

Instanton corrections of 1/6 BPS Wilson loops in ABJM theory

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ABSTRACT: We study instanton corrections to the vacuum expectation value (VEV) of 1/6 BPS Wilson loops in ABJM theory from the Fermi gas approach. We mainly consider Wilson loops in the fundamental representation and winding Wilson loops, but we also initiate the study of Wilson loops with two boundaries. We find that the membrane instanton corrections to the Wilson loop VEV are determined by the refined topological string in the Nekrasov-Shatashvili limit, and the pole cancellation mechanism between membrane instantons and worldsheet instantons works also in the Wilson loop VEVs as in the case of the partition functions.

KEYWORDS: M-Theory, Wilson, 't Hooft and Polyakov loops, AdS-CFT Correspondence

ARXIV EPRINT: [1607.06157](https://arxiv.org/abs/1607.06157)

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1 Introduction

The AdS/CFT correspondence [1] has been tested in many examples and we believe that holography is one of the fundamental principles of quantum gravity and string theory. The localization computation in supersymmetric field theories [2] opened a door to a new era

and we are now in a stage to study the holographic duality at a precise quantitative level. In particular, in the past few years we have witnessed a tremendous progress in understanding the holographic duality between M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$ and 3d $\mathcal{N} = 6$ $U(N)_k \times U(N)_{-k}$ Chern-Simons-matter theory, known as the ABJM theory [3]. From the analysis of the partition function of ABJM theory on a three-sphere obtained by the localization technique [4], the $N^{3/2}$ scaling law of the degrees of freedom on N M2-branes predicted from the gravity side [5] is correctly reproduced from the first principles computation on the field theory side [6].

Moreover, there are exponentially small corrections to the free energy [6, 7], corresponding to the M2-brane instantons wrapping some three-cycles inside S^7/\mathbb{Z}_k in the bulk M-theory side. The Fermi gas formalism [8] is a powerful technique to study such instanton corrections. In this approach, we consider the grand partition function of ABJM theory by introducing the chemical potential μ and summing over N . It turns out that the grand partition function is written as a Fredholm determinant which in turn is interpreted as a system of free fermions on a real line. The grand potential of ABJM Fermi gas system receives two types of instanton corrections: worldsheet instanton corrections of order $\mathcal{O}(e^{-4\mu/k})$ and membrane instanton corrections of order $\mathcal{O}(e^{-2\mu})$. There are also bound states of worldsheet instantons and membrane instantons, but they can be removed by introducing the “effective” chemical potential [9]. Building on a series of works [9–12], it is finally realized that the grand potential of ABJM theory is completely determined by the refined topological string on local $\mathbb{P}^1 \times \mathbb{P}^1$ [13]. In particular, the membrane instanton corrections are given by the refined topological free energy in the Nekrasov-Shatashvili limit (NS free energy), while the worldsheet instanton corrections correspond to the standard, un-refined topological string. Although the membrane instantons and the worldsheet instantons separately have poles at rational values of k , those poles are actually canceled by adding the two contributions. This *pole cancellation mechanism* guarantees that the grand partition function is well-defined for all k . For the special values $k = 1, 2, 4, 8$, the grand partition function can be written in closed forms in terms of the Jacobi theta functions [14–16].

One can also compute exactly the vacuum expectation value (VEV) of 1/6 BPS Wilson loops in ABJM theory using the localization technique [4]. The study of such BPS Wilson loops from the Fermi gas approach was initiated in [17] and further developed in [18] particularly for the 1/2 BPS Wilson loops. It is found that 1/2 BPS Wilson loops in arbitrary representations are given by a determinant of hook representations [18–20], and 1/2 BPS Wilson loops are closely related to the open topological string on local $\mathbb{P}^1 \times \mathbb{P}^1$ [20–23]. Interestingly, it is observed that there are no “pure” membrane instanton corrections in 1/2 BPS Wilson loops [18] which turns out to be a consequence of the *open-closed duality* between the 1/2 BPS Wilson loops and the grand partition functions of ABJ theory [20]. In particular, there is no pole cancellation between worldsheet instantons and membrane instantons in the VEV of 1/2 BPS Wilson loops.

On the other hand, 1/6 BPS Wilson loops in ABJM theory have no direct connection to open topological strings and the instanton corrections of 1/6 BPS Wilson loops have not been fully explored in the literature. In this paper, we initiate a study of the instanton corrections of 1/6 BPS Wilson loops mainly using the numerical analysis. We will show that

the VEV of 1/6 BPS Wilson loops can be computed numerically with high precision and we will study the instanton corrections of 1/6 BPS Wilson loops from numerical fitting. We first consider the 1/6 BPS Wilson loops in the fundamental representation. We find that the membrane instanton corrections are given by the NS free energy, and the pole cancellation mechanism works also for the 1/6 BPS Wilson loops. We find that, up to regular terms, the 1/6 BPS Wilson loops in the fundamental representation is essentially determined by the quantum volume appearing in the exact quantization condition of the spectrum [15, 24–28].

We also study the 1/6 BPS Wilson loops with winding number $n \geq 2$ and the 1/6 BPS Wilson loops with two boundaries. Our numerical study shows that the perturbative part of the 1/6 BPS Wilson loops with winding number $n \geq 2$ is different from the expression obtained in [17]. We also study the instanton corrections to those Wilson loops and find evidence that the membrane instanton corrections are again related to the NS free energy.

This paper is organized as follows. In section 2, we first review the Fermi gas approach to the 1/6 BPS Wilson loops, and explain our algorithm to compute them numerically. In section 3, we consider the perturbative part (or zero-instanton part) of the grand canonical VEV of winding Wilson loops. Our result (3.11) is different from the one found in [17] for winding number $n \geq 2$. In section 4, we study the instanton corrections to the fundamental Wilson loop numerically. We find that the grand canonical VEV of fundamental Wilson loop is closely related to the quantum volume (4.33). In section 5, we study the WKB expansion of the fundamental Wilson loop. Using the Padé approximation we confirm that the membrane instanton corrections are correctly reproduced from the WKB expansion. In section 6, we consider winding Wilson loops with winding number $n \geq 2$. We find that the membrane instanton corrections for the winding number $n = 2$ are also related to the NS free energy. We also find the first few worldsheet instanton corrections for the winding numbers $n = 3, 4$. In section 7, we study the Wilson loops with two boundaries. We show that the VEV of Wilson loops with two boundaries can be systematically computed by constructing a sequence of functions, in a similar manner as the winding Wilson loops. Then we study the instanton corrections of Wilson loops with two boundaries. Again, the membrane instanton corrections to the imaginary part of Wilson loop is related to the NS free energy. Finally, we conclude in section 8. In appendix A, we summarize the instanton corrections for some integer values of k .

2 1/6 BPS Wilson loops from Fermi gas approach

In this paper, we will consider the VEV of 1/6 BPS Wilson loops in ABJM theory on S^3 , first constructed in [29–31]

$$W_R^{(1/6)} = \text{Tr}_R P \exp \left[\int ds \left(i A_\mu \dot{x}^\mu + \frac{2\pi}{k} |\dot{x}| M_J^I C_I \bar{C}^J \right) \right]. \quad (2.1)$$

Here $x^\mu(s)$ parametrizes a great circle of S^3 , C_I ($I = 1, 2, 3, 4$) are the scalar fields in the bi-fundamental chiral multiplets, and M_J^I is a constant matrix which can be brought to the form $M = \text{diag}(1, 1, -1, -1)$ in a suitable choice of basis. A_μ is the gauge field of one of the $U(N)$ factor of the gauge group $U(N)_k \times U(N)_{-k}$ of ABJM theory. On the other

hand, the construction of 1/2 BPS Wilson loops involves the gauge fields in both of the $U(N)$ factors and utilizes the supergroup structure $U(N|N)$ [32].

The VEV of such 1/6 BPS Wilson loops can be reduced to a matrix model by the supersymmetric localization [4]

$$\langle W_R^{(1/6)} \rangle_N = \frac{1}{(N!)^2} \int \prod_{i=1}^N \frac{d\mu_i d\nu_i}{(2\pi)^2} e^{\frac{ik}{4\pi}(\mu_i^2 - \nu_i^2)} \frac{\prod_{i<j} (2 \sinh \frac{\mu_i - \mu_j}{2})^2 (2 \sinh \frac{\nu_i - \nu_j}{2})^2}{\prod_{i,j} 2 \cosh \frac{\mu_i - \nu_j}{2}} \text{Tr}_R U, \tag{2.2}$$

where U corresponds to the holonomy in one of the gauge group $U(N)$

$$U = \text{diag}(e^{\mu_1}, \dots, e^{\mu_N}). \tag{2.3}$$

One can also consider the Wilson loop in the other $U(N)$ factor, but it is related to (2.2) by the complex conjugation (or $k \rightarrow -k$). In this paper we consider the *un-normalized* VEV, i.e. we do not divide (2.2) by the partition function $Z(N, k)$.

As shown in [17], the VEV of 1/6 BPS Wilson loops in the grand canonical picture can be written as a Fermi gas system. Let us first recall the Fermi gas description of the partition function of ABJM theory on S^3 [8]. Introducing the fugacity $\kappa = e^\mu$ with chemical potential μ , we define the grand canonical partition function by summing over N

$$\Xi(\kappa, k) = 1 + \sum_{N=1}^{\infty} \kappa^N Z(N, k). \tag{2.4}$$

It turns out that the grand partition function can be written as a Fredholm determinant

$$\Xi(\kappa, k) = \text{Det}(1 + \kappa\rho), \tag{2.5}$$

where the density matrix ρ is given by

$$\rho = \frac{1}{2 \cosh \frac{x}{2}} \frac{1}{2 \cosh \frac{p}{2}}. \tag{2.6}$$

Here x and p obeys the canonical commutation relation

$$[x, p] = i\hbar, \tag{2.7}$$

and the Chern-Simons level k and the Planck constant \hbar are related by

$$\hbar = 2\pi k. \tag{2.8}$$

Similarly, we can define the grand canonical VEV of a general operator \mathcal{O} by

$$\langle \mathcal{O} \rangle^{\text{GC}} = \frac{1}{\Xi(\kappa, k)} \sum_{N=1}^{\infty} \kappa^N \langle \mathcal{O} \rangle_N. \tag{2.9}$$

The crucial observation in [17, 18] is that the grand canonical VEV of the holonomy U in (2.3) corresponds to the insertion of a quantum mechanical operator W

$$W = e^{\frac{x+p}{k}} \tag{2.10}$$

into the Fredholm determinant. More generally, the grand canonical VEV of operator $\det f(U)$ for some function f is written as

$$\langle \det f(U) \rangle^{\text{GC}} = \left\langle \prod_{i=1}^N f(e^{\mu_i}) \right\rangle^{\text{GC}} = \frac{\text{Det}(1 + \kappa \rho f(W))}{\text{Det}(1 + \kappa \rho)}. \quad (2.11)$$

For instance, setting f to

$$f(e^{\mu_i}) = 1 + \varepsilon e^{n\mu_i} \quad (2.12)$$

and picking up the term of order $\mathcal{O}(\varepsilon)$, we find that the grand canonical VEV of 1/6 BPS Wilson loops with winding number n is given by

$$\langle \text{Tr } U^n \rangle^{\text{GC}} = \text{Tr}(RW^n), \quad (2.13)$$

where we defined R as

$$R = \frac{\kappa \rho}{1 + \kappa \rho}. \quad (2.14)$$

Once we know the grand canonical VEV, one can easily find the canonical VEV with fixed N by

$$\langle \text{Tr } U^n \rangle_N = \int_{-\pi i}^{\pi i} \frac{d\mu}{2\pi i} e^{-N\mu} \Xi(\mu, k) \langle \text{Tr } U^n \rangle^{\text{GC}}. \quad (2.15)$$

For the special case $n = 1$, $\langle \text{Tr } U \rangle$ corresponds to the 1/6 BPS Wilson loop in the fundamental representation.

