Doctoral Dissertation (Shinshu University)

Composition operators and homomorphisms on Lipschitz algebras

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Abstract

As seen in Weaver's book [42], researches on Lipschitz algebras have attracted attention in recently. In this paper, we take up Lipschitz algebras taking values continuous functions and give a complete description of the homomorphism between them by composition operator (Theorem 1). We also characterize the compactness of such homomorphism (Theorem 2). Moreover, we shall give several corollaries of the above theorems. We may see that Theorem 1 and Theorem 2 are generalization of [33] and [18] (Corollary 1). Finally, we shall discuss weakly compact homomorphism between Lipschitz algebras. We also conjecture that weakly compactness leads compactness under some conditions.

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Chapter 1

Introduction

1.1 Backgrounds and Results

The studies of homomorphism between Banach algebras are kind of preserver problems of Banach algebras. Preserver problems of Banach algebras is as follows:

> When a mapping between two Banach algebras preserves some structures, does it preserve other structures?

In old times, G. Frobenius described in [6] the general form of all determinant preserving linear operators between Banach algebra of $n \times n$ complex matrices $M_n(\mathbb{C})$. From then, they were studied by many authors but were not systematic (cf.[12], [15], [20] and [25]). Studies on preserver problems attracted attention among many authors since Molnár's paper [23] and have been studied by them (cf.[24]).

We say that a linear operator T from function space $\mathcal{F}(X)$ over a set X into function space $\mathcal{F}(Y)$ over a set Y is *weighted composition operator* if for some function w on Y and some mapping φ from Y into X, T is of the form:

$$(Tf)(y) = w(y)f\big(\varphi(y)\big) \qquad \big(f \in \mathcal{F}(X), y \in Y\big).$$

In particular if $w(y) = \mathbf{1}$ for any $y \in Y$, then we say that it is *composition operator*. Preserver mappings are often characterized by weighted composition operator. For instance, surjective linear isometry between Banach algebras of $\mathbb{C}$-valued continuous functions on compact Hausdorff space, well known as Banach-Stone theorem (cf.[5, Theorem V.8.8]), was characterized by weighted composition operator. Also, multiplicative preserver mappings are often characterized by composition operator. For instance, O. Hatori, T. Miura and H. Takagi characterized in [9] multiplicative mapping between semi-simple commutative Banach algebras with unit preserving the spectrum by composition operator.

We say that a mapping between two Banach algebras with unit $\mathcal{A}$ and $\mathcal{B}$ is said to be *homomorphism* if it preserves addition, scalar multiplication and multiplication. Moreover if it maps the unit of $\mathcal{A}$ to the unit of $\mathcal{B}$, then we say that it is unital. If $\mathcal{B}$ is semi-simple commutative Banach algebra, homomorphism is continuous (cf.[29, Theorem 11.10]). To determine the form of such homomorphism is an important issue when both $\mathcal{A}$ and $\mathcal{B}$ are specific function algebras. It has been discussed by many authors and most of them are described by composition operators. For instance, N. Dunford and J. T. Schwartz characterized in [5, Theorem IV.6.26] unital homomorphism between Banach algebras of $\mathbb{C}$-valued continuous functions on compact Hausdorff space by composition operator. Also, S. Kakutani characterized in [16] isomorphism between Banach algebras of bounded analytic functions on maximal region of Riemann sphere, where a mapping between two Banach algebras is isomorphism if it is bijective homomorphism.

Since multiplicative preserver mappings between Banach algebras of functions are often described by composition operator, we need to study homomorphism between them. In this paper, we consider when $\mathcal{A}$ and $\mathcal{B}$ are the Banach algebra of all Lipschitz functions with values in commutative C^*-algebra with a unit.

Let X be a compact metric space with metric d_X and $\mathcal{A}$ be a Banach algebra with norm $\|\cdot\|_{\mathcal{A}}$. We denote by $\mathrm{Lip}(X)$ and $\mathrm{Lip}(X, \mathcal{A})$ the Banach algebras of $\mathbb{C}$-valued Lipschitz functions and $\mathcal{A}$-valued Lipschitz functions, respectively (see Definition 2.1.2). D.R. Sherbert characterized in [33] unital homomorphism from $\mathrm{Lip}(X)$ into $\mathrm{Lip}(Y)$ by the composition operator, as follows.

Theorem A (Sherbert, [33]). *Suppose that X and Y are compact metric spaces with metrics d_X and d_Y, respectively. Then T is a unital homomorphism from $\mathrm{Lip}(X)$ into $\mathrm{Lip}(Y)$ if and only if there exists a mapping $\varphi : Y \to X$ with*

$$\sup_{\substack{y,y' \in Y \\ y \neq y'}} \frac{d_X(\varphi(y), \varphi(y'))}{d_Y(y, y')} < \infty,$$

such that

$$(Tf)(y) = f(\varphi(y)) \qquad (y \in Y) \tag{1.1}$$

for all $f \in \mathrm{Lip}(X)$.

This theorem has been extended to several directions. F. Botelho and J. Jamison characterized in [2] unital homomorphisms from $\mathrm{Lip}(X, \mathbf{c})$ into $\mathrm{Lip}(Y, \mathbf{c})$, where $\mathbf{c}$ is the Banach algebra of $\mathbb{C}$-valued convergent sequences, as follows.

Theorem B (Botelho and Jamison, [2])**.** *Suppose that X is a compact metric space with metric d_X and that Y is a compact and connected metric space with metric d_Y. If T is a unital homomorphism from* $\mathrm{Lip}(X, \mathbf{c})$ *into* $\mathrm{Lip}(Y, \mathbf{c})$*, then there exist a sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ of Lipschitz mappings from Y into X and a continuous mapping $\psi : \mathbb{N}_\infty \to \mathbb{N}_\infty$ such that T is of the form*

$$\big[(Tf)(y)\big](n) = \big[f(\varphi_n(y))\big](\psi(n)) \qquad (y \in Y,\ n \in \mathbb{N})$$

for all $f \in \mathrm{Lip}(Y, \mathbf{c})$, where $\mathbb{N}_\infty$ is the one-point compactification of $\mathbb{N}$.

They also characterized unital homomorphisms from $\mathrm{Lip}(X, \ell^\infty)$ into $\mathrm{Lip}(Y, \ell^\infty)$, where ℓ^∞ is the Banach algebra of $\mathbb{C}$-valued bounded sequences.

Note that $\mathbf{c}$ and ℓ^∞ are commutative C^*-algebras with unit, respectively. According to Gel'fand-Naimark theorem (cf.[29, Theorem 11.18]), for any commutative C^*-algebra with unit $\mathcal{A}$, there exists a compact Hausdorff space K such that $\mathcal{A}$ is isometrically $*$-isomorphic to $C(K)$, where $C(K)$ is the commutative C^*-algebra of $\mathbb{C}$-valued continuous functions on K. This theorem implies that it suffices to characterize homomorphism between Lipschitz algebras with values in $C(K)$ instead of homomorphism between Lipschitz algebras with values in commutative C^*-algebra with unit $\mathcal{A}$.

In [28], as a generalization of [2], S. Oi took up $\mathrm{Lip}(X, C(K))$ and characterized unital homomorphism from $\mathrm{Lip}(X, C(K))$ into $\mathrm{Lip}(Y, C(M))$, as follow.

Theorem C (Oi, [28])**.** *Suppose that X and Y are as in Theorem B, and that K and M are compact Hausdorff spaces. Then T is a unital homomorphism from* $\mathrm{Lip}(X, C(K))$ *into* $\mathrm{Lip}(Y, C(M))$ *if and only if there exist a family $\{\varphi_\eta\}_{\eta\in M}$ of mappings from Y into X with properties (a) and (b) below and a continuous mapping $\psi : M \to K$ such that T is of the form*

$$\big[(Tf)(y)\big](\eta) = \big[f(\varphi_\eta(y))\big](\psi(\eta)) \qquad (y \in Y,\ \eta \in M) \tag{1.2}$$

for all $f \in \mathrm{Lip}(Y, C(K))$.

(a) For each $y \in Y$, the mapping $\eta \mapsto \varphi_\eta(y)$ from M into X is continuous.

(b) $\displaystyle \sup_{\eta\in M} \sup_{\substack{y,y'\in Y\\ y\neq y'}} \frac{d_X(\varphi_\eta(y), \varphi_\eta(y'))}{d_Y(y, y')} < \infty.$

In [10], as a generalization of [2] and [28], O. Hatori, S. Oi and H. Takagi also gave the conditions for unital homomorphism between Lipschitz algebras with values in unital semi-simple commutative Banach algebra to be of the form (1.2).

However, they assumed that the homomorphism is unital and Y is connected. It seems that they introduced the assumptions to make the statement simpler. In this paper, we remove these assumptions and give a complete description of homomorphism from $\mathrm{Lip}(X, C(K))$ into $\mathrm{Lip}(Y, C(M))$ by composition operator. In order to remove them and to establish a general result, we need to regard function f in $\mathrm{Lip}(X, C(K))$ as a function of two variables $x \in X$ and $\xi \in K$. So we write $f(x, \xi)$ instead of $\big[f(x)\big](\xi)$. Let $f \in \mathrm{Lip}(X, C(K))$. If $x \in X$, we write f_x instead of $f(x)$. If $\xi \in K$, f^ξ is the function defined on X by $f^\xi(x) = f(x, \xi)$. In general, for any mapping of two variables, we use the same expression: When ψ is the mapping form $Y \times M$ to K, then $\psi_y : M \to K$ and $\psi^\eta : Y \to K$ are defined by $\psi_y(\eta) = \psi(y, \eta)$ and $\psi^\eta(y) = \psi(y, \eta)$, respectively.

A subset A of a topological space G is said to be *clopen* if both A and $G \setminus A$ are open. We do not exclude the possibility that a clopen set is empty.

Theorem 1 ([11])**.** *Suppose that X and Y are compact metric spaces with metrics d_X and d_Y, respectively, and that K and M are compact Hausdorff spaces. If T is a homomorphism from $\mathrm{Lip}(X, C(K))$ into $\mathrm{Lip}(Y, C(M))$, then there exist a clopen subset $\mathcal{D}$ of $Y \times M$ and two continuous mappings $\varphi : \mathcal{D} \to X$ and $\psi : \mathcal{D} \to K$ satisfying (i) and (ii) and T is of the form:*

$$(Tf)(y, \eta) = \begin{cases} f\big(\varphi(y, \eta), \psi(y, \eta)\big) & \big((y, \eta) \in \mathcal{D}\big) \\ 0 & \big((y, \eta) \in (Y \times M) \setminus \mathcal{D}\big) \end{cases} \tag{1.3}$$

for all $f \in \mathrm{Lip}(X, C(K))$.

(i) There exists a constant $L \geq 0$ satisfying

$$\frac{d_X\big(\varphi(y, \eta), \varphi(y', \eta)\big)}{d_Y(y, y')} \leq L. \tag{1.4}$$

for $(y, \eta), (y', \eta) \in \mathcal{D}$ and $y \neq y'$.

(ii) For any $\eta \in M$, a set $\mathcal{D}^\eta = \{y \in Y : (y, \eta) \in \mathcal{D}\}$ is a union of finitely many disjoint clopen subsets $V_1^\eta, \ldots, V_{n_\eta}^\eta$ of Y such that

$$\psi^\eta \text{ is constant on } V_i^\eta \quad \text{for } i = 1, \ldots, n_\eta,$$

and

$$d_Y(V_i^\eta, V_j^\eta) \geq r \qquad (i \neq j). \tag{1.5}$$

Here r is a positive constant independent of η.

Conversely, if $\mathcal{D}$, φ, ψ are as above, and T is of the form (1.3), then T defined by (1.3) *is a homomorphism from* $\mathrm{Lip}(X, C(K))$ *into* $\mathrm{Lip}(Y, C(M))$. *Moreover, T is unital if and only if $\mathcal{D} = Y \times M$.*

Remark. In (1.5), $d_Y(A, B)$ denotes the usual distance between two sets $A, B \subset Y$, that is, $d_Y(A, B) = \inf\{d_Y(y, y') : y \in A, y' \in B\}$. (In the case that $A = \emptyset$ or $B = \emptyset$, we set $\inf \emptyset = \infty$).

If Y is connected and T is unital in Theorem 1, then we have Theorem C. Therefore Theorem 1 is generalization of [28]. Also, we may see that Theorem 1 contains [33] (see Corollary 1). Note that $\mathbf{c}$ is commutative C^*-algebra with unit. Theorem 1 is generalization of [2].

The studies of composition operators were initiated by E. A. Nordgren [27] and H. J. Schwartz [31]. Their purpose is to study the properties of composition operators (boundedness, (weakly) compactness, Fredholm property and spectrum) between various function spaces. They have been studied by many authors and their results are found in books [4], [32], [34] and [43].

Compact composition operators on various function spaces has been studied. For instance, H. Kamowitz [17], and M. Lindstoröm and J. Llavona [19] studied compact composition operator between Banach space of $\mathbb{C}$-valued continuous functions. Also, R. K. Singh and W. H. Summers studied in [35] compact composition operators between Banach spaces of continuous functions with values in Banach space. Moreover, H. Takagi studied in [36] and [37] compact composition operator between function algebras and certain subspaces of continuous functions with values in Banach space, respectively. In addition, compact composition operators between Lebesgue spaces (cf. [26], [34, Chapter II], [38]), Hardy spaces (cf.[31], [34, Chapter III], [36]) and Bergman spaces (cf.[21], [39]) were studied.

