}$$

and define an α–connection $\nabla^{(\alpha)}$ by

$$g(\nabla^{(\alpha)}_{\partial_i}\partial_j, \partial_k) = \Gamma^{(\alpha)}_{ij,k}. \tag{2.6}$$

Then the α–connection is torsion-free and $\nabla^{(-\alpha)}$ is conjugate of $\nabla^{(\alpha)}$ with respect to the Fisher metric. Thus the triple $\left(M, g, \nabla^{(\alpha)}\right)$ is a statistical manifold. $\nabla^{(0)}$ is the Levi-Civita connection with respect to the Fisher metric and is denoted by ∇. We call α–flat if the curvature tensor with respect to the α–connection vanishes identically. The α–geodesic equations on $\left(M, g, \nabla^{(\alpha)}\right)$ are defined by

$$\frac{d^2 a_k}{dt^2} + \Gamma^{(\alpha)\,k}_{ij}\,\frac{da_i}{dt}\,\frac{da_j}{dt} = 0 \qquad (k = 0, 1, \ldots, p), \tag{2.7}$$

where $\Gamma^{(\alpha)\,k}_{ij} = \Gamma^{(\alpha)}_{ij,s} g^{sk}$ and g^{sk} are components of the inverse matrix of the Fisher metric g. The solution of (2.7) is called α–geodesic. Especially, the 1–geodesic (resp. (-1)–geodesic) is called an e–geodesic (resp. m–geodesic). Generally, the 0–geodesic is a geodesic.

3 System spaces of the autoregressive process of degree 2

We consider the autoregressive process of degree 2

$$a\,x_t + b\,x_{t-1} + c\,x_{t-2} = \varepsilon_t \qquad (a \neq 0),$$

where ε_t is a normal distribution with zero mean and variance 1. Its power spectrum is given by $S(\omega; a, b, c) = (a^2 + b^2 + c^2 + 2b(a + c)\cos\omega + 2ac\cos 2\omega)^{-1}$. We put

$$M^3 = \left\{\, (a, b, c) \in \mathbb{R}^3 \,|\, a > 0,\ b^2 - 4ac > 0,\ (a - b + c)(a + b + c) > 0,\ (a - c)(a - b + c) > 0 \,\right\}.$$

Then we get components of the Fisher metric g of M from (2.4)

$$g_{aa} = \frac{2(2a^3 - ab^2 + 2a^2c + b^2c - ac^2 - c^3)}{a^2(a - c)(a - b + c)(a + b + c)},$$

$$g_{ab} = g_{ba} = g_{bc} = g_{cb} = -\frac{2b}{(a - c)(a - b + c)(a + b + c)},$$

$$g_{ac} = g_{ca} = -\frac{2(ac - b^2 + c^2)}{a(a - c)(a - b + c)(a + b + c)},$$

$$g_{bb} = g_{cc} = \frac{2(a + c)}{(a - c)(a - b + c)(a + b + c)}.$$

Because of (2.5) and (2.6), α–connections are given by

$$\begin{aligned}
\nabla^{(\alpha)}_{\partial_a}\partial_a &= \frac{1}{a}\left\{-1 + \frac{\alpha\,(2a^3 - ab^2 + 2a^2c + b^2c - ac^2 - c^3)}{(a - c)(a - b + c)(a + b + c)}\right\}\partial_a \\
&\quad + \frac{b\{a^2 - b^2 + ac + c^2 - \alpha\,(2a^2 - b^2 + c^2)\}}{a(a - c)(a - b + c)(a + b + c)}\,\partial_b + \frac{ac - b^2 + c^2 - \alpha\,(2ac - b^2 + 2c^2)}{(a - c)(a - b + c)(a + b + c)}\,\partial_c,
\end{aligned}$$

$$\begin{aligned}
\nabla^{(\alpha)}_{\partial_a}\partial_b = \nabla^{(\alpha)}_{\partial_b}\partial_a &= -\frac{\alpha\,ab}{(a - c)(a - b + c)(a + b + c)}\,\partial_a \\
&\quad - \frac{a^2 - b^2 + ac - \alpha\,(2a^2 - b^2 + ac - c^2)}{(a - c)(a - b + c)(a + b + c)}\,\partial_b + \frac{b\{a - \alpha\,(a - c)\}}{(a - c)(a - b + c)(a + b + c)}\,\partial_c,
\end{aligned}$$