2.1 Computation of trace

From the small κ expansion of the grand canonical VEV (2.13)

$$\langle \text{Tr } U^n \rangle^{\text{GC}} = \sum_{\ell=1}^{\infty} (-1)^{\ell-1} \kappa^\ell \text{Tr}(\rho^\ell W^n), \quad (2.16)$$

one can see that the Wilson loop VEV $\langle \text{Tr } U^n \rangle_N$ at fixed N can be computed from the traces $\text{Tr}(\rho^\ell W^n)$ and $\text{Tr} \rho^\ell$ with $\ell = 1, \dots, N$. Note that the trace $\text{Tr} \rho^\ell$ appears in the computation of the partition function as well. One can compute $\text{Tr} \rho^\ell$ by applying the Tracy-Widom lemma [33]. Thus our remaining task is to compute the trace $\text{Tr}(\rho^\ell W^n)$ with W^n insertion systematically. As we will show below, one can also apply the Tracy-Widom lemma to the computation of the trace $\text{Tr}(\rho^\ell W^n)$.

First we observe that, using the Baker-Campbell-Hausdorff formula, W^n is written as

$$W^n = e^{\frac{n(x+p)}{k}} = e^{\frac{np}{k}} e^{\frac{n(x+n\pi i)}{k}}. \quad (2.17)$$

Noticing that the operator $e^{\frac{np}{k}}$ shifts x by $2\pi i n$, one can show that the matrix element of ρW^n is given by¹

$$\langle x | \rho W^n | y \rangle = \langle x | \rho | y + 2\pi i n \rangle e^{\frac{n(y+n\pi i)}{k}}, \quad (2.19)$$

¹We use the following normalization of quantum mechanical states

$$\langle x | y \rangle = \delta(x - y), \quad \langle p | p' \rangle = \delta(p - p'), \quad \langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}}, \quad \langle x | \frac{1}{2 \cosh \frac{p}{2}} | y \rangle = \frac{1}{\hbar} \frac{1}{2 \cosh \frac{x-y}{2k}}. \quad (2.18)$$

where the matrix element of ρ is given by

$$\langle x|\rho|y\rangle = \frac{1}{\hbar} \frac{1}{2 \cosh \frac{x}{2}} \frac{1}{2 \cosh \frac{x-y}{2k}} = \frac{1}{\hbar} \frac{1}{2 \cosh \frac{x}{2}} \frac{e^{\frac{x+y}{2k}}}{e^{\frac{x}{k}} + e^{\frac{y}{k}}}. \quad (2.20)$$

This is exactly the form of kernel to which we can apply the Tracy-Widom lemma [33]. As shown in [33], the matrix element $\langle x|\rho^\ell|y\rangle$ can be written as

$$\langle x|\rho^\ell|y\rangle = \frac{1}{\hbar} \frac{1}{2 \cosh \frac{x}{2}} \frac{e^{\frac{x+y}{2k}}}{e^{\frac{x}{k}} + (-1)^{\ell-1} e^{\frac{y}{k}}} \sum_{j=0}^{\ell-1} (-1)^j \phi_j(x) \phi_{\ell-1-j}(y), \quad (2.21)$$

where $\phi_j(x)$ is determined recursively

$$\phi_{j+1}(x) = e^{-\frac{x}{2k}} \int \frac{dy}{\hbar} \frac{1}{2 \cosh \frac{x-y}{2k}} \frac{e^{\frac{y}{2k}}}{2 \cosh \frac{y}{2}} \phi_j(y), \quad \phi_0(x) = 1. \quad (2.22)$$

Since the trace $\text{Tr}(\rho^\ell W^n)$ with insertion of W^n is given by the integral of $\langle x|\rho^\ell|y\rangle$ with the replacement $y \rightarrow x + 2\pi i n$

$$\text{Tr}(\rho^\ell W^n) = \int dx \langle x|\rho^\ell|x + 2\pi i n\rangle e^{\frac{n(x+n\pi)}{k}}, \quad (2.23)$$

this trace can also be written in terms of the functions $\phi_j(x)$ in (2.22)

$$\text{Tr}(\rho^\ell W^n) = \frac{e^{\frac{\pi i n(n+1)}{k}}}{1 + (-1)^{\ell-1} e^{\frac{2\pi i n}{k}}} \int \frac{dx}{\hbar} \frac{e^{\frac{nx}{k}}}{2 \cosh \frac{x}{2}} \sum_{j=0}^{\ell-1} (-1)^j \phi_j(x) \phi_{\ell-1-j}(x + 2\pi i n). \quad (2.24)$$

To summarize, the computation of trace $\text{Tr}(\rho^\ell W^n)$ boils down to the construction of a sequence of functions $\phi_j(x)$ ($j = 0, 1, \dots$). We should stress that $\phi_j(x)$ is the *same function* appearing in the computation of trace $\text{Tr} \rho^\ell$. Note that for the integral (2.24) to converge, k should satisfy the condition

$$k > 2n. \quad (2.25)$$

For some integer values of k , the integral in (2.22) and (2.24) can be evaluated exactly by closing the contour and picking up the residue of poles [10, 11, 34, 35]. For instance, we have computed the exact values of $\text{Tr}(\rho^\ell W)$ with the winding number $n = 1$, for $k = 3, 4, 6$ up to certain ℓ . For example, for $k = 3$ we find

$$\begin{aligned} \text{Tr}(\rho W) &= \frac{1}{6} + i \frac{1}{2\sqrt{3}}, \\ \text{Tr}(\rho^2 W) &= \frac{2 - \sqrt{3}}{36} + i \frac{2 - \sqrt{3}}{36}, \\ \text{Tr}(\rho^3 W) &= \frac{9 - 6\pi + 2\sqrt{3}\pi}{648\pi} + i \frac{9\sqrt{3} + 9\pi - 8\sqrt{3}\pi}{216\pi}. \end{aligned} \quad (2.26)$$

For $k = 4$ we find

$$\begin{aligned} \text{Tr}(\rho W) &= \frac{1}{8\sqrt{2}} + i \frac{1}{8\sqrt{2}}, \\ \text{Tr}(\rho^2 W) &= \frac{\pi - 2}{64\sqrt{2}\pi} + i \frac{4 - \pi}{64\sqrt{2}\pi}, \\ \text{Tr}(\rho^3 W) &= \frac{-2 - 2\pi + \pi^2}{512\sqrt{2}\pi^2} + i \frac{-3 - 2\pi + \pi^2}{256\sqrt{2}\pi^2}. \end{aligned} \quad (2.27)$$

For $k = 6$ we find

$$\begin{aligned} \text{Tr}(\rho W) &= \frac{1}{12\sqrt{3}} + i\frac{1}{36}, \\ \text{Tr}(\rho^2 W) &= \frac{9 - \sqrt{3}\pi}{648\pi} + i\frac{7\pi - 12\sqrt{3}}{432\pi}, \\ \text{Tr}(\rho^3 W) &= \frac{6 - \sqrt{3}\pi}{2592\pi} + i\frac{15\sqrt{3} - 8\pi}{7776\pi}. \end{aligned} \tag{2.28}$$

We note in passing that $\text{Tr}(\rho W^n)$ can be found in a closed form for general k, n

$$\text{Tr}(\rho W^n) = \frac{e^{\frac{\pi i n^2}{k}}}{4k \cos^2 \frac{\pi n}{k}}. \tag{2.29}$$

2.2 Numerical computation

It is useful to develop a technique to compute the trace $\text{Tr}(\rho^\ell W^n)$ numerically with high precision. We use the observation that the integral appearing in the computation of $\phi_j(x)$ in (2.22) is written schematically as

$$\int \frac{dy}{\hbar} \frac{1}{2 \cosh \frac{x-y}{2k}} \psi(y) = \int \frac{dp}{2\pi} \frac{e^{\frac{ipx}{\hbar}}}{2 \cosh \frac{p}{2}} \tilde{\psi}(p), \tag{2.30}$$

where $\tilde{\psi}(p)$ is the Fourier transform of $\psi(x)$

$$\tilde{\psi}(p) = \int \frac{dy}{\hbar} e^{-\frac{ipy}{\hbar}} \psi(y). \tag{2.31}$$

Thus the integral (2.30) can be evaluated by a successive application of the Fourier transformation (FT) and the inverse Fourier transformation (FT⁻¹)

$$\int dy \frac{1}{2 \cosh \frac{x-y}{2k}} \psi(y) = \text{FT}^{-1} \left[\frac{1}{2 \cosh \frac{p}{2}} \cdot \text{FT}[\psi](p) \right] (x). \tag{2.32}$$

This can be easily done numerically by discretizing the continuous variable $x \in \mathbb{R}$ to some finite number of points in the range $-L/2 < x \leq L/2$

$$x_i = \left(\frac{i}{m} - \frac{1}{2} \right) L, \quad (i = 1, \dots, m), \tag{2.33}$$

where L is an IR cut-off. Then we can approximate the Fourier transformation by the discrete Fourier transformation.² By taking L and m sufficiently large, we can numerically evaluate the integral (2.32) with very high precision. We will use the parameters $L = 3000$ and $m = 2^{16}$ in the numerical computations below.