Note that homomorphisms between Lipschitz algebras is characterized by composition operator. We are lead the following problem:

When is a homomorphism between Lipschitz algebras compact?

In [18], H. Kamowitz and S. Scheinberg solved it as follows:

Theorem D (Kamowitz and Scheinberg, [18]). *Let T be such a unital homomorphism from* $\mathrm{Lip}(X)$ *into* $\mathrm{Lip}(Y)$ *as in Theorem A. T is compact if and only if*

$$\lim_{\substack{y, y' \in Y \\ d_Y(y, y') \to 0}} \frac{d_X(\varphi(y), \varphi(y'))}{d_Y(y, y')} = 0. \tag{1.6}$$

They defined a supercontraction mapping satisfying (1.6) and gave several examples (see [18, p.256]). In [2], F. Botelho and J. Jamison also dealt with an unital homomorphisms from $\mathrm{Lip}(X, \mathbf{c})$ into $\mathrm{Lip}(Y, \mathbf{c})$ or from $\mathrm{Lip}(X, \ell^\infty)$ into $\mathrm{Lip}(Y, \ell^\infty)$. They gave a sufficient condition for them to be compact using the supercontraction property. However, the condition is not necessary. In this paper, we shall give a necessary and sufficient condition for homomorphism from $\mathrm{Lip}(X, C(K))$ into $\mathrm{Lip}(Y, C(M))$ to be compact and get the following result:

Theorem 2 ([11])**.** *Let X, Y, K, M be as in Theorem 1, and T be a homomorphism from $\mathrm{Lip}(X, C(K))$ into $\mathrm{Lip}(Y, C(M))$ with the form* (1.3) *as in Theorem 1. T is compact if and only if the following conditions (iii) and (iv) hold.*

(iii) For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < d_Y(y, y') < \delta \text{ implies } \frac{d_X\big(\varphi(y, \eta), \varphi(y', \eta)\big)}{d_Y(y, y')} < \varepsilon \tag{1.7}$$

for $(y, \eta), (y', \eta) \in \mathcal{D}$.

(iv) For any $y \in Y$, a set $\mathcal{D}_y = \{\eta \in M : (y, \eta) \in \mathcal{D}\}$ is a union of finitely many disjoint clopen sets $\Omega_y^1, \ldots, \Omega_y^{n_y}$ such that

$$\psi_y \text{ is constant on } \Omega_y^i \text{ for } i = 1, \ldots, n_y.$$

Theorem 2 is generalization of [18] (see Corollary 1) and [2].

We say that a bounded operator T from Banach space $\mathcal{A}$ into Banach space $\mathcal{B}$ is weakly compact if $T(\mathbb{B}_{\mathcal{A}})$ is relatively weakly compact in $\mathcal{B}$, where $\mathbb{B}_{\mathcal{A}}$ is a unit ball of $\mathcal{A}$. In generally, if a linear operator between Banach spaces is compact, then it is weakly compact. So weakly compact operator is a generalization of compact operator. Other general theories of weakly compact operators may be found in Megginson's book [22].

Weakly compact composition operators on various function spaces has been studied. For instance, M. Lindstoröm and J. Llavona studied in [19] weakly compact composition operator between Banach space of $\mathbb{C}$-valued continuous functions. Also, R. K. Singh and W. H. Summers studied in [35] weakly compact composition operators between Banach spaces of continuous functions with values in Banach space. Moreover, H. Takagi and J. Wada studied in [40] weakly compact composition operators between certain subspaces of continuous functions with values in Banach space. In addition, weakly compact composition operators between Lebesgue spaces (cf.[26]).

As seen in [40], weakly compact composition operators are sometimes equivalent to compact operators. We are lead the following problem:

When is a weakly compact homomorphism between Lipschitz algebras compact?

In [13], Jiménez-Vargas solved under certain conditions (see Theorem G). In this paper, we shall give conditions for weakly compact homomorphism from $\mathrm{Lip}(X, C(K))$ into $\mathrm{Lip}(Y, C(M))$ to be compact and get the following partial result:

Theorem 3. *Let X, Y, K, M be as in Theorem 1, T be a homomorphism from* $\mathrm{Lip}(X, C(K))$ *into* $\mathrm{Lip}(Y, C(M))$ *with the form (1.3) in Theorem 1. If T is weakly compact, then ψ satisfies the condition (iv) in Theorem 2.*

1.2 Outline of the Thesis

- In chapter 2, we consider the set of all Lipschitz functions of Banach algebra valued. We also give several propositions on Lipschitz functions of continuous functions valued.
- In chapter 3, we prove Theorem 1.
- In chapter 4, we prove Theorem 2.
- In chapter 5, we give several corollaries of Theorem 1 and 2.
- In chapter 6, we prove Theorem 3 and give some remarks.

Chapter 2

Preliminaries

2.1 Basic Definitions and Properties

We recall the definition of Banach algebra.

Definition 2.1.1. *Let $\mathcal{A}$ be an algebra.*

(i) We say that $\mathcal{A}$ is normed algebra *over $\mathbb{C}$ with norm $\|\cdot\|_{\mathcal{A}}$ if $\mathcal{A}$ has the structure of normed linear space over $\mathbb{C}$ and the norm is submultiplicative, that is,*

$$\|fg\|_{\mathcal{A}} \leq \|f\|_{\mathcal{A}}\|g\|_{\mathcal{A}} \qquad (f, g \in \mathcal{A}).$$

(ii) We say that a normed algebra $\mathcal{A}$ is Banach algebra *over $\mathbb{C}$ with norm $\|\cdot\|_{\mathcal{A}}$ if it is complete respect to the norm $\|\cdot\|_{\mathcal{A}}$.*

(iii) Let $\mathcal{A}$ be a normed algebra over $\mathbb{C}$ with norm $\|\cdot\|_{\mathcal{A}}$. We say that $e \in \mathcal{A}$ is unit *if $ae = ea = a$ for all $a \in \mathcal{A}$ and $\|e\|_{\mathcal{A}} = 1$.*

(iv) We say that $\mathcal{A}$ is commutative if $ab = ba$ for all $a, b \in \mathcal{A}$.

(v) We say that Banach algebra $\mathcal{A}$ is semi-simple if radical of $\mathcal{A}$ is $\{o\}$.

Example 2.1.1. $\mathbb{C}$ is commutative Banach algebra with unit with respect to the absolute value.

Example 2.1.2. The set of all $\mathbb{C}$-valued convergent sequences $\mathbf{c}$ is unital commutative Banach algebra with pointwise operation and norm

$$\|a\|_{C(K)} = \sup_{n\in\mathbb{N}} |a_n| \qquad \big(a = \{a_n\}_{n=1}^{\infty} \in \mathbf{c}\big).$$

Example 2.1.3. Let K be a compact Hausdorff space. $C(K)$ is unital commutative Banach algebra with pointwise operation and norm

$$\|u\|_{C(K)} = \sup_{\xi \in K} |u(\xi)| \qquad \big(u \in C(K)\big).$$

Example 2.1.4. Let $\mathscr{H}$ be a Hilbert space over $\mathbb{C}$. $\mathcal{B}(\mathscr{H})$ denotes the set of all bounded operators between $\mathscr{H}$. $\mathcal{B}(\mathscr{H})$ is unital non-commutative Banach algebra with norm

$$\|T\|_{\mathcal{B}(\mathscr{H})} = \sup_{\substack{x \in \mathscr{H} \\ x \neq o}} \frac{\|Tx\|_{\mathscr{H}}}{\|x\|_{\mathscr{H}}} \qquad \big(T \in \mathcal{B}(\mathscr{H})\big).$$

Next we define Lipschitz functions.

Definition 2.1.2. *Let X be a compact metric space with metric d_X and $\mathcal{A}$ be a Banach algebra over $\mathbb{C}$ with norm $\|\cdot\|_{\mathcal{A}}$.*

(i) If an $\mathcal{A}$-valued function f on X satisfies

$$\mathcal{L}_{X,\mathcal{A}}(f) = \sup_{\substack{x,x' \in X \\ x \neq x'}} \frac{\|f(x) - f(x')\|_{\mathcal{A}}}{d_X(x,x')} < \infty,$$

then we say that f is Lipschitz and $\mathcal{L}_{X,\mathcal{A}}(f)$ is called a Lipschitz constant of f.

(ii) $\mathrm{Lip}(X,\mathcal{A})$ *denotes the set of all $\mathcal{A}$-valued Lipschitz functions on X.*

(iii) In case that $\mathcal{A} = \mathbb{C}$, we simply write $\mathrm{Lip}(X,\mathbb{C}) = \mathrm{Lip}(X)$.

Next proposition shows basic properties of $\mathrm{Lip}(X,\mathcal{A})$.

Proposition 2.1.1. *Let X be a compact metric space with metric d_X and $\mathcal{A}$ be a Banach algebra over $\mathbb{C}$ with norm $\|\cdot\|_{\mathcal{A}}$.*

(i) $\mathrm{Lip}(X,\mathcal{A}) \subset C(X,\mathcal{A})$*, where $C(X,\mathcal{A})$ is the Banach algebra of all $\mathcal{A}$-valued continuous functions on X with pointwise operations and norm*

$$\|f\|_{C(X,\mathcal{A})} = \sup_{x \in X} \|f(x)\|_{\mathcal{A}} \qquad \big(f \in C(X,\mathcal{A})\big).$$

(ii) $\mathrm{Lip}(X,\mathcal{A})$ *is the Banach algebra over $\mathbb{C}$ with pointwise operations and norm*

$$\|f\|_{\mathrm{Lip}(X,\mathcal{A})} = \|f\|_{C(X,\mathcal{A})} + \mathcal{L}_{X,\mathcal{A}}(f) \qquad \big(f \in \mathrm{Lip}(X,\mathcal{A})\big).$$

(iii) *If $\mathcal{A}$ is commutative, so is* $\mathrm{Lip}(X, \mathcal{A})$.

(iv) *If $\mathcal{A}$ has unit, so does* $\mathrm{Lip}(X, \mathcal{A})$.

We give an example of function in $\mathrm{Lip}(X)$. Put

$$\mathrm{diam}(X) = \sup_{x,x' \in X} d_X(x, x').$$

Note that $\mathrm{diam}(X) < \infty$ if X is compact.

Proposition 2.1.2. *Let $x_0 \in X$. A function $f : X \to \mathbb{C}$ defined by*

$$f(x) = d_X(x, x_0) \qquad (x \in X)$$

enjoys $f \in \mathrm{Lip}(X)$ and $\|f\|_{\mathrm{Lip}(X)} \leq \mathrm{diam}(X) + 1$.

Proof. For any $x, x' \in X$ with $x \neq x'$, by the triangle inequality, we have

$$|f(x) - f(x')| = |d(x, x_0) - d(x', x_0)| \leq d_X(x, x').$$

Thus we have

$$\mathcal{L}_{X,\mathbb{C}}(f) \leq 1.$$

Since

$$\|f\|_{C(X)} = \sup_{x \in X} |f(x)| = \sup_{x \in X} d_X(x, x_0) \leq \mathrm{diam}(X),$$

we have

$$\|f\|_{\mathrm{Lip}(X)} = \|f\|_{C(X)} + \mathcal{L}_{X,\mathbb{C}}(f) \leq \mathrm{diam}(X) + 1.$$

□

We recall the definition of homomorphism.

Definition 2.1.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras with unit $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$, respectively, and T be a mapping from $\mathcal{A}$ into $\mathcal{B}$.*

(i) *We say that T is* homomorphism *if it preserves addition, scalar multiplication and multiplication.*

(ii) *We say that T is* unital *if $Te_{\mathcal{A}} = e_{\mathcal{B}}$.*

The following theorem is well known.

Theorem E ([29])**.** *Let $\mathcal{A}$ be a Banach algebra and $\mathcal{B}$ be a semi-simple commutative Banach algebra. If T is a homomorphism from $\mathcal{A}$ into $\mathcal{B}$, then T is continuous.*

2.2 Properties of Lipschitz Algebras

In this section, we give several basic properties of $\mathrm{Lip}(X, C(K))$ to prove Theorem 1 and Theorem 2.

Since $C(K)$ is a commutative Banach algebra with unit, so is $\mathrm{Lip}(X, C(K))$.

We shall regard functions in $\mathrm{Lip}(X, C(K))$ as a function on $X \times K$.

Proposition 2.2.1. *Let f be a $\mathbb{C}$-valued function on $X \times K$. Then $f \in \mathrm{Lip}(X, C(K))$ if and only if $f \in C(X \times K)$ and*

$$\mathcal{L}_{X,C(K)}(f) = \sup_{\substack{x,x' \in X \\ x \neq x'}} \frac{\|f_x - f_{x'}\|_{C(K)}}{d_X(x, x')} < \infty. \tag{2.1}$$

Proof. Straightforward. □

Proposition 2.2.2. *For any $(x_0, \xi_0) \in X \times K$ and any open neighborhood $\mathcal{U}$ of (x_0, ξ_0), there exists an $f \in \mathrm{Lip}(X, C(K))$ and $m > 0$ such that $0 \leq f \leq 1$, $f(x_0, \xi_0) = 1$ and $f(x, \xi) \leq m < 1$ for all $(x, \xi) \in (X \times K) \setminus \mathcal{U}$.*

Proof. Let $(x_0, \xi_0) \in X \times K$ and let $\mathcal{U}$ be an open neighborhood of (x_0, ξ_0). Then there exists an open neighborhood U of x_0 in X and an open neighborhood Θ of ξ_0 in K such that $(x_0, \xi_0) \in U \times \Theta \subset \mathcal{U}$.