$$\begin{aligned}
\nabla^{(\alpha)}_{\partial_a}\partial_c = \nabla^{(\alpha)}_{\partial_c}\partial_a &= -\frac{\alpha\,(ac - b^2 + c^2)}{(a - c)(a - b + c)(a + b + c)}\,\partial_a - \frac{b\{c + \alpha\,(a - 2c)\}}{(a - c)(a - b + c)(a + b + c)}\,\partial_b \\
&\quad - \frac{a(a + c) - \alpha\,(2a^2 - b^2 + 2ac)}{(a - c)(a - b + c)(a + b + c)}\,\partial_c,
\end{aligned}$$

$$\nabla^{(\alpha)}_{\partial_b}\partial_b = \frac{\alpha\, a(a+c)}{(a-c)(a-b+c)(a+b+c)}\,\partial_a - \frac{b\{c+\alpha\,(a-2c)\}}{(a-c)(a-b+c)(a+b+c)}\,\partial_b - \frac{(a+c)\{a-\alpha\,(a-c)\}}{(a-c)(a-b+c)(a+b+c)}\,\partial_c,$$

$$\nabla^{(\alpha)}_{\partial_b}\partial_c = \nabla^{(\alpha)}_{\partial_c}\partial_b = -\frac{\alpha\, ab}{(a-c)(a-b+c)(a+b+c)}\,\partial_a + \frac{(a+c)\{c+\alpha\,(a-2c)\}}{(a-c)(a-b+c)(a+b+c)}\,\partial_b + \frac{b\{a-\alpha\,(a-c)\}}{(a-c)(a-b+c)(a+b+c)}\,\partial_c,$$

$$\nabla^{(\alpha)}_{\partial_c}\partial_c = \frac{\alpha\, a(a+c)}{(a-c)(a-b+c)(a+b+c)}\,\partial_a - \frac{b\{c+\alpha\,(a-2c)\}}{(a-c)(a-b+c)(a+b+c)}\,\partial_b + \frac{ac-b^2+c^2-\alpha\,(2ac-b^2+2c^2)}{(a-c)(a-b+c)(a+b+c)}\,\partial_c,$$

where $\partial_a = \partial/\partial a$, $\partial_b = \partial/\partial b$ and $\partial_c = \partial/\partial c$. The system space $\big(M, g, \nabla^{(\alpha)}\big)$ of AR(2) is a statistical manifold. We define the curvature tensor with respect to the α–connection by

$$R^{(\alpha)}(X,Y)Z = \nabla^{(\alpha)}_X\nabla^{(\alpha)}_Y Z - \nabla^{(\alpha)}_Y\nabla^{(\alpha)}_X Z - \nabla^{(\alpha)}_{[X,Y]}Z$$

for X, Y, $Z \in T_xM$. Then we obtain

$$R^{(\alpha)}(\partial_a,\partial_b)\partial_a = -\frac{c(\alpha)\,c}{2(a-c)}(g_{ac}\partial_b - g_{ab}\partial_c), \qquad R^{(\alpha)}(\partial_a,\partial_b)\partial_b = -\frac{c(\alpha)\,c}{2(a-c)}(g_{ab}\partial_b - g_{bb}\partial_c),$$

$$R^{(\alpha)}(\partial_a,\partial_b)\partial_c = -\frac{c(\alpha)\,c}{2(a-c)}(g_{bb}\partial_b - g_{bc}\partial_c), \qquad R^{(\alpha)}(\partial_a,\partial_c)\partial_a = \frac{c(\alpha)\,b}{2(a-c)}(g_{ac}\partial_b - g_{ab}\partial_c),$$

$$R^{(\alpha)}(\partial_a,\partial_c)\partial_b = \frac{c(\alpha)\,b}{2(a-c)}(g_{bc}\partial_b - g_{bb}\partial_c), \qquad R^{(\alpha)}(\partial_a,\partial_c)\partial_c = \frac{c(\alpha)\,b}{2(a-c)}(g_{cc}\partial_b - g_{bc}\partial_c),$$