For integer value of k , we can estimate the error of the above numerical computation by comparing with the exact values of traces obtained in the previous subsection. Let us consider the imaginary part of $\text{Tr}(\rho^\ell W^n)$ for $k = 4, n = 1$, as an example. For $k = 4$, we

²We compute the trace $\text{Tr} \rho^\ell W^n$ numerically using a `Mathematica` program implementing this algorithm, originally written by Yasuyuki Hatsuda. We are grateful to him for sharing his program with us. The numerical data of the traces are available upon request to the author.

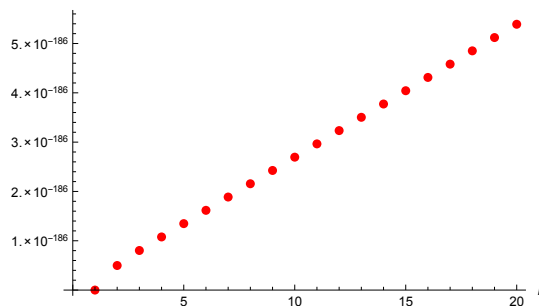


Figure 1. Plot of the numerical error e_ℓ (2.34) of the imaginary part of trace $\text{Tr}(\rho^\ell W)$ for $k = 4$.

have computed the exact values of $\text{Tr}(\rho^\ell W)$ up to $\ell = 20$ (see (2.27) for the first three terms). In figure 1, we show the plot of the relative error e_ℓ between the exact values and the numerical values of the imaginary part of $\text{Tr}(\rho^\ell W)$

$$e_\ell = \left| \frac{\text{Im Tr}(\rho^\ell W)_{\text{numerical}}}{\text{Im Tr}(\rho^\ell W)_{\text{exact}}} - 1 \right|. \tag{2.34}$$

As we can see from figure 1, the numerical error e_ℓ is extremely small

$$e_\ell < 6 \times 10^{-186}, \quad (\ell \leq 20). \tag{2.35}$$

On the other hand, the order of worldsheet 1-instanton correction and membrane 1-instanton correction for $k = 4, N = 20$ are given by (see (4.9))

$$\begin{aligned} \text{worldsheet 1-instanton} : & \quad e^{-2\pi\sqrt{\frac{2N}{k}}} \approx 2.35 \times 10^{-9}, \\ \text{membrane 1-instanton} : & \quad e^{-2\pi\sqrt{\frac{kN}{2}}} \approx 5.52 \times 10^{-18}. \end{aligned} \tag{2.36}$$

Comparing (2.35) and (2.36), we conclude that our numerical calculation has enough accuracy to study instanton corrections.

3 Perturbative part of winding Wilson loops

We will study the large μ expansion of the grand canonical VEV of 1/6 BPS winding Wilson loops. First we consider the perturbative part (or zero-instanton sector) of this expansion.

In the large μ limit, the grand canonical VEV of 1/6 Wilson loops with winding number n behaves as [17]

$$\langle \text{Tr } U^n \rangle^{\text{GC}} \sim e^{\frac{2n\mu}{k}}. \tag{3.1}$$

We would like to understand the large μ behavior of 1/6 Wilson loops in more detail. It is shown in [17] that $\langle \text{Tr } U^n \rangle^{\text{GC}}$ can be decomposed as

$$\langle \text{Tr } U^n \rangle^{\text{GC}} = i^{n-1} \left(\frac{1}{2} |\mathcal{W}_n^{(1/2)}| + i\mathcal{W}_n \right). \tag{3.2}$$

The first term is the absolute value of the 1/2 BPS Wilson loop which was studied extensively in the literature [17, 18]. The perturbative part of 1/2 BPS Wilson loop is given by [17]

$$|\mathcal{W}_n^{(1/2),\text{pert}}| = \frac{e^{-\frac{2n\mu}{k}}}{2 \sin \frac{2\pi n}{k}}. \tag{3.3}$$

It is known that the 1/2 BPS Wilson loops are related to the open topological string on local $\mathbb{P}^1 \times \mathbb{P}^1$ [20–23], and the worldsheet instanton corrections to the 1/2 BPS Wilson loops can be readily obtained from the results of open topological string.

On the other hand, \mathcal{W}_n in (3.2) is less understood. Note that \mathcal{W}_n is written as

$$\mathcal{W}_n = \text{Im} \left[i^{1-n} \langle \text{Tr } U^n \rangle^{\text{GC}} \right]. \tag{3.4}$$

We will loosely call \mathcal{W}_n “the imaginary part of 1/6 BPS Wilson loop” with the understanding that the precise meaning is (3.4). In this paper, we will study the instanton corrections in \mathcal{W}_n . To do that, first we compute the canonical VEV $\mathcal{W}_n(N, k)$ numerically with high precision using the algorithm in section 2.2. Then we can find the instanton corrections by numerically fitting with the expansion

$$\mathcal{W}_n(N, k) = e^{A(k)} C(k)^{-\frac{1}{3}} \sum_{j,w} a_{j,w} (-\partial_N)^j \text{Ai} \left[C(k)^{-\frac{1}{3}} \left(N - B(k) - \frac{2n}{k} - w \right) \right], \tag{3.5}$$

where $a_{j,w}$ is the coefficient in the instanton expansion of the grand canonical VEV

$$\Xi(\mu, k) \mathcal{W}_n(\mu, k) = e^{J_{\text{pert}}(\mu, k) + \frac{2n\mu}{k}} \sum_{j,w} a_{j,w} \mu^j e^{-w\mu}. \tag{3.6}$$

Here $J_{\text{pert}}(\mu, k)$ denotes the perturbative part of the grand potential of ABJM theory

$$J_{\text{pert}}(\mu, k) = \frac{C(k)\mu^3}{3} + B(k)\mu + A(k), \tag{3.7}$$

and the coefficients $C(k)$, $B(k)$ and $A(k)$ are given by

$$\begin{aligned} C(k) &= \frac{2}{\pi k^2}, & B(k) &= \frac{1}{3k} + \frac{k}{24}, \\ A(k) &= -\frac{k^2 \zeta(3)}{8\pi^2} + 4 \int_0^\infty dx \frac{x}{e^x - 1} \log \left(2 \sinh \frac{2\pi x}{k} \right). \end{aligned} \tag{3.8}$$

It is found [36–38] that $A(k)$ is a certain resummation of the constant map contribution of topological string on local $\mathbb{P}^1 \times \mathbb{P}^1$. The expansion (3.5) in the canonical picture and the grand canonical picture (3.6) are related by the integral transformation

$$\mathcal{W}_n(N, k) = \int_{\mathcal{C}} \frac{d\mu}{2\pi i} e^{J(\mu, k)} \mathcal{W}_n(\mu, k), \tag{3.9}$$

where \mathcal{C} is the contour in the μ -plane from $e^{-\frac{\pi i}{3}} \infty$ to $e^{\frac{\pi i}{3}} \infty$, and $J(\mu, k)$ is known as the modified grand potential which is related to the grand partition function by [11]

$$\Xi(\mu, k) = \sum_{n \in \mathbb{Z}} e^{J(\mu + 2\pi i n, k)}. \tag{3.10}$$

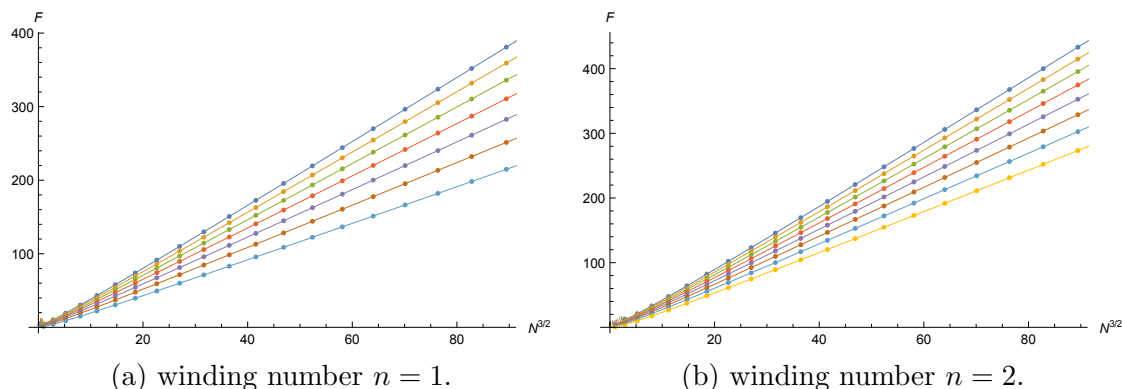


Figure 2. This is the plot of “free energy” $F = -\log \mathcal{W}_n(N, k)$ for (a) $n = 1$ and (b) $n = 2$ of the imaginary part of $1/6$ BPS Wilson loop VEV. Note that the horizontal axis is $N^{3/2}$. We plot the free energy for $k = 3, 4, \dots, 9$ in (a) and for $k = 5, 6, \dots, 12$ in (b). k increases from the bottom curve to the top curve in both (a) and (b). The dots are the numerical values while the solid curves represent the perturbative free energy given by the Airy function (3.12).

Note that the original integral in (2.15) is along the finite segment, but we can extend the contour to an infinite path \mathcal{C} in (3.9) because of the summation over the $2\pi i$ -shift (3.10).

From the numerical fitting, we find that the perturbative part of the imaginary part of winding Wilson loop is given by

$$\mathcal{W}_n^{\text{pert}}(\mu, k) = \frac{e^{\frac{2n\mu}{k}}}{k\pi \sin \frac{2\pi n}{k}} \left(\mu - \pi \sum_{j=1}^n \cot \frac{2\pi j}{k} \right). \quad (3.11)$$

Using (3.5), the corresponding canonical VEV is written as

$$\mathcal{W}_n^{\text{pert}}(N, k) = \frac{e^{A(k)} C(k)^{-\frac{1}{3}}}{k\pi \sin \frac{2\pi n}{k}} \left(-\partial_N - \pi \sum_{j=1}^n \cot \frac{2\pi j}{k} \right) \text{Ai} \left[C(k)^{-\frac{1}{3}} \left(N - B(k) - \frac{2n}{k} \right) \right]. \quad (3.12)$$

In figure 2, we show the plot of “free energy” $F = -\log \mathcal{W}_n(N, k)$ for $n = 1, 2$ as a function of $N^{3/2}$. One can clearly see that the numerical value of $\mathcal{W}_n(N, k)$ computed from the algorithm in section 2.2 exhibits a nice agreement with the perturbative part given by the Airy function and its derivative (3.12).