Let h be a function on X defined by

$$h(x) = 1 - \frac{d_X(x, x_0)}{\mathrm{diam}(X)} \qquad (x \in X).$$

We easily see that $h \in \mathrm{Lip}(X)$, $0 \leq h \leq 1$, $h(x_0) = 1$ and $h(x) < 1$ for all $x \in X \setminus \{x_0\}$.

By Urysohn's lemma, there exists an $u \in C(K)$ such that $0 \leq u \leq 1$, $u(\xi_0) = 1$ and $u(\xi) = 0$ for all $\xi \in K \setminus \Theta$.

Put $f(x, \xi) = h(x)u(\xi)$ for $(x, \xi) \in X \times K$. Then we can verify that f has the desired properties. Here m may be taken as the maximum of f on the compact set $(X \times K) \setminus \mathcal{U}$. □

Next, we consider properties of $\mathrm{Lip}(X, C(K))$ as a Banach algebra. We denote by $\mathfrak{M}_{\mathrm{Lip}(X,C(K))}$ the maximal ideal space of $\mathrm{Lip}(X, C(K))$, that is, the set of all multiplicative linear functionals on $\mathrm{Lip}(X, C(K))$.

Definition 2.2.1. *Let* $(x,\xi) \in X \times K$*. We define the evaluation functional at* (x,ξ) $\tau_{(x,\xi)} \in \mathfrak{M}_{\mathrm{Lip}(X,C(K))}$ *by*

$$\tau_{(x,\xi)}(f) = f(x,\xi) \qquad \big(f \in \mathrm{Lip}(X, C(K))\big).$$

Proposition 2.2.3. *The Banach algebra* $\mathrm{Lip}(X, C(K))$ *is semi-simple.*

Proof. Let $\hat{f}$ be the Gel'fand representation of f. To show the proposition, by [3, p.28], it is enough to show $\hat{f} = 0$ implies $f = 0$.

Now assume $\hat{f}(\Lambda) = 0$ for any $\Lambda \in \mathfrak{M}_{\mathrm{Lip}(X,C(K))}$ and take $(x,\xi) \in X \times K$. Since $\tau_{(x,\xi)} \in \mathfrak{M}_{\mathrm{Lip}(X,C(K))}$, it is clear

$$f(x,\xi) = \tau_{(x,\xi)}(f) = \hat{f}\big(\tau_{(x,\xi)}\big) = 0.$$

□

Here we define a mapping τ from $X \times K$ into $\mathfrak{M}_{\mathrm{Lip}(X,C(K))}$ by

$$\tau(x,\xi) = \tau_{(x,\xi)} \qquad \big((x,\xi) \in X \times K\big).$$

Then we have the following proposition:

Proposition 2.2.4. τ *is bijective from* $X \times K$ *to* $\mathfrak{M}_{\mathrm{Lip}(X,C(K))}$.

Proof. We first show that τ is injective. Let $(x,\xi), (x',\xi') \in X \times K$ with $(x,\xi) \neq (x',\xi')$. Put $\mathcal{U} = (X \times K) \smallsetminus \{(x',\xi')\}$. Since $\mathcal{U}$ is open neighborhood of (x,ξ), by Proposition 2.2.2, there exists an $f \in \mathrm{Lip}(X, C(K))$ such that

$$f(x,\xi) = 1 \text{ and } f(x',\xi') < 1.$$

Since

$$\tau_{(x',\xi')}(f) = f(x',\xi') < 1 = f(x,\xi) = \tau_{(x,\xi)}(f),$$

we have $\tau_{(x,\xi)} \neq \tau_{(x',\xi')}$.

Next we show that τ is surjective. Let us assume that τ is not surjective. Then it is easy to see that there exists $\Lambda \in \mathfrak{M}_{\mathrm{Lip}(X,C(K))} \smallsetminus \tau(X \times K)$ such that for any $(x,\xi) \in X \times K$ and $f_{(x,\xi)} \in \mathrm{Lip}(X, C(K))$ satisfying

$$\tau_{(x,\xi)}(f_{(x,\xi)}) \neq 0 \text{ and } \Lambda(f_{(x,\xi)}) = 0. \tag{2.2}$$

Put

$$\mathcal{U}_{(x,\xi)} = \big\{(y,\eta) \in X \times K \,:\, f_{(x,\xi)}(y,\eta) \neq 0\big\}.$$

By Proposition 2.2.1, $\mathcal{U}_{(x,\xi)}$ is open neighborhood of (x,ξ).

Note that

$$X \times K = \bigcup_{(x,\xi)\in X\times K} \mathcal{U}_{(x,\xi)}$$

and $X \times K$ is compact. We can select finitely many $(x_1,\xi_1),\dots,(x_n,\xi_n) \in X \times K$ such that

$$X \times K = \bigcup_{i=1}^{n} \mathcal{U}_{(x_i,\xi_i)}. \tag{2.3}$$

For any $g \in \mathrm{Lip}(X, C(K))$, we have $\overline{g} \in \mathrm{Lip}(X, C(K))$ where $\overline{g}(x,\xi) = \overline{g(x,\xi)}$. Put

$$g(x,\xi) = \sum_{i=1}^{n} \left| f_{(x_i,\xi_i)}(x,\xi) \right|^2 \qquad \big((x,\xi) \in X \times K\big).$$

Since $g = \sum_{i=1}^{n} f_{(x_i,\xi_i)} \overline{f_{(x_i,\xi_i)}}$ and $f_{(x_1,\xi_1)},\dots,f_{(x_n,\xi_n)} \in \mathrm{Lip}(X, C(K)$, we have $g \in \mathrm{Lip}(X, C(K))$. By (2.2), we have

$$\Lambda(g) = \sum_{i=1}^{n} \Lambda(f_{(x_i,\xi_i)}) \Lambda(\overline{f_{(x_i,\xi_i)}}) = 0.$$

However, by Proposition 2.2.1, there exists a point $(x_0,\xi_0) \in X \times K$ such that

$$0 \le g(x_0,\xi_0) \le g(x,\xi) \qquad \big((x,\xi) \in X \times K\big).$$

By (2.3), there exists $i \in \{1,\dots n\}$ such that $(x_0,\xi_0) \in \mathcal{U}_{(x_i,\xi_i)}$. Since $f_{(x_i,\xi_i)}(x_0,\xi_0) \neq 0$, we have

$$g(x,\xi) = \sum_{i=1}^{n} \left| f_{(x_i,\xi_i)}(x,\xi) \right|^2 \ge \left| f_{(x_i,\xi_i)}(x_0,\xi_0) \right|^2 > 0.$$

Set $\frac{1}{g}(x,\xi) = \frac{1}{g(x,\xi)}$. Then $\frac{1}{g} \in C(X \times K)$, and we have

$$\begin{aligned}
\sup_{\substack{x,x' \in X \\ x \neq x'}} \frac{\left\| \left(\frac{1}{g}\right)_x - \left(\frac{1}{g}\right)_{x'} \right\|_{C(K)}}{d_X(x,x')} &= \sup_{\substack{x,x' \in X \\ x \neq x'}} \sup_{\xi \in K} \frac{\left| \frac{1}{g(x,\xi)} - \frac{1}{g(x',\xi)} \right|}{d_X(x,x')} \\
&= \sup_{\substack{x,x' \in X \\ x \neq x'}} \sup_{\xi \in K} \frac{\left| \frac{g(x,\xi) - g(x',\xi)}{g(x,\xi) g(x',\xi)} \right|}{d_X(x,x')} \\
&\leq \frac{1}{g(x_0,\xi_0)^2} \sup_{\substack{x,x' \in X \\ x \neq x'}} \frac{\|g_x - g_{x'}\|_{C(K)}}{d_X(x,x')} \\
&= \frac{1}{g(x_0,\xi_0)^2} \mathcal{L}_{X,C(K)}(g) < \infty.
\end{aligned}$$

By Proposition 2.2.1, we have $\frac{1}{g} \in \mathrm{Lip}(X, C(K))$. Since $g \cdot \frac{1}{g} = \mathbf{1}$, where $\mathbf{1}$ is unit in $\mathrm{Lip}(X, C(K))$, we have

$$\Lambda(g)\Lambda\left(\frac{1}{g}\right) = \Lambda\left(g \cdot \frac{1}{g}\right) = \Lambda(\mathbf{1}) = 1.$$

This is a contradiction. $\square$

Chapter 3

Proof of Theorem 1

In this chapter, we prove Theorem 1.

3.1 Proof of Sufficiency

We first prove the sufficiency part of Theorem 1 under the following conditions:

(A-1) There exists a clopen subset $\mathcal{D}$ of $Y \times M$.

(A-2) There exist two continuous mappings $\varphi : \mathcal{D} \to X$ and $\psi : \mathcal{D} \to K$

(A-3) T is of the form (1.3).

(A-4) φ satisfies the condition (i).

(A-5) ψ satisfies the condition (ii).

Lemma 3.1.1. *Suppose (A-1)–(A3). Then, for any* $f \in \mathrm{Lip}(X, C(K))$*,* $Tf \in C(Y \times M)$

Proof. Take f from $\mathrm{Lip}(X, C(K))$. By proposition 2.2.1, we have $f \in C(X \times K)$. Therefore, by condition (A-2) and (A-3),

$$(Tf)(y, \eta) = f\big(\varphi(y, \eta), \psi(y, \eta)\big) \qquad \big((y, \eta) \in \mathcal{D}\big)$$

is continuous on $\mathcal{D}$. Since

$$(Tf)(y, \eta) = 0 \qquad \big((y, \eta) \in (Y \times M) \setminus \mathcal{D}\big),$$

Tf is continuous on $(Y \times M) \setminus \mathcal{D}$. Noting that $\mathcal{D}$ is clopen, we see that Tf is continuous on $Y \times M$. □

Lemma 3.1.2. *Suppose (A-1). Then*

$$\inf\{d_Y(y,y') \,:\, (y,\eta)\in\mathcal{D} \text{ and } (y',\eta)\in(Y\times M)\setminus\mathcal{D} \text{ for some } \eta\in M\}>0,$$

might be infinity if there are no such pairs (y,η) *and* (y',η).

We shall denote by ρ the above positive value.

Proof. Assume $\rho=0$. Then for each $n=1,2,\ldots$, there exist $(y_n,\eta_n)\in\mathcal{D}$ and $(y'_n,\eta_n)\in(Y\times M)\setminus\mathcal{D}$ such that $d_Y(y_n,y'_n)<1/n$. Since $\mathcal{D}$ is compact, there exist a subnet $\{(y_{n_\alpha},\eta_{n_\alpha})\}$ of $\{(y_n,\eta_n)\}$ and $(y,\eta)\in\mathcal{D}$ such that $y_{n_\alpha}\to y$ and $\eta_{n_\alpha}\to\eta$. Then $d_Y(y_{n_\alpha},y'_{n_\alpha})<1/n_\alpha\to 0$. Hence $y_{n_\alpha}\to y$ and so $(y'_{n_\alpha},\eta_{n_\alpha})\to(y,\eta)$. Since $(Y\times M)\setminus\mathcal{D}$ is closed, we have $(y,\eta)\in(Y\times M)\setminus\mathcal{D}$. This contradict the fact $(y,\eta)\in\mathcal{D}$. □

Lemma 3.1.3. *Suppose (A-1)–(A-5). Then, for any* $f\in\mathrm{Lip}(X,C(K))$, $Tf\in\mathrm{Lip}(Y,C(M))$.

Proof. By Lemma 3.1.1, we have $Tf\in C(Y\times M)$. By proposition 2.2.1, it is sufficient to show

$$\mathcal{L}_{Y,C(M)}(Tf)=\sup_{\substack{y,y'\in Y\\ y\neq y'}}\frac{\|(Tf)_y-(Tf)_{y'}\|_{C(M)}}{d_Y(y,y')}<\infty. \tag{3.1}$$

To this end, choose $y,y'\in Y$ with $y\neq y'$ and let $\eta\in M$. We consider three cases.

[Case 1] $(y,\eta),(y',\eta)\in\mathcal{D}$: By (ii), we can choose $i,i'\in\{1,\ldots,n_\eta\}$ for $y\in V_i^\eta$ and $y'\in V_{i'}^\eta$. We first consider the case $i=i'$. Then $y,y'\in V_i^\eta$. Since ψ^η is constant on V_i^η, $\psi(y,\eta)=\psi^\eta(y)=\psi^\eta(y')=\psi(y',\eta)$. Put $x=\varphi(y,\eta)$, $x'=\varphi(y',\eta)$ and $\xi=\psi(y,\eta)$. Using (1.3), we compute

$$\begin{aligned}|(Tf)(y,\eta)-(Tf)(y',\eta)| &= |f(\varphi(y,\eta),\psi(y,\eta))-f(\varphi(y',\eta),\psi(y',\eta))|\\ &= |f(x,\xi)-f(x',\xi)|\\ &= |f_x(\xi)-f_{x'}(\xi)|\\ &\le \|f_x-f_{x'}\|_{C(K)}\\ &\le \mathcal{L}_{X,C(K)}(f)\,d_X(x,x')\\ &= \mathcal{L}_{X,C(K)}(f)\,d_X(\varphi(y,\eta),\varphi(y',\eta))\\ &\le \mathcal{L}_{X,C(K)}(f)\,L\,d_Y(y,y'),\end{aligned} \tag{3.2}$$

where the fifth and last lines follow from (2.1) and (1.4), respectively.