$$R^{(\alpha)}(\partial_b,\partial_c)\partial_a = -\frac{c(\alpha)\,a}{2(a-c)}(g_{ac}\partial_b - g_{ab}\partial_c), \qquad R^{(\alpha)}(\partial_b,\partial_c)\partial_b = -\frac{c(\alpha)\,a}{2(a-c)}(g_{bc}\partial_b - g_{bb}\partial_c),$$

$$R^{(\alpha)}(\partial_b,\partial_c)\partial_c = -\frac{c(\alpha)\,a}{2(a-c)}(g_{cc}\partial_b - g_{bc}\partial_c),$$

where $c(\alpha) = (1-\alpha)(1+\alpha)$. Hence we have

Lemma 1. *We find*

(1) *curvature tensors with respect to α–connection are spanned by ∂_b and ∂_c.*

(2) $R^{(\alpha)} = R^{(-\alpha)}$ *holds.*

(3) $R^{(\pm 1)} = 0$, *that is,* $\big(M, g, \nabla^{(\pm 1)}\big)$ *is ± 1–flat, respectively.*

Theorem 1. *In the statistical manifold* $\big(M, g, \nabla^{(\alpha)}\big)$, *we get*

$$ab\,R^{(\alpha)}(\partial_a,\partial_b) = bc\,R^{(\alpha)}(\partial_b,\partial_c) = ca\,R^{(\alpha)}(\partial_c,\partial_a).$$

Next we obtain components of the Ricci tensor $\mathrm{Ric}^{(\alpha)}$ and the scalar curvature $r^{(\alpha)}$ with respect to the α–connection

$$\mathrm{Ric}^{(\alpha)}_{aa} = -\frac{c(\alpha)(ab^2 - b^2c + ac^2 + c^3)}{a(a-c)^2(a-b+c)(a+b+c)},$$

$$\mathrm{Ric}^{(\alpha)}_{ab} = \mathrm{Ric}^{(\alpha)}_{ba} = \mathrm{Ric}^{(\alpha)}_{bc} = \mathrm{Ric}^{(\alpha)}_{cb} = \frac{c(\alpha)\,ab}{(a-c)^2(a-b+c)(a+b+c)},$$

$$\mathrm{Ric}^{(\alpha)}_{ac} = \mathrm{Ric}^{(\alpha)}_{ca} = \frac{c(\alpha)(ac-b^2+c^2)}{(a-c)^2(a-b+c)(a+b+c)},$$

$$\mathrm{Ric}^{(\alpha)}_{bb} = \mathrm{Ric}^{(\alpha)}_{cc} = -\frac{c(\alpha)\,a(a+c)}{(a-c)^2(a-b+c)(a+b+c)} \qquad \text{and} \qquad r^{(\alpha)} = -\frac{c(\alpha)\,a}{a-c}.$$

Thus we find $\mathrm{Ric}^{(\alpha)}$ is symmetric, moreover, $\mathrm{Ric}^{(\alpha)} = \mathrm{Ric}^{(-\alpha)}$ and $\mathrm{Ric}^{(\pm 1)} = 0$. Also, if we put $\|\, R^{(\alpha)} \,\|^2 = g^{hi} g^{jk} g^{pq} g_{rs} R^{(\alpha)\, r}_{hjp} R^{(\alpha)\, s}_{ikq}$ and $\|\, \mathrm{Ric}^{(\alpha)} \,\|^2 = g^{rs} g^{pq} \,\mathrm{Ric}^{(\alpha)}_{rp} \,\mathrm{Ric}^{(\alpha)}_{sq}$, then we find $\|\, R^{(\alpha)} \,\| = |\, r^{(\alpha)} \,|$ and $\|\, \mathrm{Ric}^{(\alpha)} \,\| = \frac{|\, r^{(\alpha)} \,|}{\sqrt{2}}$. Also, we define the derivative of the Ricci tensor by