Comparison with KMSS. In [17] (which we refer to as KMSS), the perturbative part of winding Wilson loop was obtained as

$$\langle \text{Tr } U^n \rangle^{\text{GC, pert}} = \frac{2\pi n e^{\frac{2n\mu}{k}}}{k \sin \frac{2\pi n}{k}} \left[\left(\mu + \frac{k}{2n} - \pi \cot \frac{2\pi n}{k} \right) A + B \right], \quad (3.13)$$

with

$$A = \frac{i^n}{2\pi^2 n}, \quad B = i^{n-1} \frac{k}{4\pi^2 n} \left(\frac{\pi}{2} - iH_n \right). \quad (3.14)$$

Here H_n denotes the harmonic number. We can recast (3.13) in the form of our decomposition in (3.2)

$$\langle \text{Tr } U^n \rangle^{\text{GC, pert}} = i^{n-1} \left(\frac{1}{2} |W_n^{(1/2)}| + iW_n^{\text{KMSS}} \right). \quad (3.15)$$

The first term is the same as (3.3) while the second term is given by

$$\mathcal{W}_n^{\text{KMSS}} = \frac{e^{\frac{2n\mu}{k}}}{k\pi \sin \frac{2\pi n}{k}} \left(\mu - \pi \cot \frac{2\pi n}{k} - \frac{k}{2} H_{n-1} \right), \quad (3.16)$$

which is different from our result (3.11) for general $n \geq 2$. For the fundamental representation $n = 1$, (3.16) agrees with (3.11) since $H_0 = 0$.

In the 't Hooft limit

$$k, \mu \rightarrow \infty \quad \text{with} \quad \frac{\mu}{k} : \text{fixed} \quad (3.17)$$

the genus-zero part of (3.11) is equal to (3.16) for general n , but the higher genus parts are different. Note that, for $n \geq 2$ the agreement of (3.16) and the matrix model computation has been checked in [17] only for the genus-zero part.³ To see whether our conjecture (3.11) is correct or not, we should therefore consider the higher genus corrections. It is likely that the approximation used in [17] to derive (3.16) misses some $1/k$ corrections.⁴ Our numerical study strongly suggests that (3.11) is the correct perturbative part.

4 Fundamental Wilson loop

In this section, we will consider the instanton corrections of the imaginary part of 1/6 BPS Wilson loop $\mathcal{W}_1(\mu, k)$ in the fundamental representation, i.e. the winding number $n = 1$. Using the instanton expansion (3.5) in terms of the Airy function and its derivatives, one can find the coefficient $a_{j,w}$ in (3.5) by fitting the value of canonical VEV $\mathcal{W}_1(N, k)$ computed either exactly (see section 2.1) or numerically (see section 2.2). The expansion coefficients $a_{j,w}$ becomes simple numbers for some integer k , and one can guess the exact value of the coefficients from the numerical fitting. The results are summarized in appendix A.1. We have also computed the instanton expansion for fractional k using the numerical computation of the trace $\text{Tr}(\rho^\ell W)$ explained in section 2.2. It turns out that the grand canonical VEV of the imaginary part of 1/6 BPS Wilson loop in the fundamental representation has the following expansion

$$\mathcal{W}_1(\mu, k) = \mathcal{W}_1^{\text{pert}}(\mu, k) + \sum_{\ell, m} \mathcal{W}_1^{(\ell, m)}(\mu, k) \quad (4.1)$$

where $\mathcal{W}_1^{(\ell, m)}(\mu, k)$ denotes the contribution of the bound state of membrane ℓ -instanton and worldsheet m -instanton

$$e^{-\frac{2\mu}{k}} \mathcal{W}_1^{(\ell, m)}(\mu, k) \propto e^{-2\ell\mu - \frac{4\mu m}{k}}. \quad (4.2)$$

³For the fundamental representation, the agreement between (3.16) and the matrix model was checked for higher genus corrections as well. While for $n > 1$, the authors of [17] used (3.16) to predict the higher genus corrections for general winding.

⁴We would like to thank Marcos Mariño for raising this possibility.

Worldsheet instantons. First, let us consider the worldsheet instanton corrections (and the perturbative part)

$$\mathcal{W}_1^{\text{WS}}(\mu, k) = \mathcal{W}_1^{\text{pert}}(\mu, k) + \sum_{m=1}^{\infty} \mathcal{W}_1^{(0,m)}(\mu, k). \quad (4.3)$$

From the numerical fitting, we find that the worldsheet instanton corrections are given by

$$\begin{aligned} & e^{-\frac{2\mu}{k}} \mathcal{W}_1^{\text{WS}}(\mu, k) \\ &= \frac{\mu}{k\pi \sin \frac{2\pi}{k}} \left[1 + 2e^{-\frac{4\mu}{k}} - e^{-\frac{8\mu}{k}} + 2e^{-\frac{12\mu}{k}} - 7e^{-\frac{16\mu}{k}} + \mathcal{O}(e^{-\frac{20\mu}{k}}) \right] \\ & - \frac{\cos \frac{2\pi}{k}}{k \sin^2 \frac{2\pi}{k}} \left[1 + 0 \cdot e^{-\frac{4\mu}{k}} - \left(3 + \frac{1}{\cos^2 \frac{2\pi}{k}} \right) e^{-\frac{8\mu}{k}} \right. \\ & \quad + \frac{2 \sin \frac{2\pi}{k}}{\sin \frac{6\pi}{k}} \left(32 \cos^2 \frac{2\pi}{k} - 10 + \frac{1}{\cos^2 \frac{2\pi}{k}} \right) e^{-\frac{12\mu}{k}} \\ & \quad \left. + \left(\frac{8 \sin \frac{2\pi}{k}}{\sin \frac{6\pi}{k}} - \frac{7}{2 \cos^2 \frac{2\pi}{k}} - \frac{1}{\cos \frac{4\pi}{k}} - 43 - 32 \cos \frac{4\pi}{k} \right) e^{-\frac{16\mu}{k}} + \mathcal{O}(e^{-\frac{20\mu}{k}}) \right]. \end{aligned} \quad (4.4)$$

Note that the worldsheet instanton in (4.4) has poles at $k = 4, 6$, which should be canceled by the membrane instanton and the bound state. Thus, we expect that there are membrane instanton corrections in the imaginary part of 1/6 BPS Wilson loop. This is in contrast to the 1/2 BPS Wilson loops where the “pure” membrane instanton corrections are absent except for the bound state corrections coming from the quantum mirror map $\mu \rightarrow \mu_{\text{eff}}$ [18, 20].

Membrane instantons and bound states. From the numerical fitting, we find that the membrane 1-instanton and the (1, 1) bound state are given by

$$\begin{aligned} e^{-\frac{2\mu}{k}} \mathcal{W}_1^{(1,0)} &= e^{-2\mu} \frac{\cos \frac{\pi k}{2}}{k\pi \sin \frac{2\pi}{k}} \left[\left(4 - \frac{4}{k} \right) \left(\mu - \pi \cot \frac{2\pi}{k} \right) - 2 - \pi k \cot \frac{\pi k}{2} \right], \\ e^{-\frac{2\mu}{k}} \mathcal{W}_1^{(1,1)} &= e^{-\frac{4\mu}{k} - 2\mu} \frac{\cos \frac{\pi k}{2}}{k\pi \sin \frac{2\pi}{k}} \left[\left(8 + \frac{8}{k} \right) \mu - 4 - 2\pi k \cot \frac{\pi k}{2} \right]. \end{aligned} \quad (4.5)$$

As expected, one can show that the poles at $k = 4, 6$ are canceled between worldsheet instantons (4.4) and membrane instantons or bound states (4.5), and the remaining finite part reproduces the result in appendix A.1

$$\begin{aligned} \lim_{k \rightarrow 4} e^{-\frac{2\mu}{k}} \left[\mathcal{W}_1^{(1,0)} + \mathcal{W}_1^{(0,2)} \right] &= \frac{3\mu - 1}{2\pi} e^{-2\mu}, \\ \lim_{k \rightarrow 4} e^{-\frac{2\mu}{k}} \left[\mathcal{W}_1^{(1,1)} + \mathcal{W}_1^{(0,3)} \right] &= \frac{5\mu - 1}{\pi} e^{-3\mu}, \\ \lim_{k \rightarrow 6} e^{-\frac{2\mu}{k}} \left[\mathcal{W}_1^{(1,0)} + \mathcal{W}_1^{(0,3)} \right] &= \left[\frac{2(-8\mu + 3)}{9\sqrt{3}\pi} - \frac{20}{27} \right] e^{-2\mu}, \\ \lim_{k \rightarrow 6} e^{-\frac{2\mu}{k}} \left[\mathcal{W}_1^{(1,1)} + \mathcal{W}_1^{(0,4)} \right] &= \left[\frac{-73\mu + 12}{9\sqrt{3}\pi} + \frac{35}{9} \right] e^{-\frac{8\mu}{3}}. \end{aligned} \quad (4.6)$$

This pole cancellation mechanism was originally found in the grand potential of ABJM theory [11]. Here we find that the similar pole cancellation mechanism works also in the VEV of 1/6 BPS Wilson loops.