On the other hand, if $i \neq i'$, then (1.5) yields $d_Y(y, y') \geq d_Y(V_i^\eta, V_{i'}^\eta) \geq r$. Hence

$$\frac{\left|(Tf)(y,\eta) - (Tf)(y',\eta)\right|}{d_Y(y,y')} \leq \frac{\left|(Tf)(y,\eta)\right| + \left|(Tf)(y',\eta)\right|}{r} \leq \frac{2\,\|f\|_{C(X\times K)}}{r}. \tag{3.3}$$

[Case 2] $(y, \eta) \in \mathcal{D}$ and $(y', \eta) \in (Y \times K) \smallsetminus \mathcal{D}$: Then Lemma 3.1.2 asserts that $d_Y(y, y') \geq \rho > 0$. By (1.3),

$$\frac{\left|(Tf)(y,\eta) - (Tf)(y',\eta)\right|}{d_Y(y,y')} \leq \frac{\left|f(\varphi(y,\eta), \psi(y,\eta)) - 0\right|}{\rho} \leq \frac{\|f\|_{C(X\times K)}}{\rho}. \tag{3.4}$$

[Case 3] $(y, \eta), (y', \eta) \in (Y \times K) \smallsetminus \mathcal{D}$: By (1.3),

$$(Tf)(y,\eta) - (Tf)(y',\eta) = 0. \tag{3.5}$$

Combining (3.2)–(3.5), we can arrive at (3.1). By setting

$$C = \max\left\{L, \frac{2}{r}, \frac{1}{\rho}\right\},$$

we obtain

$$\mathcal{L}_{Y,C(M)}(Tf) = \sup_{\substack{y,y'\in Y\\ y\neq y'}} \sup_{\eta\in M} \frac{\left|(Tf)(y,\eta) - (Tf)(y',\eta)\right|}{d_Y(y,y')} \leq C\,\|f\|_{\mathrm{Lip}(X,C(K))} \tag{3.6}$$

since $\mathcal{L}_{X,C(K)}(f) \leq \|f\|_{\mathrm{Lip}(X,C(K))}$ and $\|f\|_{C(X\times K)} \leq \|f\|_{\mathrm{Lip}(X,C(K))}$. □

Lemma 3.1.3 says that T maps $\mathrm{Lip}(X, C(K))$ into $\mathrm{Lip}(Y, C(M))$ if the conditions (A-1)–(A-5) are satisfied.

Finally, we show following lemma:

Lemma 3.1.4. *Suppose (A-1)–(A-5). Then T is a homomorphism.*

Proof. Let $f, g \in \mathrm{Lip}(X, C(K))$. By (1.3), if $(y, \eta) \in \mathcal{D}$, then we have

$$\begin{aligned}\big(T(f+g)\big)(y,\eta) &= (f+g)(\varphi(y,\eta), \psi(y,\eta)) \\ &= f(\varphi(y,\eta), \psi(y,\eta)) + g(\varphi(y,\eta), \psi(y,\eta)) \\ &= (Tf)(y,\eta) + (Tg)(y,\eta) = (Tf+Tg)(y,\eta).\end{aligned}$$

If $(y, \eta) \in (Y \times M) \smallsetminus \mathcal{D}$, then we have

$$\big(T(f+g)\big)(y,\eta) = 0 = (Tf)(y,\eta) + (Tg)(y,\eta) = (Tf+Tg)(y,\eta).$$

Therefore we have

$$T(f+g) = Tf + Tg.$$

Similarly, if $f, g \in \mathrm{Lip}(X, C(K))$ and $\alpha \in \mathbb{C}$, then we have

$$T(\alpha f) = \alpha(Tf), \quad T(fg) = (Tf)(Tg).$$

□

3.2 Proof of Necessity

We turn to the proof of the necessity part of Theorem 1, that is, if T is homomorphism, then the conditions (A-1)–(A-5) in previous section are satisfied. This is the main part of the proof of Theorem 1. Suppose that T is a homomorphism from $\mathrm{Lip}(X, C(K))$ into $\mathrm{Lip}(Y, C(M))$. Since $\mathrm{Lip}(Y, C(M))$ is semi-simple by Proposition 2.2.3, T is continuous.

If $T = O$, then (1.3) is trivial by taking $\mathcal{D} = \emptyset$. So, we assume that $T \neq O$.

Lemma 3.2.1. *There exist a clopen subset $\mathcal{D}$ of $Y \times M$ and two mappings $\varphi : \mathcal{D} \to X$ and $\psi : \mathcal{D} \to K$ such that* (1.3) *holds.*

Proof. Since $(T\mathbf{1})(T\mathbf{1}) = T(\mathbf{1} \cdot \mathbf{1}) = T\mathbf{1}$, we have $(T\mathbf{1})(y, \eta) \in \{1, 0\}$ for all $(y, \eta) \in Y \times M$. Put

$$\mathcal{D} = \{(y, \eta) \in Y \times M : (T\mathbf{1})(y, \eta) = 1\}. \tag{3.7}$$

Then

$$(Y \times M) \setminus \mathcal{D} = \{(y, \eta) \in Y \times M : (T\mathbf{1})(y, \eta) = 0\}.$$

Since $T\mathbf{1}$ is continuous on $Y \times M$, both $\mathcal{D}$ and $(Y \times M) \setminus \mathcal{D}$ are closed. Hence $\mathcal{D}$ is clopen.

Let $(y, \eta) \in \mathcal{D}$ be arbitrary. Define the functional Λ on $\mathrm{Lip}(X, C(K))$ by

$$\Lambda(f) = (Tf)(y, \eta) \qquad (f \in \mathrm{Lip}(X, C(K))).$$

Let $f, g \in \mathrm{Lip}(X, C(K))$. Since T is homomorphism, we have

$$\begin{aligned}\Lambda(f+g) &= \big(T(f+g)\big)(y, \eta) \\ &= \big(Tf + Tg\big)(y, \eta) = (Tf)(y, \eta) + (Tg)(y, \eta) = \Lambda(f) + \Lambda(g).\end{aligned}$$

Similarly, if $f, g \in \mathrm{Lip}(X, C(K))$ and $\alpha \in \mathbb{C}$, then we have

$$\Lambda(\alpha f) = \alpha\Lambda(f), \quad \Lambda(fg) = \big(\Lambda(f)\big)\big(\Lambda(g)\big).$$

In addition,

$$\Lambda(\mathbf{1}) = (T\mathbf{1})(y, \eta) = 1.$$

Hence we have $\Lambda \in \mathfrak{M}_{\mathrm{Lip}(X,C(K))}$. By Proposition 2.2.4, there exists a unique point $(x, \xi) \in X \times K$ such that

$$\Lambda = \tau_{(x,\xi)}.$$

Put $\varphi(y, \eta) = x$ and $\psi(y, \eta) = \xi$. We determine the mappings $\varphi : \mathcal{D} \to X$ and $\psi : \mathcal{D} \to K$. Then, for any $f \in \mathrm{Lip}(X, C(K))$,

$$(Tf)(y, \eta) = \Lambda(f) = \tau_{(x,\xi)}(f) = f(x, \xi) = f\big(\varphi(y, \eta), \psi(y, \eta)\big). \tag{3.8}$$

Finally, if $(y, \eta) \in (Y \times M) \setminus \mathcal{D}$, then $(T\mathbf{1})(y, \eta) = 0$. Thus, for any $f \in \mathrm{Lip}(X, C(K))$, $Tf = T(f\mathbf{1}) = (Tf)(T\mathbf{1})$ and so

$$(Tf)(y, \eta) = (Tf)(y, \eta)\,(T\mathbf{1})(y, \eta) = 0.$$

Together with (3.8), we establish (1.3). □

Lemma 3.2.2. *The mappings $\varphi : \mathcal{D} \to X$ and $\psi : \mathcal{D} \to K$ are continuous.*

Proof. Define the mapping $\Phi : \mathcal{D} \to X \times K$ by

$$\Phi(y, \eta) = \big(\varphi(y, \eta), \psi(y, \eta)\big) \qquad \big((y, \eta) \in \mathcal{D}\big).$$

We prove the lemma by verifying that Φ is continuous at each point $(y_0, \eta_0) \in \mathcal{D}$. Let $\mathcal{U}$ be an arbitrary open neighborhood of $\Phi(y_0, \eta_0)$ in $X \times K$. By Proposition 2.2.2, there exist an $f \in \mathrm{Lip}(X, C(K))$ and $m > 0$ such that $0 \le f \le 1$, $f\big(\Phi(y_0, \eta_0)\big) = 1$ and

$$0 \le f(x, \xi) \le m < 1. \tag{3.9}$$

Put

$$\mathcal{V} = \{(y, \eta) \in \mathcal{D} \,:\, |(Tf)(y, \eta)| > m\}.$$

Since Tf is continuous on $Y \times M$, $\mathcal{V}$ is open. Also, $(y_0, \eta_0) \in \mathcal{V}$ because

$$(Tf)(y_0, \eta_0) = f\big(\varphi(y_0, \eta_0), \psi(y_0, \eta_0)\big) = f(\Phi(y_0, \eta_0)) = 1 > m.$$

Moreover, if $(y, \eta) \in \mathcal{V}$, then

$$|f(\Phi(y, \eta))| = |f\big(\varphi(y, \eta), \psi(y, \eta)\big)| = |(Tf)(y, \eta)| > m.$$

and (3.9) forces that $\Phi(y, \eta) \in \mathcal{U}$. Hence $\Phi(\mathcal{V}) \subset \mathcal{U}$. Thus Φ is continuous at (y_0, η_0), as desired. □

Lemma 3.2.3. *φ satisfies the condition (i).*

Proof. Let $(y,\eta),(y',\eta)\in\mathcal{D}$ with $y\neq y'$. Put $x_0=\varphi(y',\eta)$ and

$$f(x)=d_X(x,x_0)\qquad (x\in X).$$

By Proposition 2.1.2, $f\in \mathrm{Lip}(X)$ and $\|f\|_{\mathrm{Lip}(X)}\leq \mathrm{diam}(X)+1$. Define the function $\check{f}$ on $X\times K$ by

$$\check{f}(x,\xi)=f(x)\qquad \big((x,\xi)\in X\times K\big).$$

Clearly $\check{f}\in\mathrm{Lip}(X,C(K))$ and $\|\check{f}\|_{\mathrm{Lip}(X,C(K))}=\|f\|_{\mathrm{Lip}(X)}$. Moreover, we have

$$\begin{aligned}
d_X\big(\varphi(y,\eta),\varphi(y',\eta)\big) &= \big|d_X\big(\varphi(y,\eta),x_0\big)-d_X\big(\varphi(y',\eta),x_0\big)\big|\\
&= \big|f\big(\varphi(y,\eta)\big)-f\big(\varphi(y',\eta)\big)\big|\\
&= \big|\check{f}\big(\varphi(y,\eta),\psi(y,\eta)\big)-\check{f}\big(\varphi(y',\eta),\psi(y',\eta)\big)\big|\\
&= \big|(T\check{f})(y,\eta)-(T\check{f})(y',\eta)\big|\\
&= \big|(T\check{f})_y(\eta)-(T\check{f})_{y'}(\eta)\big|\\
&\leq \big\|(T\check{f})_y-(T\check{f})_{y'}\big\|_{C(M)}\\
&\leq \mathcal{L}_{Y,C(M)}(T\check{f})\, d_Y(y,y').
\end{aligned}$$

Since

$$\mathcal{L}_{Y,C(M)}(T\check{f})\leq \|T\check{f}\|_{\mathrm{Lip}(Y,C(M))}\leq \|T\|\,\|\check{f}\|_{\mathrm{Lip}(X,C(K))}\leq \|T\|\big(\mathrm{diam}(X)+1\big),$$

we obtain

$$\frac{d_X\big(\varphi(y,\eta),\varphi(y',\eta)\big)}{d_Y(y,y')}\leq \|T\|\big(\mathrm{diam}(X)+1\big).$$

By setting $L=\|T\|\big(\mathrm{diam}(X)+1\big)$, we obtain (i). □

Lemma 3.2.4. *If $(y,\eta),(y',\eta)\in\mathcal{D}$ satisfies $d_Y(y,y')<1/\|T\|$, then we have $\psi^\eta(y)=\psi^\eta(y')$.*