$$\left(\nabla^{(\alpha)}_X \mathrm{Ric}^{(\alpha)}\right)(Y, Z) = X\left(\mathrm{Ric}^{(\alpha)}(Y, Z)\right) - \mathrm{Ric}^{(\alpha)}\left(\nabla^{(\alpha)}_X Y, Z\right) - \mathrm{Ric}^{(\alpha)}\left(Y, \nabla^{(\alpha)}_X Z\right)$$

for $X,\, Y,\, Z \in T_x M$. If $\alpha = 0$, then we have

LEMMA 2. $\nabla_X \mathrm{Ric} = (X \log |\, r \,|)\, \mathrm{Ric}$ *holds, that is,* (M, g) *is the Ricci recurrent manifold.*

From (2.7), we get 1–geodesic equations

$$\begin{aligned}
&\frac{d^2 a}{dt^2} + \frac{a(a+c)}{(a-c)(a-b+c)(a+b+c)} \left(\frac{da}{dt} + \frac{db}{dt} + \frac{dc}{dt}\right)^2 \\
&- \frac{2a}{(a-c)(a-b+c)} \frac{db}{dt} \left(\frac{da}{dt} + \frac{dc}{dt}\right) - \frac{2}{a-c} \frac{da}{dt} \frac{dc}{dt} = 0, \\
&\frac{d^2 b}{dt^2} - \frac{b}{(a-b+c)(a+b+c)} \left(\frac{da}{dt} + \frac{db}{dt} + \frac{dc}{dt}\right)^2 + \frac{2}{a-b+c} \frac{db}{dt} \left(\frac{da}{dt} + \frac{dc}{dt}\right) = 0, \\
&\frac{d^2 c}{dt^2} - \frac{c(a+c)}{(a-c)(a-b+c)(a+b+c)} \left(\frac{da}{dt} + \frac{db}{dt} + \frac{dc}{dt}\right)^2 \\
&+ \frac{2c}{(a-c)(a-b+c)} \frac{db}{dt} \left(\frac{da}{dt} + \frac{dc}{dt}\right) + \frac{2}{a-c} \frac{da}{dt} \frac{dc}{dt} = 0.
\end{aligned}$$

Hence we have

THEOREM 2. *The* 1*–geodesic* $(a(t), b(t), c(t))$ *of the system space* $\left(M, g, \nabla^{(1)}\right)$ *is given by*

$$\begin{aligned}
a(t) &= \frac{1}{4}\left\{\sqrt{A_1 t + B_1} + \sqrt{A_2 t + B_2} + \sqrt{\left(\sqrt{A_1 t + B_1} + \sqrt{A_2 t + B_2}\right)^2 - 16(A_3 t + B_3)}\right\}, \\
b(t) &= \frac{1}{2}\left(\sqrt{A_1 t + B_1} - \sqrt{A_2 t + B_2}\right), \\
c(t) &= \frac{1}{4}\left\{\sqrt{A_1 t + B_1} + \sqrt{A_2 t + B_2} - \sqrt{\left(\sqrt{A_1 t + B_1} + \sqrt{A_2 t + B_2}\right)^2 - 16(A_3 t + B_3)}\right\},
\end{aligned}$$

where A_i*,* B_i $(i = 1, 2, 3)$ *are constants.*

COROLLARY. *The* 1*–geodesic of the system space* $\left(M, g, \nabla^{(1)}\right)$ *is given by an intersect curve of two surfaces* $A_2\{(a+b+c)^2 - B_1\} = A_1\{(a-b+c)^2 - B_2\}$ *and* $A_3\{(a+b+c)^2 - B_1\} = A_1(ac - B_3)$*.*

REFERENCES

[1] S. Amari and H. Nagaoka, Methods of Information Geometry, AMS & Oxford University Press, 2000.

[2] N. Ravishanker, E. L. Melnick and C.-L. Tsai, Differential geometry of ARMA models, *J. Time Ser. Anal.* **11** (1990), 259–274.

[3] K. Takano and Z. Şentürk, Geometric properties of system spaces of autoregressive process of degree 1, preprint.

[4] F. Tanaka and F. Komaki, The sectional curvature of AR model manifolds, *Tensor, N. S.* **64** (2003), 131–143.

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