Let us see that our conjecture of instanton corrections (4.4) and (4.5) correctly reproduces the behavior of $\mathcal{W}_1(N, k)$ for non-integer k as well. We take $k = 2.7$ as an example. For $k = 2.7$, the instanton factors have the following ordering

$$e^{-\frac{4\mu}{k}} > e^{-2\mu} > e^{-\frac{8\mu}{k}} > e^{-\frac{4\mu}{k}-2\mu} > \dots \quad (4.7)$$

Once we know the instanton correction in the grand canonical picture $\mathcal{W}_1^{(\ell,m)}(\mu, k)$, we can easily translate it to the canonical picture $\mathcal{W}_1^{(\ell,m)}(N, k)$ using (3.5), and as a consequence the instanton correction $\mathcal{W}_1^{(\ell,m)}(N, k)$ can be written as an Airy function and its derivatives. From (4.7), we define the quantity $\delta_{\ell,m}$ by subtracting the instanton corrections up to the order $e^{-2\ell\mu - \frac{4m\mu}{k}}$ (we do not subtract the term $\mathcal{W}_1^{(\ell,m)}(N, k)$ in defining $\delta_{\ell,m}$)

$$\begin{aligned} \delta_{0,1} &= \frac{\mathcal{W}_1(N, k) - \mathcal{W}_1^{\text{pert}}(N, k)}{\mathcal{W}_1^{\text{pert}}(N, k)} e^{\frac{4\mu_*}{k}}, \\ \delta_{1,0} &= \frac{\mathcal{W}_1(N, k) - \mathcal{W}_1^{\text{pert}}(N, k) - \mathcal{W}_1^{(0,1)}(N, k)}{\mathcal{W}_1^{\text{pert}}(N, k)} e^{2\mu_*}, \\ \delta_{0,2} &= \frac{\mathcal{W}_1(N, k) - \mathcal{W}_1^{\text{pert}}(N, k) - \mathcal{W}_1^{(0,1)}(N, k) - \mathcal{W}_1^{(1,0)}(N, k)}{\mathcal{W}_1^{\text{pert}}(N, k)} e^{\frac{8\mu_*}{k}}, \\ \delta_{1,1} &= \frac{\mathcal{W}_1(N, k) - \mathcal{W}_1^{\text{pert}}(N, k) - \mathcal{W}_1^{(0,1)}(N, k) - \mathcal{W}_1^{(1,0)}(N, k) - \mathcal{W}_1^{(0,2)}(N, k)}{\mathcal{W}_1^{\text{pert}}(N, k)} e^{2\mu_* + \frac{4\mu_*}{k}}. \end{aligned} \quad (4.8)$$

We have included the exponential factor $e^{2\ell\mu_* + \frac{4m\mu_*}{k}}$ in the definition of $\delta_{\ell,m}$ with μ_* being the saddle point value of the chemical potential⁵

$$\mu_* = \sqrt{\frac{N}{C(k)}} = \pi \sqrt{\frac{kN}{2}}. \quad (4.10)$$

The canonical VEV $\mathcal{W}_1(N, k)$ in (4.8) can be evaluated numerically with high precision using the method in section 2.2 even for a non-integer value of $k = 2.7$. We expect that $\delta_{\ell,m}$ is approximated by

$$\delta_{\ell,m} \approx \frac{\mathcal{W}_1^{(\ell,m)}(N, k)}{\mathcal{W}_1^{\text{pert}}(N, k)} e^{2\ell\mu_* + \frac{4m\mu_*}{k}}. \quad (4.11)$$

As shown in figure 3, we find a nice agreement between (4.8) and (4.11), as expected. This confirms the validity of our conjecture of instanton corrections (4.4) and (4.5), for $k = 2.7$. We have performed similar checks for various values of k .

⁵Note that this relation (4.10) implies that the worldsheet instanton and the membrane instanton factors in the canonical picture are given by

$$e^{-\frac{4\mu_*}{k}} = e^{-2\pi\sqrt{\frac{2N}{k}}}, \quad e^{-2\mu_*} = e^{-2\pi\sqrt{\frac{kN}{2}}}. \quad (4.9)$$

We have already used this relation in (2.36).

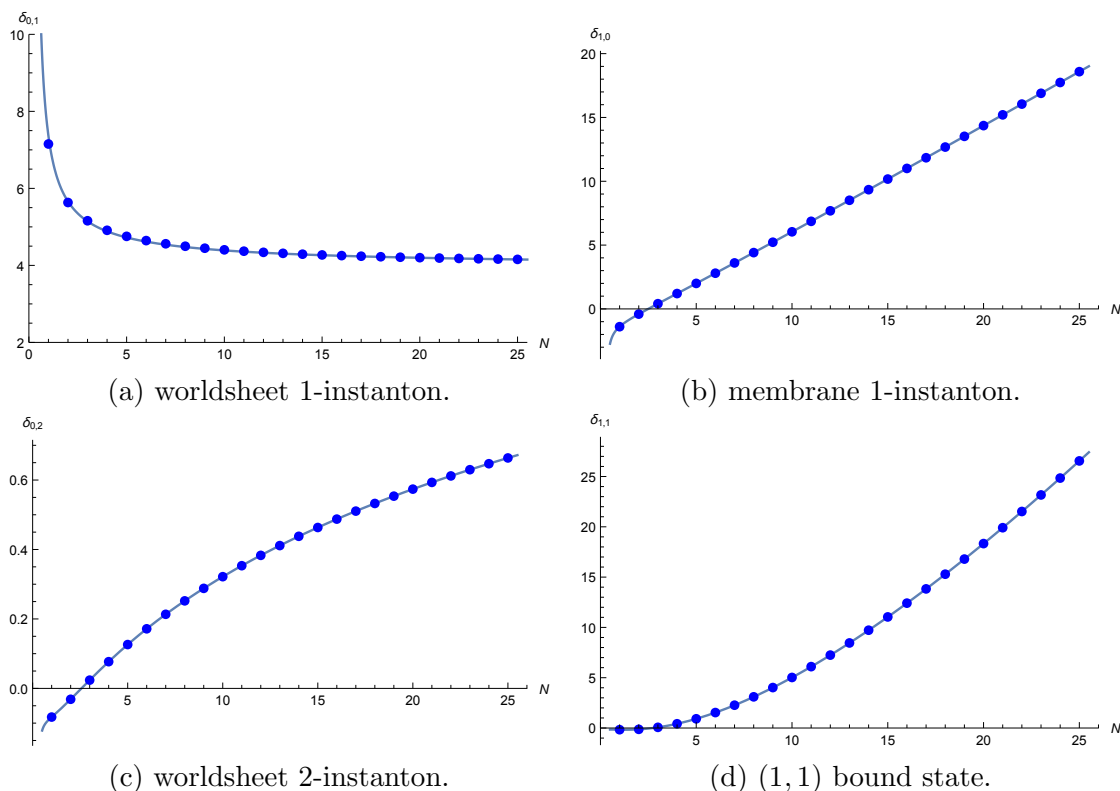


Figure 3. This is the plot of (a) $\delta_{0,1}$ (b) $\delta_{1,0}$ (c) $\delta_{0,2}$ and (d) $\delta_{1,1}$ against N , for $k = 2.7$. The dots are the numerical values of (4.8) while the solid curves represent the expected behavior of (ℓ, m) instanton given by the Airy function and its derivatives (4.11).

Rewriting in terms of μ_{eff} . Let us rewrite $\mathcal{W}_1(\mu, k)$ in terms of the “effective” chemical potential μ_{eff} , which was first introduced in [11]

$$\mu_{\text{eff}} = \mu - 2 \cos \frac{\pi k}{2} e^{-2\mu} - (4 + 5 \cos \pi k) e^{-4\mu} + \dots \quad (4.12)$$

This is interpreted as the quantum mirror map of local $\mathbb{P}^1 \times \mathbb{P}^1$ [13] and the coefficient of this expansion can be easily obtained from the formal solution of the wavefunction annihilated by the quantized mirror curve of local $\mathbb{P}^1 \times \mathbb{P}^1$ [39].

After rewriting the instanton corrections in (4.4) and (4.5) in terms of μ_{eff} , somewhat miraculously, we find that $\mathcal{W}_1(\mu, k)$ is completely factorized up to the (1, 1) bound state

$$\mathcal{W}_1 = \frac{e^{\frac{2\mu_{\text{eff}}}{k}}}{k\pi \sin \frac{2\pi}{k}} (1+2Q_w) \left(1+4 \cos \frac{\pi k}{2} Q_m\right) \left[\mu_{\text{eff}} - \pi \cot \frac{2\pi}{k} (1-2Q_w) - \pi k \cos \frac{\pi k}{2} \cot \frac{\pi k}{2} Q_m \right], \quad (4.13)$$

where we have introduced the worldsheet instanton factor Q_w and the membrane instanton factor Q_m by

$$Q_w = e^{-\frac{4\mu_{\text{eff}}}{k}}, \quad Q_m = e^{-2\mu_{\text{eff}}}. \quad (4.14)$$

Assuming that this factorized structure holds for higher instanton numbers, we can continue the numerical fitting of instanton coefficients. In this way, we find that \mathcal{W}_1 is

written as

$$\mathcal{W}_1 = \frac{e^{\frac{2\mu_{\text{eff}}}{k}}}{k\pi \sin \frac{2\pi}{k}} f_w f_m \left(\mu_{\text{eff}} - \pi \cot \frac{2\pi}{k} V_w - \pi k V_m \right), \quad (4.15)$$

where

$$\begin{aligned} f_w = & 1 + 2Q_w - Q_w^2 + 2Q_w^3 - 7Q_w^4 + 16 \cos^2 \frac{2\pi}{k} \left(3 - \cos \frac{4\pi}{k} \right) Q_w^5 + \\ & + \left(-72 \cos \left(\frac{4\pi}{k} \right) - 24 \cos \left(\frac{8\pi}{k} \right) + 8 \cos \left(\frac{12\pi}{k} \right) + 6 \cos \left(\frac{16\pi}{k} \right) - 90 \right) Q_w^6 + \mathcal{O}(Q_w^7), \quad (4.16) \\ V_w = & 1 - 2Q_w + \frac{\cos \frac{4\pi}{k}}{\cos^2 \frac{2\pi}{k}} Q_w^2 + 4 \left(2 + \frac{\sin \frac{2\pi}{k}}{\sin \frac{6\pi}{k}} \right) Q_w^3 - 2 \left(23 + \frac{(1 + 4 \cos \frac{4\pi}{k})^2 \sin \frac{6\pi}{k}}{\cos \frac{2\pi}{k} \sin \frac{8\pi}{k}} \right) Q_w^4 \\ & + \left(322 + 152 \cos \left(\frac{4\pi}{k} \right) + 108 \cos \left(\frac{8\pi}{k} \right) + \frac{8 \sin \left(\frac{2\pi}{k} \right) \cos \left(\frac{4\pi}{k} \right)}{\sin \left(\frac{10\pi}{k} \right)} \right) Q_w^5 \\ & + \left(-1428 - 1416 \cos \left(\frac{4\pi}{k} \right) - 816 \cos \left(\frac{8\pi}{k} \right) - 248 \cos \left(\frac{12\pi}{k} \right) - 84 \cos \left(\frac{16\pi}{k} \right) \right. \\ & \left. - \frac{82}{3 \cos^2 \left(\frac{2\pi}{k} \right)} + \frac{4 \left(13 \sin \left(\frac{4\pi}{k} \right) - 14 \sin \left(\frac{8\pi}{k} \right) \right)}{3 \sin \left(\frac{12\pi}{k} \right)} \right) Q_w^6 + \mathcal{O}(Q_w^7), \quad (4.17) \end{aligned}$$

and

$$\begin{aligned} f_m = & 1 + 4 \cos \frac{\pi k}{2} Q_m + 4(2 + 3 \cos \pi k) Q_m^2 + 8 \cos \frac{\pi k}{2} (5 + 9 \cos \pi k + 3 \cos 2\pi k) Q_m^3 + \mathcal{O}(Q_m^4), \\ V_m = & \cos \frac{\pi k}{2} \cot \frac{\pi k}{2} Q_m + (4 + 5 \cos \pi k) \cot \pi k Q_m^2 \\ & + 2 \cot \frac{3\pi k}{2} \cos \frac{\pi k}{2} (13 + 19 \cos \pi k + 9 \cos 2\pi k) Q_m^3 + \mathcal{O}(Q_m^4). \quad (4.18) \end{aligned}$$