Proof. Take so that $0<r\leq 1/\|T\|$. Choose $(y,\eta),(y',\eta)\in\mathcal{D}$ with $d_Y(y,y')<r$ and assume that $\psi^\eta(y)\neq\psi^\eta(y')$. By Urysohn's lemma, there exists a function $u\in C(K)$ such that $0\leq u\leq 1$, $u(\psi^\eta(y))=1$ and $u(\psi^\eta(y'))=0$. Define the function $\tilde{u}:X\times K\to\mathbb{C}$ by

$$\tilde{u}(x,\xi)=u(\xi)\qquad \big((x,\xi)\in X\times K\big).$$

Clearly and $\|\tilde{u}\|_{\mathrm{Lip}(X,C(K))} = \|u\|_{C(K)} = 1$. Moreover, we have

$$\begin{aligned}
1 &= |u(\psi^\eta(y)) - u(\psi^\eta(y'))| \\
&= |u(\psi(y,\eta)) - u(\psi(y',\eta))| \\
&= |\tilde{u}(\varphi(y,\eta),\psi(y,\eta)) - \tilde{u}(\varphi((y',\eta)),\psi(y',\eta))| \\
&= |(T\tilde{u})(y,\eta) - (T\tilde{u})(y',\eta)| \\
&= |(T\tilde{u})_y(\eta) - (T\tilde{u})_{y'}(\eta)| \\
&\le \|(T\tilde{u})_y - (T\tilde{u})_{y'}\|_{C(M)} \\
&\le \mathcal{L}_{Y,C(M)}(T\tilde{u})\, d_Y(y,y') \\
&< \|T\tilde{u}\|_{\mathrm{Lip}(X,C(K))}\, r \\
&\le \|T\|\,\|u\|_{C(K)}\, r = \|T\|\, r \le 1,
\end{aligned}$$

which is a contradiction. Hence $\psi^\eta(y) = \psi^\eta(y')$. □

Lemma 3.2.5. *ψ satisfies the condition (ii).*

Proof. Fix any $\eta \in M$ and put

$$\mathcal{D}^\eta = \{y \in Y \ : \ (y,\eta) \in \mathcal{D}\}.$$

Since $\mathcal{D}$ is clopen, $\mathcal{D}^\eta$ is a clopen subset of Y. For any $y \in \mathcal{D}^\eta$, put

$$V_y = \{z \in \mathcal{D}^\eta \ : \ \psi^\eta(z) = \psi^\eta(y)\}. \tag{3.10}$$

Put $c = \psi^\eta(y)$. If $z \in V_y$, then we have

$$\psi^\eta(z) = \psi^\eta(y) = c.$$

Therefore ψ^η is constant on V_y. Also, we have

$$V_y \cap V_{y'} \neq \emptyset \implies V_y = V_{y'}. \tag{3.11}$$

Since ψ^η is continuous by Lemma 3.2.2, V_y is closed subset of $\mathcal{D}^\eta$. To see that V_y is open subset of $\mathcal{D}^\eta$, let $z \in V_y$ and consider an r-ball

$$B(r;z) = \{w \in \mathcal{D}^\eta \ : \ d_Y(w,z) < r\},$$

where $r \le 1/\|T\|$ is given in Lemma 3.2.4. If $w \in B(r;z)$, then $(w,\eta),(z,\eta) \in \mathcal{D}$ and $d_Y(w,z) < r$. Hence Lemma 3.2.4 implies that

$$\psi^\eta(w) = \psi^\eta(z) = \psi^\eta(y),$$

so that $w \in V_y$. Therefore $B(r; z) \subset V_y$. Thus V_y is an open subset of $\mathcal{D}^\eta$. Consequently, V_y is a clopen subset of $\mathcal{D}^\eta$.

Note that,

$$\mathcal{D}^\eta = \bigcup_{y \in \mathcal{D}^\eta} V_y.$$

Since $\mathcal{D}^\eta$ is compact, we can select finitely many $y_1, \ldots, y_n \in \mathcal{D}^\eta$ such that

$$\mathcal{D} = \bigcup_{i=1}^{n} V_{y_i}.$$

By (3.11), we may assume that $V_{y_1}, \ldots, V_{y_n}$ are disjoint.

To complete the proof, we show that $d_Y\big(V_{y_i}, V_{y_j}\big) \geq r$ $(i \neq j)$. Assume that $d_Y\big(V_{y_i}, V_{y_j}\big) < r$. Then there exist $z_i \in V_{y_i}$ and $z_j' \in V_{y_j}$ such that $d_Y(z_i, z_j) < r$. By Lemma 3.2.4, $\psi^\eta(z_i) = \psi^\eta(z_j)$, and hence (3.10) and (3.11) yield $V_{y_i} = V_{y_j}$. Since $V_{y_1} \ldots, V_{y_n}$ are disjoint, we must have $d_Y\big(V_{y_i}, V_{y_j}\big) \geq r$ $(i \neq j)$.

By setting $n_\eta = n$ and writing V_i^η for V_i $(i = 1, \ldots n_\eta)$, we obtain (ii). □

In Lemma 3.2.1 and Lemma 3.2.2, we have shown that the conditions (A-1)–(A-3) are satisfied. Also, in Lemma 3.2.3 and Lemma 3.2.5, we have shown that the conditions (A-4) and (A-5) are satisfied. Thus the proof of Theorem 1 is completed.

Chapter 4

Proof of Theorem 2

In this chapter, we prove Theorem 2. Through this chapter, T is assumed to be a homomorphism from $\mathrm{Lip}(X, C(K))$ into $\mathrm{Lip}(Y, C(M))$ with the form (1.3) as in Theorem 1. We also the set $\mathcal{D}$ and the mapping φ and ψ are as in Theorem 1. Since T is bounded, we may denote by $\|T\|$ its norm. Let $\mathbb{B}_{\mathrm{Lip}(X,C(K))}$ be the unit ball of $\mathrm{Lip}(X, C(K))$, namely,

$$\mathbb{B}_{\mathrm{Lip}(X,C(K))} = \big\{ f \in \mathrm{Lip}(X, C(K)) : \|f\|_{\mathrm{Lip}(X,C(K))} \leq 1 \big\}.$$

4.1 Proof of Sufficiency

We first prove the "if" part in Theorem 2 under the following conditions:

(B-1) φ satisfies the condition (iii).

(B-2) ψ satisfies the condition (iv).

Here we may assume that $T \neq O$ without loss of generality.

Lemma 4.1.1. *Suppose (B-2). Let $(y_0, \eta_0) \in \mathcal{D}$. For any $\varepsilon > 0$, there exists an open neighborhood Θ of η_0 in $\mathcal{D}_{y_0}$ such that*

$$\eta \in \Theta \quad \textit{implies} \sup_{f \in \mathbb{B}_{\mathrm{Lip}(X,C(K))}} \big|(Tf)(y_0, \eta) - (Tf)(y_0, \eta_0)\big| < \varepsilon. \tag{4.1}$$

Proof. Let $\varepsilon > 0$. Put $\mathcal{D}_{y_0} = \{\eta \in M \ : \ (y_0, \eta) \in \mathcal{D}\}$. Since $\eta_0 \in \mathcal{D}_{y_0}$, by (ii), there exists $j \in \{i, \ldots, n_{y_0}\}$ such that $\eta \in \Omega^j_{y_0}$. Then $\Omega^j_{y_0}$ is clopen subset on which ψ_{y_0} is constant. Hence if $\eta \in \Omega^j_{y_0}$,

$$\psi(y_0, \eta) = \psi_{y_0}(\eta) = \psi_{y_0}(\eta_0) = \psi(y_0, \eta_0).$$

Put

$$\Theta = \{\eta \in \Omega^j_{y_0} \ : \ d_X\big(\varphi_{y_0}(\eta), \varphi_{y_0}(\eta_0)\big) < \varepsilon\}.$$

Since $\varphi_{y_0} : \mathcal{D}_{y_0} \to X$ is continuous, Θ is an open neighborhood of η_0 in $\mathcal{D}_{y_0}$. For any $\eta \in \Theta$, put $x = \varphi(y_0, \eta)$, $x_0 = \varphi(y_0, \eta_0)$ and $\xi = \psi(y_0, \eta) = \psi(y_0, \eta_0)$. Then, for any $f \in \mathbb{B}_{\mathrm{Lip}(X,C(K))}$, we have

$$\begin{aligned}
\big|(Tf)(y_0,\eta) - (Tf)(y_0,\eta_0)\big| &= \big|f\big(\varphi(y_0,\eta),\psi(y_0,\eta)\big) - f\big(\varphi(y_0,\eta_0),\psi(y_0,\eta_0)\big)\big| \\
&= \big|f(x,\xi) - f(x_0,\xi)\big| \\
&= \big|f_x(\xi) - f_{x_0}(\xi)\big| \\
&\le \|f_x - f_{x_0}\|_{C(K)} \\
&\le \mathcal{L}_{X,C(K)}(f)\, d_X(x, x_0) \\
&= \mathcal{L}_{X,C(K)}(f)\, d_X\big(\varphi(y_0,\eta), \varphi(y_0,\eta_0)\big) \\
&= \mathcal{L}_{X,C(K)}(f)\, d_X\big(\varphi_{y_0}(\eta), \varphi_{y_0}(\eta_0)\big) \\
&\le \|f\|_{\mathrm{Lip}(X,C(K))}\, \varepsilon \le \varepsilon.
\end{aligned}$$

Hence we obtain (4.1). □

Lemma 4.1.2. *Suppose (B-2). Then* $T\big(\mathbb{B}_{\mathrm{Lip}(X,C(K))}\big)$ *is relatively compact in* $C(Y \times M)$.

Proof. According to Arzelá-Ascoli Theorem (cf. [5, Theorem IV.6.7]), we show that $T\big(\mathbb{B}_{\mathrm{Lip}(X,C(K))}\big)$ is bounded and equicontinuous on $Y \times M$.

The boundedness follows from an easy computation:

$$|(Tf)(y,\eta)| \le \|Tf\|_{C(Y\times M)} \le \|Tf\|_{\mathrm{Lip}(Y,C(M))} \le \|T\|\,\|f\|_{\mathrm{Lip}(X,C(K))} \le \|T\|$$

for all $f \in \mathbb{B}_{\mathrm{Lip}(X,C(K))}$ and all $(y,\eta) \in Y \times M$.

The equicontinuity will be shown as follows: Clearly, $T\big(\mathbb{B}_{\mathrm{Lip}(X,C(K))}\big)$ is equicontinuous on the clopen set $(Y \times M) \setminus \mathcal{D}$, because $Tf = 0$ on $(Y \times M) \setminus \mathcal{D}$ for all $f \in \mathbb{B}_{\mathrm{Lip}(X,C(K))}$, by (1.3) in Theorem 1. To show that $T\big(\mathbb{B}_{\mathrm{Lip}(X,C(K))}\big)$ is equicontinuous at each $(y_0,\eta_0) \in \mathcal{D}$, we take $\varepsilon > 0$ arbitrary. Take an open neighborhood Θ of η_0 in M as Lemma 4.1.1, and put

$$V = \left\{y \in Y \ : \ d_Y(y, y_0) < \frac{\varepsilon}{\|T\|}\right\}.$$

Define an open neighborhood $\mathcal{W}$ of (y_0,η_0) in $Y \times M$ as

$$\mathcal{W} = \big(V \times \Theta\big) \cap \mathcal{D}.$$

Then, for any $(y, \eta) \in \mathcal{W}$ and $f \in \mathbb{B}_{\mathrm{Lip}(X,C(K))}$, we have

$$\begin{aligned}
\big|(Tf)(y,\eta) - (Tf)(y_0,\eta)\big| &\le \big\|(Tf)_y - (Tf)_{y_0}\big\|_{C(M)} \\
&\le \mathcal{L}_{Y,C(M)}(Tf)\, d_Y(y,y_0) \\
&< \big\|Tf\big\|_{\mathrm{Lip}(Y,C(M))} \frac{\varepsilon}{\|T\|} \\
&\le \|T\|\, \|f\|_{\mathrm{Lip}(X,C(K))} \frac{\varepsilon}{\|T\|} \le \varepsilon,
\end{aligned}$$

because $y \in V$. Since $\eta \in \Theta$, Lemma 4.1.1 implies

$$\big|(Tf)(y_0,\eta) - (Tf)(y_0,\eta_0)\big| < \varepsilon.$$

Hence the triangle inequality shows that $(y, \eta) \in \mathcal{W}$ implies

$$\sup_{f \in \mathbb{B}_{\mathrm{Lip}(X,C(K))}} \big|(Tf)(y,\eta) - (Tf)(y,\eta_0)\big| < 2\varepsilon.$$

Thus we conclude that $T\big(\mathbb{B}_{\mathrm{Lip}(X,C(K))}\big)$ is equicontinuous on $Y \times M$. □

Lemma 4.1.3. *Suppose (B-1). Then, for any $\varepsilon > 0$, there exists a constant $c_\varepsilon > 0$ such that*

$$\|Tf\|_{\mathrm{Lip}(Y,C(M))} \le \varepsilon + c_\varepsilon \|Tf\|_{C(Y\times M)} \tag{4.2}$$

for all $f \in \mathbb{B}_{\mathrm{Lip}(X,C(K))}$.

Proof. Fix $\varepsilon > 0$. By (iii), there exists a $\delta_\varepsilon > 0$ such that

$$0 < d_Y(y,y') < \delta_\varepsilon \text{ implies } \frac{d_X\big(\varphi(y,\eta),\varphi(y',\eta)\big)}{d_Y(y,y')} < \varepsilon. \tag{4.3}$$

for $(y,\eta), (y',\eta) \in \mathcal{D}$. Let $f \in \mathbb{B}_{\mathrm{Lip}(X,C(K))}$, and choose $(y,\eta), (y',\eta) \in Y \times M$ with $y \ne y'$. We consider three cases.

[Case 1] $(y,\eta), (y',\eta) \in \mathcal{D}$: By (ii) in Theorem 1, $y \in V_i^\eta$ and $y' \in V_{i'}^\eta$ for some $i, i' \in \{1, \ldots, n_\eta\}$. We first consider the case $i = i'$. If $d_Y(y,y') < \delta_\varepsilon$, then the computation (3.2) using (4.3) instead of (1.4) gives

$$\begin{aligned}
\big|(Tf)(y,\eta) - (Tf)(y',\eta)\big| &\le \mathcal{L}_{X,C(K)}(f)\, d_X\big(\varphi(y,\eta),\varphi(y',\eta)\big) \\
&\le \mathcal{L}_{X,C(K)}(f)\, \varepsilon\, d_Y(y,y') \\
&\le \|f\|_{\mathrm{Lip}(X,C(K))}\, \varepsilon\, d_Y(y,y') \le \varepsilon\, d_Y(y,y').
\end{aligned} \tag{4.4}$$

On the other hand, if $d_Y(y,y') \geq \delta_\varepsilon$, then

$$\frac{\left|(Tf)(y,\eta)-(Tf)(y',\eta)\right|}{d_Y(y,y')} \leq \frac{|(Tf)(y,\eta)|+|(Tf)(y',\eta)|}{\delta_\varepsilon} \leq \frac{2\|Tf\|_{C(Y\times M)}}{\delta_\varepsilon}. \tag{4.5}$$

In case that $i \neq i'$, we have $d_Y(y,y') \geq r$ by (1.5), and so

$$\frac{\left|(Tf)(y,\eta)-(Tf)(y',\eta)\right|}{d_Y(y,y')} \leq \frac{2\|Tf\|_{C(Y\times M)}}{r}. \tag{4.6}$$

[Case 2] $(y,\eta) \in \mathcal{D}$ and $(y',\eta) \in (Y \times M) \smallsetminus \mathcal{D}$: Then Lemma 3.1.1 says that $d_Y(y,y') \geq \rho$ and so

$$\frac{\left|(Tf)(y,\eta)-(Tf)(y',\eta)\right|}{d_Y(y,y')} \leq \frac{|(Tf)(y,\eta)|}{\rho} \leq \frac{\|Tf\|_{C(Y\times M)}}{\rho}. \tag{4.7}$$

[Case 3] $(y,\eta), (y',\eta) \in (Y \times M) \smallsetminus \mathcal{D}$: By (1.3),

$$(Tf)(y,\eta)-(Tf)(y',\eta) = 0. \tag{4.8}$$

Now, put

$$\tilde{c}_\varepsilon = \max\left\{\frac{2}{\delta_\varepsilon}, \frac{2}{r}, \frac{1}{\rho}\right\}.$$

We combine (4.4)–(4.8) to get

$$\mathcal{L}_{Y,C(M)}(Tf) = \sup_{\eta\in M}\sup_{\substack{y,y'\in Y\\ y\neq y'}} \frac{\left|(Tf)(y,\eta)-(Tf)(y',\eta)\right|}{d_Y(y,y')} \leq \max\bigl\{\varepsilon, \tilde{c}_\varepsilon\,\|Tf\|_{C(Y\times M)}\bigr\}.$$

Therefore we obtain

$$\|Tf\|_{\mathrm{Lip}(Y,C(M))} \leq \varepsilon + (\tilde{c}_\varepsilon + 1)\|Tf\|_{C(Y\times M)}.$$

By setting $c_\varepsilon = \tilde{c}_\varepsilon + 1$, we obtain (4.2). □

Lemma 4.1.4. *Suppose (B-1) and (B-2). Then* $T\bigl(\mathbb{B}_{\mathrm{Lip}(X,C(K))}\bigr)$ *is relatively compact in* $\mathrm{Lip}(Y, C(M))$.

Proof. Let $\{f_n\}$ be an arbitrary sequence in $\mathbb{B}_{\mathrm{Lip}(X,C(K))}$. By Lemma 4.1.2, there exist a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ and a function $g \in C(Y \times M)$ such that

$$\bigl\|Tf_{n_i} - g\bigr\|_{C(Y\times M)} \to 0 \text{ as } i \to \infty. \tag{4.9}$$

Since $\operatorname{Lip}(Y, C(M))$ is Banach algebra, we see that $\{Tf_{n_i}\}$ is Cauchy sequence in $\operatorname{Lip}(Y, C(M))$. Let $\varepsilon > 0$ be arbitrary. Since $\{Tf_{n_i}\}$ is Cauchy sequence in $C(Y \times M)$ by (4.9), there exists an N such that $i, j \geq N$ implies

$$\|Tf_{n_i} - Tf_{n_j}\|_{C(Y\times M)} < \frac{\varepsilon}{c_\varepsilon}$$

Substituting $f = (f_{n_i} - f_{n_j})/2$ in (4.2), we obtain that $i, j \geq N$ implies

$$\|Tf_{n_i} - Tf_{n_j}\|_{\operatorname{Lip}(Y,C(M))} \leq 2\varepsilon + c_\varepsilon \|Tf_{n_i} - Tf_{n_j}\|_{C(Y\times M)} < 3\varepsilon.$$

Hence $\{Tf_{n_i}\}$ is a Cauchy sequence in $\operatorname{Lip}(Y, C(M))$. Thus we conclude that $T(\mathbb{B}_{\operatorname{Lip}(X,C(K))})$ is relatively compact in $\operatorname{Lip}(Y, C(M))$. □

Lemma 4.1.4 says that T is compact, and the "if" part was proved.

4.2 Proof of Necessity

In this section, we show the "only if" part in Theorem 2, that is, φ and ψ satisfy the conditions (iii) and (iv), respectively. Through this section, we suppose that T is compact.

Lemma 4.2.1. *φ satisfies the condition (iii).*

Proof. Let us assume that φ does not satisfy (iii). Then there exists an $\varepsilon_0 > 0$ and two sequences $\{(y_n, \eta_n)\}$ and $\{(y'_n, \eta_n)\}$ in $\mathcal{D}$ such that

$$0 < d_Y(y_n, y'_n) < \frac{1}{n^2} \qquad \text{and} \qquad \frac{d_X(\varphi(y_n, \eta_n), \varphi(y'_n, \eta_n))}{d_Y(y_n, y'_n)} \geq \varepsilon_0.$$

Put $z_n = \varphi(y_n, \eta_n)$ and $z'_n = \varphi(y'_n, \eta_n)$ for $n = 1, 2, \ldots$ For each $n = 1, 2, \ldots$, we define the function $\chi_n : [0, \infty) \to \mathbb{R}$ by

$$\chi_n(t) = \frac{1}{2n}(1 - e^{-nt}) \qquad (t \in [0, \infty)).$$

Clearly, for each $n = 1, 2, \ldots$, $0 \leq \chi_n \leq 1/(2n)$ and χ_n is differentiable and $\chi'_n(t) = e^{-nt}/2$. For each $n = 1, 2, \ldots$, we define the function $f_n : X \to \mathbb{C}$ by

$$f_n(x) = \chi_n(d_X(x, z'_n)) \qquad (x \in X).$$

For any $x, x' \in X$ with $x \neq x'$, the mean value theorem gives a point s_n between $d_X(x, z_n')$ and $d_X(x', z_n')$ such that

$$\chi_n\big(d_X(x, z_n')\big) - \chi_n\big(d_X(x', z_n')\big) = \chi_n'(s_n)\big(d_X(x, z_n') - d_X(x', z_n')\big).$$

By the triangle inequality, we have

$$|f_n(x) - f_n(x')| = |\chi_n'(s_n)|\big|d_X(x, z_n') - d_X(x', z_n')\big| \leq \frac{e^{-nt}}{2}\, d_X(x, x') \leq \frac{1}{2}\, d_X(x, x').$$

Hence $f_n \in \mathrm{Lip}(X)$ and

$$\|f_n\|_{\mathrm{Lip}(X)} = \|f\|_{C(X)} + \mathcal{L}_{X,\mathbb{C}}(f_n) \leq \frac{1}{2n} + \frac{1}{2} \leq 1.$$

Now we define a function $\check{f}_n$ on $X \times K$ by

$$\check{f}_n(x, \xi) = f_n(x) \qquad \big((x, \xi) \in X \times K\big)$$

Then $\check{f}_n \in \mathrm{Lip}(X, C(K))$ and $\|\check{f}_n\|_{\mathrm{Lip}(X,C(K))} \leq 1$, that is, $\check{f}_n \in \mathbb{B}_{\mathrm{Lip}(X,C(K))}$.

Next we estimate the norm $\|T\check{f}_n\|_{\mathrm{Lip}(Y,C(M))}$. For each $n = 1, 2, \ldots$, we use the mean value theorem again for χ_n, there exists a point σ_n with $0 \leq \sigma_n \leq d_X(z_n, z_n')$ such that

$$\chi_n\big(d_X(z_n, z_n')\big) - \chi_n(0) = \frac{e^{-n\sigma_n}}{2}\big(d_X(z_n, z_n') - 0\big)$$

Then, by (1.3) in Theorem 1, we have

$$\begin{aligned}
\big|T\check{f}_n(y_n, \eta_n) - T\check{f}_n(y_n', \eta_n)\big| &= \big|\check{f}_n\big(\varphi(y_n, \eta_n), \psi(y_n, \eta_n)\big) - \check{f}_n\big(\varphi(y_n', \eta_n), \psi(y_n', \eta_n)\big)\big| \\
&= \big|f_n\big(\varphi(y_n, \eta_n)\big) - f_n\big(\varphi(y_n', \eta_n)\big)\big| \\
&= \big|\chi_n\big(d_X(z_n, z_n')\big) - \chi_n(0)\big| \\
&= |\chi_n'(\sigma_n)|\, |d_X(z_n, z_n') - 0| \\
&= \frac{e^{-n\sigma_n}}{2}\, d_X\big(\varphi(y_n, \eta_n), \varphi(y_n', \eta_n)\big) \\
&\geq \frac{e^{-n\sigma_n}}{2}\, \varepsilon_0\, d_Y(y_n, y_n').
\end{aligned}$$

Hence

$$\big\|T\check{f}_n\big\|_{\mathrm{Lip}(Y,C(M))} \geq \mathcal{L}_{Y,C(M)}(T\check{f}_n) \geq \frac{\|(T\check{f}_n)_{y_n} - (T\check{f}_n)_{y_n'}\|_{C(M)}}{d_Y(y_n, y_n')} \geq \frac{e^{-n\sigma_n}}{2}\, \varepsilon_0. \quad (4.10)$$

While (ii) in Theorem 1 implies

$$0 \le \sigma_n \le d_X(z_n, z'_n) = d_X\big(\varphi(y_n, \eta_n), \varphi(y'_n, \eta_n)\big) \le L\, d_Y(y_n, y'_n) \le L\frac{1}{n^2},$$

and so $n\sigma_n \to 0$ as $n \to \infty$. Thus (4.10) implies

$$\liminf_{n\to\infty} \big\|T\check{f}_n\big\|_{\mathrm{Lip}(Y,C(M))} \ge \frac{\varepsilon_0}{2}. \tag{4.11}$$

On the other hand, since T is compact and $\{\check{f}_n\} \subset \mathbb{B}_{\mathrm{Lip}(X,C(K))}$, there exist a subsequence $\{\check{f}_{n_i}\}$ of $\{\check{f}_n\}$ and $g \in \mathrm{Lip}(Y, C(M))$ such that

$$\big\|T\check{f}_{n_i} - g\big\|_{\mathrm{Lip}(Y,C(M))} \to 0 \text{ as } i \to \infty.$$

Since $\big\|T\check{f}_{n_i} - g\big\|_{C(Y\times M)} \le \big\|T\check{f}_{n_i} - g\big\|_{\mathrm{Lip}(Y,C(M))}$, we have

$$\big|T\check{f}_{n_i}(y,\eta) - g(y,\eta)\big| \le \big\|T\check{f}_{n_i} - g\big\|_{C(Y\times M)} \le \big\|T\check{f}_{n_i} - g\big\|_{\mathrm{Lip}(Y,C(M))} \to 0 \text{ as } i \to \infty$$

for any $(y, \eta) \in Y \times M$. If $(y, \eta) \in \mathcal{D}$, then

$$\big|\big(T\check{f}_{n_i}\big)(y,\eta)\big| = \big|\check{f}_{n_i}\big(\varphi(y,\eta), \psi(y,\eta)\big)\big| = \big|f_{n_i}\big(\varphi(y,\eta)\big)\big| \le \frac{1}{2n_i} \to 0 \text{ as } i \to \infty.$$

If $(y, \eta) \in (Y \times M) \smallsetminus \mathcal{D}$, then $\big(T\check{f}_{n_i}\big)(y,\eta) = 0$ for $i = 1, 2, \ldots$ As a result, we have $g(u, \eta) = 0$ for all $(y, \eta) \in Y \times M$, and so we have

$$\big\|T\check{f}_{n_i}\big\|_{\mathrm{Lip}(Y,C(M))} \to 0 \text{ as } i \to \infty,$$

which contradicts (4.11). □

Fix $y \in Y$ and put

$$\mathcal{D}_y = \{\eta \in M \ :\ (y, \eta) \in \mathcal{D}\}.$$

Lemma 4.2.2. *For any $\eta_0 \in \mathcal{D}_y$, there exists an open neighborhood Θ of η_0 in $\mathcal{D}_y$ such that*

$$\psi_y \text{ is constant on } \Theta.$$

This lemma can be proved tracing a method in [17].

Proof. Since $\mathcal{D}_y$ is a compact subset of M, we can treat the Banach algebra $C(\mathcal{D}_y)$. We define a mapping P from $\mathrm{Lip}(Y, C(M))$ into $C(\mathcal{D}_y)$ by

$$(Pg)(\eta) = g(y,\eta) \qquad (g \in \mathrm{Lip}(Y, C(M)), \eta \in \mathcal{D}_y). \tag{4.12}$$

Clearly, P is bounded operator from $\mathrm{Lip}(Y, C(M))$ into $C(\mathcal{D}_y)$.