We observe that the coefficient of V_m is proportional to the membrane instanton coefficient $\tilde{b}_\ell(k)$ in the modified grand potential of ABJM theory [11]

$$J(\mu, k) = J_{\text{WS}}(\mu_{\text{eff}}) + \sum_{\ell=1}^{\infty} \left[\mu_{\text{eff}} \tilde{b}_\ell(k) - k^2 \frac{\partial}{\partial k} \left(\frac{\tilde{b}_\ell(k)}{2\ell k} \right) \right] e^{-2\ell\mu_{\text{eff}}}. \quad (4.19)$$

From this observation, we conjecture that V_m is given by

$$V_m = -\frac{\pi}{4} \frac{\partial \tilde{J}_b}{\partial \mu_{\text{eff}}}, \quad \tilde{J}_b = \sum_{\ell=1}^{\infty} \tilde{b}_\ell(k) e^{-2\ell\mu_{\text{eff}}}. \quad (4.20)$$

We also observe that f_m in (4.18) is equal to the derivative of the effective chemical potential

$$f_m = \frac{\partial \mu_{\text{eff}}}{\partial \mu}. \quad (4.21)$$

Relation to quantum volume. As shown in [13], the membrane instanton part \tilde{J}_b of the modified grand potential of ABJM theory is given by the NS free energy on local

$\mathbb{P}^1 \times \mathbb{P}^1$. From this relation, V_m in (4.20) is further rewritten as⁶

$$V_m = -\frac{1}{2} \frac{\partial^2}{\partial t^2} F_{\text{NS}} \left(Q_m q^{\frac{1}{2}}, Q_m q^{-\frac{1}{2}}; q \right), \quad (4.23)$$

where we defined

$$Q_m = e^{-t}, \quad t = 2\mu_{\text{eff}}, \quad q = e^{i\hbar_{\text{top}}}, \quad \hbar_{\text{top}} = \pi k. \quad (4.24)$$

Note that the normalization of the Planck constant \hbar_{top} in topological string is different from the \hbar of ABJM Fermi gas (2.8) by a factor of 2, which comes from rewriting the operator ρ^{-1} into the canonical form of the mirror curve of local $\mathbb{P}^1 \times \mathbb{P}^1$ [24]. Appearance of the NS free energy in (4.23) suggests that the imaginary part of 1/6 BPS Wilson loop is closely related to the quantum volume Ω of the phase space, which determines the exact quantization condition of the operator ρ

$$\Omega = 2\pi \left(n + \frac{1}{2} \right). \quad (4.25)$$

It is found in [24, 26–28] that the quantum volume Ω has two pieces

$$\Omega = \Omega^{\text{WKB}} + \Omega^{\text{np}}, \quad (4.26)$$

where the WKB part Ω^{WKB} is given by the NS free energy

$$\Omega^{\text{WKB}} = \frac{t^2}{\hbar_{\text{top}}} - \frac{2\pi^2}{3\hbar_{\text{top}}} + \frac{\hbar_{\text{top}}}{12} + 2 \frac{\partial}{\partial t} F_{\text{NS}} \left(Q q^{\frac{1}{2}}, Q q^{-\frac{1}{2}}; q \right), \quad (4.27)$$

while the non-perturbative part Ω^{np} of quantum volume is given by the ‘‘S-dual’’ of Ω^{WKB}

$$\Omega^{\text{np}} = 2 \frac{\partial}{\partial t_D} F_{\text{NS}}(-Q_w, -Q_w; q_D), \quad (4.28)$$

where the ‘‘S-dual’’ variables are given by

$$Q_w = e^{-t_D}, \quad t_D = \frac{2\pi}{\hbar_{\text{top}}} t, \quad q_D = e^{i\hbar_{\text{top}}^D}, \quad \hbar_{\text{top}}^D = \frac{4\pi^2}{\hbar_{\text{top}}}. \quad (4.29)$$

We find that the membrane instanton part of $\mathcal{W}_1(\mu, k)$ is related to Ω^{WKB} by

$$\mu_{\text{eff}} - \pi k V_m = \frac{\pi k}{4} \partial_t \Omega^{\text{WKB}}. \quad (4.30)$$

It is found that the pole cancellation at rational value of k is guaranteed by the combination of $\Omega^{\text{WKB}} + \Omega^{\text{np}}$. Since the pole cancellation also occurs in the imaginary part of 1/6

⁶For a general local Calabi-Yau, the NS free energy is written as

$$F_{\text{NS}}(\mathbf{Q}; e^{i\hbar}) = \sum_{j_L, j_R, \mathbf{d}} N_{j_L, j_R}^{\mathbf{d}} \sum_{w=1}^{\infty} \frac{\sin \frac{\hbar w}{2} (2j_L + 1) \sin \frac{\hbar w}{2} (2j_R + 1)}{2w^2 \sin^3 \frac{\hbar w}{2}} \prod_I Q_I^{w d_I}. \quad (4.22)$$

BPS Wilson loop $\mathcal{W}_1(\mu, k)$, it is natural to expect that the worldsheet instanton correction V_w in (4.17) is related to the S-dual of NS free energy in (4.28). In fact, if we define

$$\partial_{t_D} \tilde{\Omega}^{\text{np}} = -2 \cot \frac{2\pi}{k} V_w, \tag{4.31}$$

we find that the $\partial_{t_D} \tilde{\Omega}^{\text{np}}$ and $\partial_{t_D} \Omega^{\text{np}}$ have the same singularity structure, and their difference is regular in the convergence region $k > 2$ of $\mathcal{W}_1(\mu, k)$

$$\partial_{t_D} (\tilde{\Omega}^{\text{np}} - \Omega^{\text{np}}) = 2 \cot \frac{2\pi}{k} \left[-1 + 8 \cos \frac{4\pi}{k} Q_w^2 - 4 \left(7 + 9 \cos \frac{8\pi}{k} \right) Q_w^3 + \dots \right]. \tag{4.32}$$

Putting all together, we arrive at a surprisingly simple expression of the imaginary part of the 1/6 BPS Wilson loop in the fundamental representation

$$\mathcal{W}_1(\mu, k) = \frac{e^{\frac{2\mu_{\text{eff}}}{k}} f_w}{4 \sin \frac{2\pi}{k}} \frac{\partial t}{\partial(2\mu)} \frac{\partial}{\partial t} (\Omega^{\text{WKB}} + \tilde{\Omega}^{\text{np}}) = \frac{e^{\frac{2\mu_{\text{eff}}}{k}} f_w}{8 \sin \frac{2\pi}{k}} \frac{\partial}{\partial \mu} (\Omega^{\text{WKB}} + \tilde{\Omega}^{\text{np}}). \tag{4.33}$$

Currently we do not have a clear understanding of the physical meaning of the regular part in (4.32). The factor f_w might be interpreted as a part of the definition of the ‘‘open flat coordinate’’ [40] for the 1/6 BPS Wilson loops. It would be very interesting to understand the deep reason of the connection between the 1/6 BPS Wilson loops and the quantum volume.

4.1 Genus expansion of 1/6 BPS Wilson loops at large λ

Using (4.4), we can predict the 't Hooft expansion of 1/6 BPS Wilson loop at large 't Hoot coupling $\lambda = N/k$. Let us consider the genus expansion of the canonical VEV of the imaginary part of 1/6 BPS Wilson loop in the fundamental representation, normalized by the partition function

$$\frac{\mathcal{W}_1(N, k)}{Z(N, k)} = \frac{e^{\frac{1}{2}s}}{\pi i} \sum_{g=0}^{\infty} g_s^{2g-1} \mathcal{W}_1^{(g)}, \tag{4.34}$$

where the string coupling g_s and the parameter s are defined by

$$g_s = \frac{4\pi i}{k}, \quad s = 2\pi \sqrt{2\hat{\lambda}}, \quad \hat{\lambda} = \lambda - \frac{1}{24}. \tag{4.35}$$

From the planar solution of the resolvent of ABJM matrix model, the genus-zero part of the fundamental Wilson loop is written as [6, 21]

$$\lim_{N, k \rightarrow \infty} \frac{\frac{1}{N} \langle \text{Tr } U \rangle_N}{Z(N, k)} = \frac{1}{2\pi^2 i \lambda} \int_{-a}^a dx e^x \arctan \sqrt{\frac{\sinh^2 \frac{a}{2} - \sinh^2 \frac{x}{2}}{\cosh^2 \frac{x}{2} - \sinh^2 \frac{a}{2}}}, \tag{4.36}$$

where the end-point of the cut a is given by

$$i\kappa = 4 \sinh^2 \frac{a}{2}, \tag{4.37}$$

and λ and κ are related by

$$\lambda = \frac{\kappa}{8\pi} {}_3F_2 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; -\frac{\kappa^2}{16} \right). \tag{4.38}$$

We find that the imaginary part of 1/6 BPS Wilson loop at genus-zero can be written in a simple form

$$\mathcal{W}_1^{(g=0)} = -\pi^2 \int d\kappa \kappa \frac{d\lambda}{d\kappa} = 4E\left(-\frac{\kappa^2}{16}\right) - \frac{(\kappa^2 + 16)}{4} K\left(-\frac{\kappa^2}{16}\right), \quad (4.39)$$

where $K(k^2)$ and $E(k^2)$ denote the complete elliptic integrals of the first and the second kinds, respectively.

The worldsheet instanton corrections of $\mathcal{W}_1(\mu, k)$ in (4.4) in the grand canonical picture can be easily translated to the canonical picture in the 't Hooft limit

$$\begin{aligned} \mathcal{W}_1^{(g=0)} &= 1 - \frac{s}{2} - (1+s)e^{-s} + \left(\frac{s}{2} - \frac{1}{s} + \frac{1}{4}\right)e^{-2s} + \left(\frac{4}{3s^3} + \frac{10}{3s^2} - s + \frac{5}{2s} - \frac{23}{18}\right)e^{-3s} + \mathcal{O}(e^{-4s}), \\ \mathcal{W}_1^{(g=1)} &= -\frac{1}{24} \left[1 - \frac{s}{2} - (1+s)e^{-s} + \left(\frac{s}{2} - \frac{1}{s} + \frac{1}{4}\right)e^{-2s} + \left(\frac{4}{3s^3} + \frac{10}{3s^2} - s + \frac{5}{2s} - \frac{23}{18}\right)e^{-3s} + \mathcal{O}(e^{-4s}) \right], \\ \mathcal{W}_1^{(g=2)} &= \frac{5}{192s^3} - \frac{7}{384s^2} - \frac{7s}{11520} + \frac{1}{144s} - \frac{1}{5760} \\ &\quad + \left(-\frac{5}{16s^5} - \frac{11}{96s^4} - \frac{43}{576s^3} + \frac{19}{576s^2} - \frac{7s}{5760} + \frac{11}{1440s} - \frac{31}{5760}\right)e^{-s} \\ &\quad + \left(\frac{25}{8s^7} + \frac{209}{48s^6} + \frac{719}{192s^5} + \frac{755}{384s^4} + \frac{8629}{11520s^3} - \frac{169}{1920s^2} + \frac{7s}{11520} - \frac{1483}{5760s} + \frac{47}{2560}\right)e^{-2s} + \mathcal{O}(e^{-3s}). \end{aligned} \quad (4.40)$$

One can show that the genus-zero and the genus-one amplitudes in (4.40) are consistent with the matrix model results [6]. Interestingly, from (4.40) we observe that the genus-zero and the genus-one amplitudes are proportional to each other

$$\mathcal{W}_1^{(g=1)} = -\frac{1}{24} \mathcal{W}_1^{(g=0)}. \quad (4.41)$$

It would be interesting to prove this relation directly from the matrix model.

5 WKB expansion

In this section, we study the membrane instanton corrections of 1/6 BPS Wilson loops in the fundamental representation from the WKB expansion of Fermi gas. As discussed in [41], the WKB expansion can be systematically analyzed using the spectral trace and the Mellin-Barnes representation.

First, we rewrite the small κ expansion of $\langle \text{Tr } U \rangle^{\text{GC}}$ in (2.16) into the Mellin-Barnes type integral representation

$$\langle \text{Tr } U \rangle^{\text{GC}} = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \frac{\pi}{\sin \pi s} \text{Tr}(\rho^s W) e^{s\mu}, \quad (5.1)$$

where c is a constant in the range $2/k < c < 1$. Picking up the pole at $s = \ell \in \mathbb{N}$, we recover the small κ expansion (2.16). On the other hand, closing the contour in the direction $\text{Re}(s) < 0$ and picking up the poles along the negative real s -axis we find the large μ expansion of $\langle \text{Tr } U \rangle^{\text{GC}}$. Hence the non-trivial information of instanton corrections is contained in the pole structure of the *spectral trace* $\text{Tr}(\rho^s W)$. In the classical limit $\hbar \rightarrow 0$, the spectral trace $\text{Tr}(\rho^s W)$ can be approximated by the phase space integral

$$Z_0(s) = \int \frac{dx dp}{2\pi \hbar} \frac{e^{\frac{x+p}{k}}}{(2 \cosh \frac{x}{2} \cdot 2 \cosh \frac{p}{2})^s} = \frac{\Gamma\left(\frac{s}{2} + \frac{1}{k}\right)^2 \Gamma\left(\frac{s}{2} - \frac{1}{k}\right)^2}{2\pi \hbar \Gamma(s)^2}. \quad (5.2)$$

Note that $Z_0(s)$ has poles at $s = 2/k - 2\ell$ ($\ell \in \mathbb{N}$) corresponding to the membrane instanton corrections. We consider the WKB expansion of the imaginary part of $\text{Tr}(\rho^s W)$

$$\text{Im Tr}(\rho^s W) = Z_0(s)D(s), \quad (5.3)$$

where $Z_0(s)$ is given by (5.2) and $D(s)$ represents the \hbar -corrections

$$D(s) = \sum_{n=1}^{\infty} \hbar^{2n-1} D_{2n-1}(s). \quad (5.4)$$

Then the imaginary part of 1/6 BPS Wilson loop $\mathcal{W}_1(\mu, k)$ is written as a Mellin-Barnes integral

$$\mathcal{W}_1(\mu, k) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \frac{\pi}{\sin \pi s} Z_0(s) D(s) e^{s\mu}. \quad (5.5)$$

From the explicit form of the first few terms in the expansion (5.4), we find that $D_{2n-1}(s)$ in (5.4) has the form

$$D_{2n-1}(s) = \frac{p_{4n-5}(s)}{\prod_{j=0}^{2n-3} (s+j)}, \quad (n \geq 2) \quad (5.6)$$

where $p_{4n-5}(s)$ is a $(4n-5)^{\text{th}}$ order polynomial of s . One can easily fix the coefficients in this polynomial $p_{4n-5}(s)$ by matching the value at integer s . When $s = \ell$ is an integer, the trace is given by

$$\text{Tr}(\rho^\ell W) = \int \frac{dx dp}{2\pi \hbar} \left(\frac{1}{2 \cosh \frac{x}{2}} \star \frac{1}{2 \cosh \frac{p}{2}} \right)_{\star}^{\ell} \star e^{\frac{x+p}{k}}, \quad (5.7)$$

where the star-product is defined by

$$f \star g = f \exp \left[\frac{i\hbar}{2} \left(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x \right) \right] g. \quad (5.8)$$

From this expression, we can easily compute the \hbar -expansion of $\text{Tr}(\rho^\ell W)$. In this way, we have computed $D_{2n-1}(s)$ up to $n_{\text{max}} = 18$. The first few terms of $D_{2n-1}(s)$ read

$$\begin{aligned} D_1(s) &= \frac{1}{2k^2 s}, \\ D_3(s) &= \frac{k^4 s^3 - (k^2 - 8) k^2 s^2 - 8k^2 s - 32}{768k^6 s(s+1)}, \\ D_5(s) &= \left(-21k^8 s^7 + 6(8 - 17k^2) k^6 s^6 + 6(19k^4 + 88k^2 + 64) k^4 s^4 \right. \\ &\quad - 64(10k^4 + 19k^2 + 88) k^2 s^2 + 1024(k^4 + 4k^2 + 13) + (-87k^8 + 256k^6 + 320k^4) s^5 \\ &\quad \left. + 32(3k^6 + 6k^4 - 52k^2 - 80) k^2 s^3 - 128(3k^6 - 9k^4 - 32k^2 - 40) s \right) \\ &\quad / \left(2949120k^{10} s(s+1)(s+2)(s+3) \right). \end{aligned} \quad (5.9)$$

Perturbative part. The perturbative part of \mathcal{W}_1 comes from the pole at $s = 2/k$. The residue of the pole at $s = 2/k$ is given by

$$\begin{aligned} \text{Res}_{s=\frac{2}{k}} & \left[\frac{\pi}{\sin \pi s} Z_0(s) D(s) e^{s\mu} \right] \\ &= \frac{e^{\frac{2\mu}{k}}}{\pi k \sin \frac{2\pi}{k}} \left[D(2/k) \left(\mu - \pi \cot \frac{2\pi}{k} - \psi(2/k) - \gamma \right) + D'(2/k) \right], \end{aligned} \quad (5.10)$$

where $\psi(2/k)$ denotes the digamma function and $D'(s) = \partial_s D(s)$. From the explicit form of the coefficients $D_{2n-1}(s)$ in (5.9), we find that the WKB expansion of $D(2/k)$ becomes

$$D(2/k) = \frac{\hbar}{4k} - \frac{1}{3!} \left(\frac{\hbar}{4k} \right)^3 + \frac{1}{5!} \left(\frac{\hbar}{4k} \right)^5 + \dots, \quad (5.11)$$

from which we can easily guess the exact form of $D(2/k)$

$$D(2/k) = \sin \left(\frac{\hbar}{4k} \right). \quad (5.12)$$

We have checked that the WKB expansion of $D(2/k)$ agrees with (5.12) up to $\mathcal{O}(\hbar^{2n_{\max}-1}) = \mathcal{O}(\hbar^{35})$. Plugging the relation $\hbar = 2\pi k$ into (5.12), we find

$$D(2/k) = 1. \quad (5.13)$$

This correctly reproduces the coefficient of μ in the perturbative part (3.11) for the winding number $n = 1$.

In order to reproduce the $\mathcal{O}(\mu^0)$ term in (3.11), we need to show the relation

$$D'(2/k) = \psi(2/k) + \gamma. \quad (5.14)$$

We do not have an analytic proof of this relation, but we can check this using the diagonal Padé approximation

$$D'(2/k) \approx \sum_{n=1}^{n_{\max}} D'_{2n-1}(2/k) \hbar^{2n-1} \approx \hbar \frac{a_0 + a_1 \hbar^2 + \dots + a_{\frac{n_{\max}-1}{2}} \hbar^{n_{\max}-2}}{1 + b_1 \hbar^2 + \dots + b_{\frac{n_{\max}-1}{2}} \hbar^{n_{\max}-2}}. \quad (5.15)$$

As we can see from figure 4, the Padé approximation of $D'(2/k)$ nicely agrees with the expected function (5.14).

Membrane 1-instanton. The membrane 1-instanton term comes from the pole at $s = 2/k - 2$. In order to reproduce the result in (4.5), we find that $D(2/k - 2)$ and $D'(2/k - 2)$ should satisfy

$$\begin{aligned} \left(1 - \frac{1}{k}\right) D\left(\frac{2}{k} - 2\right) &= \cos \frac{\pi k}{2}, \\ 4\left(1 - \frac{1}{k}\right)^2 D'\left(\frac{2}{k} - 2\right) &= -\left(2 + \pi k \cot \frac{\pi k}{2}\right) \cos \frac{\pi k}{2} \\ &\quad + 4\left(1 - \frac{1}{k}\right) \left(-1 + \gamma + 2\psi(2/k - 2) - \psi(2/k - 1)\right) \cos \frac{\pi k}{2}. \end{aligned} \quad (5.16)$$

As in the case of perturbative part, we can check these relations numerically by the Padé approximation. From figure 5, one can see that the Padé approximation of $D(2/k - 2)$ and $D'(2/k - 2)$ agrees with the exact functions (5.16).