Put $S = PT$. Since T is compact, S is a compact operator from $\mathrm{Lip}(X, C(K))$ into $C(\mathcal{D}_y)$. Hence Arzelá-Ascoli theorem implies that $S\big(\mathbb{B}_{\mathrm{Lip}(X,C(K))}\big)$ is equicontinuous on $\mathcal{D}_y$. Therefore there exists an open neighborhood of η_0 in $\mathcal{D}_y$ such that $\eta \in \Theta$ implies

$$\sup_{f \in \mathbb{B}_{\mathrm{Lip}(X,C(K))}} \big|(Sf)(\eta) - (Sf)(\eta_0)\big| < \frac{1}{2}. \tag{4.13}$$

Let us show that ψ_y is constant on Θ. Conversely, assume that ψ_y is not constant on Θ. Then, there exists a $\eta_1 \in \Theta$ such that $\psi_y(\eta_1) \neq \psi_y(\eta_0)$. By Urysohn's lemma, there exists a $u \in C(K)$ such that $0 \leq u \leq 1$, $u\big(\psi_y(\eta_1)\big) = 1$ and $\psi_y(\eta_0) = 0$. We define a function $\tilde{u}$ on $X \times K$ by

$$\tilde{u}(x,\xi) = u(\xi) \qquad \big((x,\xi) \in X \times K\big).$$

Then $\tilde{u} \in \mathrm{Lip}(X, C(K))$ and $\|\tilde{u}\|_{\mathrm{Lip}(X,C(K))} = \|u\|_{C(K)}$. Since $\|u\|_{C(K)} \leq 1$ and $\eta_1 \in \Theta$, by (4.13), we have

$$\big|\big(S\tilde{u}\big)(\eta_1) - \big(S\tilde{u}\big)(\eta_0)\big| < \frac{1}{2}. \tag{4.14}$$

However, by (1.3) in Theorem 1,

$$\begin{aligned}
\big|\big(S\tilde{u}\big)(\eta_1) - \big(S\tilde{u}\big)(\eta_0)\big| &= \big|\big(PT\tilde{u}\big)(\eta_1) - \big(PT\tilde{u}\big)(\eta_0)\big| \\
&= \big|\big(T\tilde{u}\big)(y,\eta_1) - \big(T\tilde{u}\big)(y,\eta_0)\big| \\
&= \big|\tilde{u}\big(\varphi(y,\eta_1), \psi(y,\eta_1)\big) - \tilde{u}\big(\varphi(y,\eta_0), \psi(y,\eta_0)\big)\big| \\
&= \big|u\big(\psi(y,\eta_1)\big) - u\big(\psi(y,\eta_0)\big)\big| \\
&= \big|u\big(\psi_y(\eta_1)\big) - u\big(\psi_y(\eta_0)\big)\big| = 1 > \frac{1}{2}.
\end{aligned}$$

This contradicts (4.14). Thus we conclude that ψ_y is constant on Θ. □

Lemma 4.2.3. *ψ satisfies the condition (iv).*

Proof. For any $\eta \in \mathcal{D}_y$, put

$$\Omega^\eta = \big\{ \zeta \in \mathcal{D}_y \;:\; \psi_y(\zeta) = \psi_y(\eta) \big\}.$$

Put $c = \psi_y(\eta)$. If $\zeta \in \Omega^\eta$, then we have

$$\psi_y(\zeta) = \psi_y(\eta) = c.$$

Therefore ψ_y is constant on Ω^η. Also, we have

$$\Omega^\eta \cap \Omega^{\eta'} \neq \emptyset \implies \Omega^\eta = \Omega^{\eta'}. \tag{4.15}$$

Since ψ_y is continuous by Theorem 1, Ω^η is closed subset of $\mathcal{D}_y$. Let us show that Ω^η is open subset of $\mathcal{D}_y$. For any $\zeta \in \Omega^\eta$, by Lemma 4.2.2, there exists an open neighborhood of Θ_ζ of ζ such that

$$\psi_y \text{ is constant on } \Theta_\zeta.$$

Clearly, $\Theta_\zeta \subset \Omega^\eta$. If we consider the open set Θ_ζ for any $\zeta \in \Omega^\eta$, then we have

$$\Omega^\eta = \bigcup_{\zeta \in \Omega^\eta} \Theta_\zeta.$$

Thus Ω^η is open subset of $\mathcal{D}_y$.

Note that

$$\mathcal{D}_y = \bigcup_{\eta \in \mathcal{D}_y} \Omega^\eta.$$

Since $\mathcal{D}_y$ is compact, we can select finitely many $\eta_1, \ldots, \eta_n \in \mathcal{D}_y$ such that

$$\mathcal{D}_y = \bigcup_{i=1}^{n} \Omega^{\eta_i}.$$

By (4.15), we may assume that $\Omega^{\eta_1}, \ldots, \Omega^{\eta_n}$ are disjoint. By setting $n_y = n$ and writing $\Omega_y^i = \Omega^{\eta_i}$, we obtain (iv). □

In Lemma 4.2.1, we have shown that φ satisfies the condition (iii). Also, in Lemma 4.2.3, we have shown that ψ satisfies the condition (iv). Thus the proof of Theorem 2 is completed.

Chapter 5

Corollaries

In this chapter, we give corollaries of Theorem 1 and 2. If K is a one-point set, then $\mathrm{Lip}(X, C(K))$ is isometrically isomorphic to $\mathrm{Lip}(X)$ by Gel'fand-Mazur theorem (cf.[3, Corollary 1.2.6]). On the other hand, if X is a one-point set, then $\mathrm{Lip}(X, C(K))$ is isometrically isomorphic to $C(K)$ by [28, p.213].

Corollary 1. *Suppose that X and Y are compact metric spaces with metrics d_X and d_Y, respectively.*

(I) *If T is a homomorphism from $\mathrm{Lip}(X)$ into $\mathrm{Lip}(Y)$, then there exist a clopen subset Y_0 of Y and a continuous mapping $\varphi : Y_0 \to X$ with*

$$\sup_{\substack{y,y' \in Y \\ y \neq y'}} \frac{d_X(\varphi(y), \varphi(y'))}{d_Y(y, y')} < \infty$$

such that T has the form :

$$(Tf)(y) = \begin{cases} f\big(\varphi(y)\big) & \big(y \in Y_0\big) \\ 0 & \big(y \in Y \setminus Y_0\big) \end{cases} \tag{5.1}$$

for all $f \in \mathrm{Lip}(X)$. Conversely, if Y_0 and φ are given as above, then T defined by (5.1) *is a homomorphism from $\mathrm{Lip}(X)$ into $\mathrm{Lip}(Y)$. Moreover, T is unital if and only if $Y_0 = Y$.*

(II) *Suppose that T is a homomorphism from $\mathrm{Lip}(X)$ into $\mathrm{Lip}(Y)$ with the form* (5.1). *Then T is compact if and only if*

$$\lim_{\substack{y,y' \in Y_0 \\ d_Y(y,y') \to 0}} \frac{d_X(\varphi(y), \varphi(y'))}{d_Y(y, y')} = 0.$$

Therefore Theorem 1 and Theorem 2 are generalization of Theorem A and Theorem D, respectively.

Corollary 2. *Suppose that K and M are compact Hausdorff spaces.*

(I) *If T is a homomorphism from $C(K)$ into $C(M)$, then there exist a clopen subset M_0 of M and a continuous mapping $\psi : M_0 \to K$ such that T has the form :*

$$(Tf)(\eta) = \begin{cases} f\big(\psi(\eta)\big) & \big(\eta \in M_0\big) \\ 0 & \big(\eta \in M \setminus M_0\big) \end{cases} \tag{5.2}$$

for all $f \in C(K)$. Conversely, if M_0 and ψ are given as above, then T defined by (5.2) is a homomorphism from $C(K)$ into $C(M)$. Moreover, T is unital if and only if $M_0 = M$.

(II) *Suppose that T is a homomorphism from $C(K)$ into $C(M)$ with the form (5.2). Then T is compact if and only if M_0 is a union of finitely many clopen subsets $M_1, \ldots, M_n$ such that ψ is constant on each M_i for $i = 1, \ldots, n$. Moreover, T is compact if and only if T has a finite rank.*

Corollary 3. *Suppose that X is a compact metric space with metric d_X and that M is a compact Hausdorff space.*

(I) *If T is a homomorphism from $\mathrm{Lip}(X)$ into $C(M)$, then there exist a clopen subset M_0 of M and a continuous mapping $\varphi : M_0 \to X$ such that T has the form :*

$$(Tf)(\eta) = \begin{cases} f\big(\varphi(\eta)\big) & \big(\eta \in M_0\big) \\ 0 & \big(\eta \in M \setminus M_0\big) \end{cases} \tag{5.3}$$

for all $f \in \mathrm{Lip}(X)$. Conversely, if M_0 and φ are given as above, then T defined by (5.3) is a homomorphism from $\mathrm{Lip}(X)$ into $C(M)$. Moreover, T is unital if and only if $M_0 = M$.

(II) *Every homomorphism from $\mathrm{Lip}(X)$ into $C(M)$ is compact.*

Corollary 4. *Suppose that Y is a compact metric space with metric d_Y and that K is a compact Hausdorff space.*

(I) *If T is a homomorphism from $C(K)$ into $\mathrm{Lip}(Y)$, then Y is a union of finitely many disjoint clopen subsets $Y_0, Y_1, \ldots, Y_n$ and there exist constant mappings $\psi_i : Y_i \to K$ $(i = 1, \ldots, n)$ such that T has the form :*

$$(Tf)(y) = \begin{cases} f\big(\psi_i(y)\big) & \big(y \in Y_i, i = 1, \ldots, n\big) \\ 0 & \big(y \in Y_0\big) \end{cases} \tag{5.4}$$

for all $f \in C(K)$*. Conversely, if* $Y_0, Y_1, \ldots, Y_n$ *and* $\psi_1, \ldots, \psi_n$ *are given as above, then* T *defined by* (5.4) *is a homomorphism from* $C(K)$ *into* $\mathrm{Lip}(Y)$*. Moreover,* T *is unital if and only if* $Y_0 = \emptyset$*.*

(II) *Every homomorphism from* $C(K)$ *into* $\mathrm{Lip}(Y)$ *has a finite rank.*

Chapter 6

Proof of Theorem 3 and some remarks

In this chapter, we prove Theorem 3 and give some remarks and conjectures.

6.1 Proof of Theorem 3

Through this section, we suppose that T is weakly compact.

Lemma 6.1.1. *Let $\eta_0 \in \mathcal{D}_y$. If, for each open neighborhood Θ of η_0, ψ_y is not constant on Θ, then there exists an net $\{\eta_\alpha\}_\alpha$ in $\mathcal{D}_y$ with $\eta_\alpha \to \eta_0$ such that $\psi_y(\eta_\alpha) \neq \psi(\eta_0)$ for any α.*

Proof. Since ψ_y is not constant on Θ, it is easy to see that there exists an η_Θ such that $\psi_y(\eta_\Theta) \neq \psi_y(\eta_0)$. We denote by $\mathcal{O}(\eta_0)$ the collection of open neighborhoods of η_0. Let $\mathcal{O}(\eta_0)$ directed by defining $\Theta_1 \succeq \Theta_2$ to mean that $\Theta_1 \subseteq \Theta_2$. It is clear that $\{\eta_\Theta\}_{\Theta \in \mathcal{O}(\eta_0)}$ is net in $\mathcal{D}_y$ with $\eta_\alpha \to \eta_0$ and $\psi_y(\eta_\Theta) \neq \psi(\eta_0)$ for any $\Theta \in \mathcal{O}(\eta_0)$. □

Lemma 6.1.2. *For any $\eta_0 \in \mathcal{D}_y$, there exists an open neighborhood Θ of η_0 in $\mathcal{D}_y$ such that*

$$\psi_y \text{ is constant on } \Theta.$$

Proof. Conversely, assume that for each open neighborhood Θ of η_0 in $\mathcal{D}_y$, ψ_y is not constant on Θ. By Lemma 6.1.1, there exists an net $\{\eta_\alpha\}$ in $\mathcal{D}_y$ with $\eta_\alpha \to \eta_0$ such that $\psi_y(\eta_\alpha) \neq \psi(\eta_0)$ for any α.

Suppose that P is (4.12) as in proof of Lemma 4.2.2. Put $S = PT$. Since T is weakly compact, by [22, Proposition 3.5.11], S is a weakly compact operator from $\operatorname{Lip}(Y, C(M))$ into $C(\mathcal{D}_y)$. By [5, Theorem IV.6.14], $S\big(\mathbb{B}_{\operatorname{Lip}(X,C(K))}\big)$ is quasi-equicontinuous on $\mathcal{D}_y$. Therefore, for any α_0, there exists a finite set of indexes $\alpha_i \succeq \alpha_0 (i = 1, \dots, n)$ such that for each $f \in \mathbb{B}_{\operatorname{Lip}(X,C(K))}$,

$$\min_{1\le i\le n}\big|(Sf)(\eta_{\alpha_i}) - (Sf)(\eta_0)\big| < \frac{1}{2}.$$

Then there exists an $i_0 \in \{1, \dots, n\}$ such that for each $f \in \mathbb{B}_{\operatorname{Lip}(X,C(K))}$,

$$\big|(Sf)(\eta_{\alpha_{i_0}}) - (Sf)(\eta_0)\big| < \frac{1}{2}. \tag{6.1}$$

However, since $\psi_y(\eta_{\alpha_{i_0}}) \ne \psi(\eta_0)$, by Urysohn's lemma, there exists an $u \in C(K)$ such that $0 \le u \le 1$, $u\big(\psi_y(\eta_{\alpha_{i_0}})\big) = 1$ and $u\big(\psi_y(\eta_0)\big) = 0$. We define a function $\tilde{u}$ on $X \times K$ by

$$\tilde{u}(x, \xi) = u(\xi) \qquad \big((x, \xi) \in X \times K\big).$$

Then $\tilde{u} \in \operatorname{Lip}(X, C(K))$ and $\|\tilde{u}\|_{\operatorname{Lip}(X,C(K))} = \|u\|_{C(K)}$. Since $\|u\|_{C(K)} \le 1$, by (6.1), we have

$$\big|\big(S\tilde{u}\big)(\eta_{\alpha_{i_0}}) - \big(S\tilde{u}\big)(\eta_0)\big| < \frac{1}{2}. \tag{6.2}$$

By (1.3) in Theorem 1, we have

$$\begin{aligned}
\big|\big(S\tilde{u}\big)(\eta_{\alpha_{i_0}}) - \big(S\tilde{u}\big)(\eta_0)\big| &= \big|\big(PT\tilde{u}\big)(\eta_{\alpha_{i_0}}) - \big(PT\tilde{u}\big)(\eta_0)\big| \\
&= \big|\big(T\tilde{u}\big)(y, \eta_{\alpha_{i_0}}) - \big(T\tilde{u}\big)(y, \eta_0)\big| \\
&= \big|\tilde{u}\big(\varphi(y, \eta_{\alpha_{i_0}}), \psi(y, \eta_{i_0})\big) - \tilde{u}\big(\varphi(y, \eta_0), \psi(y, \eta_0)\big)\big| \\
&= \big|u\big(\psi(y, \eta_{\alpha_{i_0}})\big) - u\big(\psi(y, \eta_0)\big)\big| \\
&= \big|u\big(\psi_y(\eta_{\alpha_{i_0}})\big) - u\big(\psi_y(\eta_0)\big)\big| = 1 > \frac{1}{2}.
\end{aligned}$$

This contradicts (6.2). □

Lemma 6.1.3. *ψ satisfies the condition (iv) in Theorem 2.*

Proof. Same as proof of Lemma 4.2.3. □

Same as Chapter 5, if X and Y are one-point set, then we have following corollary.

Corollary 5. *Let K and M be compact Hausdorff spaces, and T be a homomorphism from $C(K)$ into $C(M)$ with the form (5.2) as in Corollary 2. Then the following are equivalent:*

(a) T *is compact.*

(b) T *is weakly compact.*

(c) M_0 *is a union of finitely many clopen subsets* $M_1, \ldots, M_n$ *such that* ψ *is constant on each* M_i *for* $i = 1, \ldots, n$*.*

6.2 Some Remarks and Conjectures

In this section, we state some remarks and conjectures.

We have not yet discussed whether φ satisfies the condition (iii) in Theorem 2 if T is weakly compact. In fact, we do not understand that φ satisfies the condition (iii) in Theorem 2. On the other hand, in [13], A. Jiménez-Vargas proved that weakly compact composition operator between $\mathrm{Lip}(X)$ is compact under certain conditions. To state it, we give some definitions.

Definition 6.2.1. *Let* $0 < \alpha \leq 1$*,* X *be a compact metric space with metric* d_X^α *and* $\mathcal{A}$ *be a Banach space over* $\mathbb{C}$ *with norm* $\|\cdot\|_{\mathcal{A}}$*.*

(i) If an $\mathcal{A}$*-valued function* f *on* X *satisfies*

$$\sup_{\substack{x,x' \in X \\ x \neq x'}} \frac{\|f(x) - f(x')\|_{\mathcal{A}}}{d_X(x, x')^\alpha} < \infty,$$

then we say that f *is* α*-Lipschitz or Hölder.*

(ii) $\mathrm{Lip}_\alpha(X, \mathcal{A})$ *denotes the set of all* $\mathcal{A}$*-valued* α*-Lipschitz functions on* X*.*

(iii) If $f \in \mathrm{Lip}(X, \mathcal{A})$ *satisfies the condition*

$$\lim_{d_X(x,x') \to 0} \frac{\|f(x) - f(x')\|_{\mathcal{A}}}{d_X(x, x')^\alpha} = 0,$$

then we say that f *is little* α*-Lipschitz. In particular, if* $\alpha = 1$*, then we say that* f *is little Lipschitz.*

(iv) $\mathrm{lip}_\alpha(X, \mathcal{A})$ *denotes the set of all* $\mathcal{A}$*-valued little* α*-Lipschitz functions on* X*.*

(v) In case that $\mathcal{A} = \mathbb{C}$*, we simply write* $\mathrm{Lip}_\alpha(X, \mathbb{C}) = \mathrm{Lip}_\alpha(X)$ *and* $\mathrm{lip}_\alpha(X, \mathbb{C}) = \mathrm{lip}_\alpha(X)$*, respectively.*

(vi) In case that $\alpha = 1$*, we simply write* $\mathrm{lip}_1(X, \mathcal{A}) = \mathrm{lip}(X, \mathcal{A})$.

Of course, $\mathrm{Lip}(X, \mathcal{A})$ is in case that $\alpha = 1$. Note that little Lipschitz function is special case of supercontraction mapping.

Next proposition shows basic properties of $\mathrm{Lip}_\alpha(X, \mathcal{A})$ and $\mathrm{lip}_\alpha(X, \mathcal{A})$.

Proposition 6.2.1. *Let* $0 < \alpha \leq 1$*,* X *be a metric space with metric* d_X^α *and* $\mathcal{A}$ *be a Banach space over* $\mathbb{C}$ *with norm* $\|\cdot\|_{\mathcal{A}}$.

(a) $\mathrm{lip}_\alpha(X, \mathcal{A})$ *is a closed subspace of* $\mathrm{Lip}_\alpha(X, \mathcal{A})$*. In particular, if* $\mathcal{A}$ *is a Banach algebra, then* $\mathrm{lip}_\alpha(X, \mathcal{A})$ *is a closed subalgebra of* $\mathrm{Lip}_\alpha(X, \mathcal{A})$.

(b) If $0 < \alpha < 1$*, then* $\mathrm{Lip}(X, \mathcal{A}) \subset \mathrm{lip}_\alpha(X, \mathcal{A})$

We give the simplest example of $\mathrm{lip}(X)$.

Example 6.2.1 ([42])**.** Let $X = [0, 1]$ with usual metric. Rademacher's theorem (cf.[42, Theorem 1.41]) implies that $\mathrm{lip}(X)$ consists only of the constant functions.

The following non-trivial example was given by H. Kamowitz and S. Scheinberg in [18].

Example 6.2.2 ([18])**.** Let $X = [\frac{1}{4}, 1]$ and define $d_X : X \times X \to [0, \infty)$ by

$$\begin{aligned} d_X(x, y) &= \sqrt{|x - y|} && \left(\tfrac{1}{4} \leq x, y \leq \tfrac{1}{2}\right) \\ d_X(x, y) = d_X(y, x) &= \sqrt{\tfrac{1}{2} - x} + \left(y - \tfrac{1}{2}\right) && \left(\tfrac{1}{4} \leq x \leq \tfrac{1}{2} \leq y \leq 1\right) \\ d_X(x, y) &= |x - y| && \left(\tfrac{1}{2} \leq x, y \leq 1\right) \end{aligned}$$

Then X is compact and connected metric space with metric d_X. A function f defined by

$$f(x) = \begin{cases} 2x & \left(\frac{1}{4} \leq x \leq \frac{1}{2}\right) \\ 1 & \left(\frac{1}{2} \leq x \leq 1\right) \end{cases}$$

is little Lipschitz function.

N. Weaver defined in [41] the following property to get a Stone-Weierstrass type theorem.

Definition 6.2.2 (Weaver, [41])**.** *Let X be a compact metric space and $\mathcal{B}$ be a subalgebra of* $\mathrm{Lip}(X)$*. We say that $\mathcal{B}$ separates points uniformly if there exists a constant $a > 1$ such that for any $x, x' \in X$ with $x \neq x'$, some $f \in \mathcal{B}$ satisfies $\mathcal{L}_{X,\mathbb{C}}(f) \leq a$ and $|f(x) - f(x')| = d_X(x, x')$.*

Example 6.2.3 ([42])**.** Let X be a compact metric space with metric d_X. If X is countable, then $\mathrm{lip}(X)$ separates points uniformly.

Example 6.2.4 ([42])**.** If X is the middle-third Cantor set, then $\mathrm{lip}(X)$ separates points uniformly.

Also, in [1], W. G. Bade, P. C. Curtis Jr. and H. G. Dales showed that $\mathrm{lip}_\alpha(X)$ separates points uniformly for $0 < \alpha < 1$.

L. G. Hanin [8] and N. Weaver [41] showed following theorem:

Theorem F (Hanin [8] and Weaver [41])**.** *Let X be a compact metric space with metric d_X. The following are equivalent:*

(a) $\mathrm{lip}(X)$ *separates points uniformly.*

(b) There exists $b > 1$ such that for any $g \in \mathrm{Lip}(X)$ and any finite subset $S \subset X$, some $f \in \mathrm{lip}(X)$ satisfies $\mathcal{L}_{X,\mathbb{C}}(f) \leq b\mathcal{L}_{X,\mathbb{C}}(g)$ and $f(x) = g(x)$ for $x \in S$.

(c) The second dual space of $\mathrm{lip}(X)$ *is isometrically isomorphic to* $\mathrm{Lip}(X)$*, via the mapping $\Lambda : \mathrm{lip}(X)^{**} \to \mathrm{Lip}(X)$ defined by*

$$(\Lambda f^{**})(x) = f^{**}(\tau_x) \qquad \big(f^{**} \in \mathrm{lip}(X)^{**}, x \in X\big),$$

where τ_x is evaluation functional at x.

Also, J. A. Johnson [14], and W. G. Bade, P. C. Curtis Jr. and H. G. Dales [1] showed that $\mathrm{lip}_\alpha(X)^{**}$ is isometrically isomorphic to $\mathrm{Lip}_\alpha(X)$, via the mapping $\Lambda_\alpha : \mathrm{lip}_\alpha(X)^{**} \to \mathrm{Lip}_\alpha(X)$ defined by

$$(\Lambda_\alpha f^{**})(x) = f^{**}(\tau_x) \qquad \big(f^{**} \in \mathrm{lip}_\alpha(X)^{**}, x \in X\big).$$

In [13], A. Jiménez-Vargas prove following theorem using the Theorem F.

Theorem G (Jiménez-Vargas, [13])**.** *Let X be a compact metric space with metric d_X. Suppose that* $\mathrm{lip}(X)$ *separates points uniformly, that φ is Lipschitz mapping between X and that T is a composition operator between* $\mathrm{Lip}(X)$*, that is,*

$$(Tf)(x) = f\big(\varphi(x)\big) \qquad \big(f \in \mathrm{Lip}(X)\big).$$

If T is weakly compact, then T is compact, that is, φ satisfies the supercontraction property.

In [7], as a generalization of [13], A. Golbaharan also proved that weakly compact weighted composition operator between $\mathrm{Lip}(X)$ is compact under the same assumptions as in Theorem G.

They proved their results using the condition (c) in Theorem F and Gantmacher's Theorem (cf.[22, Theorem 3.5.8]). Therefore we need to research on $\mathrm{lip}(X, C(K))^{**}$. Author studied $\mathrm{lip}(X, C(K))^{**}$, but gave no results. On the other hand, in [14], J. A. Johnson proved following theorem:

Theorem H (Johnson, [14])**.** *Let $0 < \alpha < 1$, X be a compact metric space with metric d_X^α and $\mathcal{A}$ be a Banach space with norm $\|\cdot\|_{\mathcal{A}}$. If either $\mathrm{lip}_\alpha(X)^*$ or $\mathcal{A}^*$ has the approximation property, then $\mathrm{lip}_\alpha(X, \mathcal{A})^{**}$ is isometrically isomorphic to $\mathrm{Lip}_\alpha(X, \mathcal{A}^{**})$.*

He proved this theorem using the tensor product of $\mathrm{lip}_\alpha(X)$ and $\mathcal{A}$. Note that $C(K)^*$ has the approximation property (cf.[30, p.74]). We conjecture following two statements:

Conjecture 1. *Let X be a compact metric space with metric d_X and K be a compact Hausdorff space. If $\mathrm{lip}(X)$ separates points uniformly, then $\mathrm{lip}(X, C(K))^{**}$ is isometrically isomorphic to $\mathrm{Lip}(X, C(K)^{**})$.*

Conjecture 2. *Let X, Y, K, M be as in Theorem 1, and T be a homomorphism from $\mathrm{Lip}(X, C(K))$ into $\mathrm{Lip}(Y, C(M))$ with the form (1.3) as in Theorem 1. Suppose that $\mathrm{lip}(X)$ separates points uniformly. Then the following are equivalent:*

(a) T is compact.

(b) T is weakly compact.

(c) φ and ψ satisfy the condition (iii) and (iv) in Theorem 2, respectively.

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