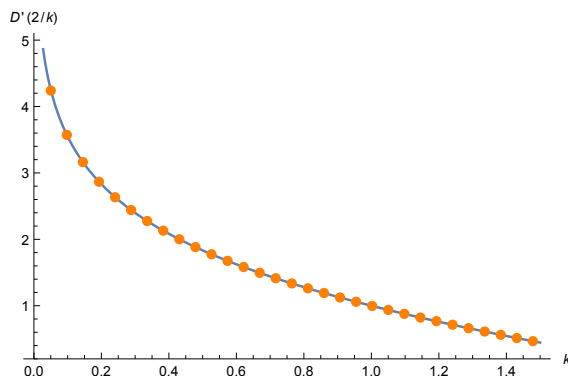


Figure 4. This is the plot of $D'(2/k)$ against k . The orange dots are the numerical values obtained from the Padé approximation, while the solid curve is the exact function $\psi(2/k) + \gamma$ in (5.14).

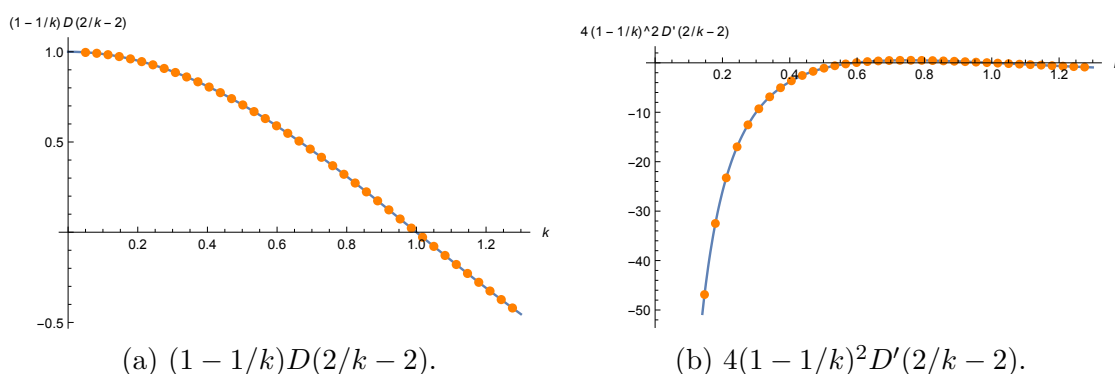


Figure 5. We show the plot of (a) $(1 - 1/k)D(2/k - 2)$ and (b) $4(1 - 1/k)^2 D'(2/k - 2)$. The orange dots are the numerical values obtained from the Padé approximation, while the solid curves represent the exact functions in (5.16).

Membrane 2-instanton. We can repeat the same analysis for the membrane 2-instanton. In order to reproduce the result in the previous section, we need the following relations

$$\begin{aligned}
 \frac{k^2 \Gamma\left(\frac{2}{k} - 2\right)^2}{4 \Gamma\left(\frac{2}{k} - 4\right)^2} D\left(\frac{2}{k} - 4\right) &= 2(2k - 1)(4k - 2 + (5k - 2) \cos \pi k), \\
 \frac{k^2 \Gamma\left(\frac{2}{k} - 2\right)^2}{4 \Gamma\left(\frac{2}{k} - 4\right)^2} D'\left(\frac{2}{k} - 4\right) &= -\frac{1}{2} \pi k^2 \left((4 - 18k) \sin \pi k + k \tan \frac{\pi k}{2} + (25k - 8) \cot \frac{\pi k}{2} \right) \\
 &\quad - k(8k - 4 + (9k - 4) \cos \pi k) \\
 &\quad + 2(2k - 1)(4k - 2 + (5k - 2) \cos \pi k) \\
 &\quad \times (3 - 2\gamma - 4\psi(2/k - 4) + 2\psi(2/k - 2)).
 \end{aligned} \tag{5.17}$$

Again, the Padé approximation of the WKB expansion of $D(2/k - 4)$ and $D'(2/k - 4)$ correctly reproduces the expected analytic functions on the right hand side of (5.17) (see figure 6).

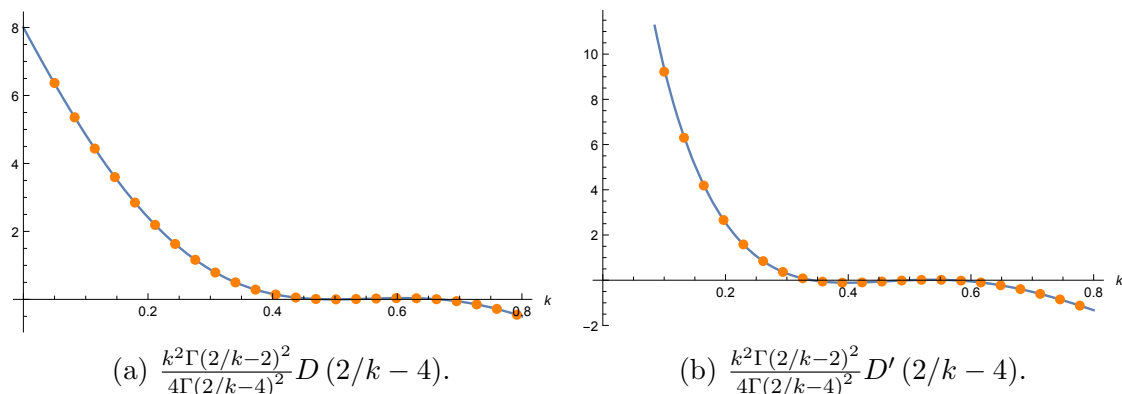


Figure 6. We show the plot of (a) $\frac{k^2 \Gamma(2/k-2)^2}{4 \Gamma(2/k-4)^2} D(2/k-4)$ and (b) $\frac{k^2 \Gamma(2/k-2)^2}{4 \Gamma(2/k-4)^2} D'(2/k-4)$. The orange dots are the numerical values obtained from the Padé approximation, while the solid curves represent the exact functions in (5.17).

5.1 Comment on winding Wilson loops

One can in principle compute the WKB expansion of winding Wilson loops $\mathcal{W}_n(\mu, k)$ with $n \geq 2$ in a similar manner as the $n = 1$ case studied above. Then $\mathcal{W}_n(\mu, k)$ is written as a Mellin-Barnes type integral

$$\mathcal{W}_n(\mu, k) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \frac{\pi}{\sin \pi s} Z_0^{(n)}(s) D^{(n)}(s) e^{s\mu}, \quad (5.18)$$

where the classical trace $Z_0^{(n)}(s)$ is given by

$$Z_0^{(n)}(s) = \frac{\Gamma\left(\frac{s}{2} + \frac{n}{k}\right) \Gamma\left(\frac{s}{2} - \frac{n}{k}\right)}{2\pi \hbar \Gamma(s)^2}, \quad (5.19)$$

and $D^{(n)}(s)$ represents the \hbar -corrections. Let us consider the perturbative part coming from the pole at $s = 2n/k$

$$\begin{aligned} & \text{Res}_{s=\frac{2n}{k}} \left[\frac{\pi}{\sin \pi s} Z_0^{(n)}(s) D^{(n)}(s) e^{s\mu} \right] \\ &= \frac{e^{\frac{2n\mu}{k}}}{\pi k \sin \frac{2\pi n}{k}} \left[D^{(n)}\left(\frac{2n}{k}\right) \left(\mu - \pi \cot \frac{2\pi n}{k} - \psi\left(\frac{2n}{k}\right) - \gamma \right) + D^{(n)'}\left(\frac{2n}{k}\right) \right]. \end{aligned} \quad (5.20)$$

One can easily show that $D^{(n)}(2n/k) = 1$ as in the case of $n = 1$, and hence the coefficient of μ in the perturbative part (3.11) is correctly reproduced from the WKB expansion. In order to reproduce the $\mathcal{O}(\mu^0)$ term in (3.11), we have to show that

$$D^{(n)'}\left(\frac{2n}{k}\right) = \psi\left(\frac{2n}{k}\right) + \gamma - \pi \sum_{j=1}^{n-1} \cot \frac{2\pi j}{k}. \quad (5.21)$$

However, it is not so straightforward to reproduce this behavior from the Padé approximation of WKB expansion, since the function on the right hand side of (5.21) has a dense set of poles at rational values of k , accumulating around $k = 0$. Note that this problem

does not occur for $n = 1$. At present we could not succeed in reproducing the perturbative part of winding Wilson loop from the WKB expansion for $n \geq 2$. It would be interesting to derive the perturbative part (3.11) of winding Wilson loop analytically.

6 Winding Wilson loops

In this section, we will consider the instanton corrections of the imaginary part of 1/6 BPS winding Wilson loops $\mathcal{W}_n(\mu, k)$ with the winding number $n \geq 2$. We can study the instanton corrections of $\mathcal{W}_n(\mu, k)$ for $n \geq 2$ by the same method of numerical fitting we used for the fundamental representation in section 4.

6.1 Winding number $n = 2$

Let us first consider the Wilson loop with winding number $n = 2$. After rewriting $\mathcal{W}_2(\mu, k)$ in terms of the effective chemical potential μ_{eff} , we find that the factorized structure (4.15) found in the fundamental representation also appears in the $n = 2$ case

$$\mathcal{W}_2 = \frac{e^{\frac{4\mu_{\text{eff}}}{k}}}{k\pi \sin \frac{4\pi}{k}} \frac{\partial \mu_{\text{eff}}}{\partial \mu} f_w \left[\mu_{\text{eff}} - \pi \cot \frac{2\pi}{k} V_w - \pi \cot \frac{4\pi}{k} f_w^{-1} - \pi k V_m \right], \quad (6.1)$$

where

$$\begin{aligned} f_w &= 1 - 4 \cos^2 \frac{2\pi}{k} Q_w + \frac{2 \sin \frac{6\pi}{k}}{\sin \frac{2\pi}{k}} Q_w^2 - 16 \cos^2 \frac{2\pi}{k} Q_w^3 \\ &\quad + \left(33 + 40 \cos \frac{4\pi}{k} + 6 \cos \frac{8\pi}{k} \right) Q_w^4 + \mathcal{O}(Q_w^5), \\ V_w &= 1 + 2 \cos \frac{4\pi}{k} Q_w + 4 \cos^2 \frac{4\pi}{k} Q_w^2 + 8 \cos^2 \frac{4\pi}{k} \left(\cos \frac{4\pi}{k} + \frac{2 \sin \frac{2\pi}{k}}{\sin \frac{6\pi}{k}} \right) Q_w^3 \\ &\quad + \left(9 - 4 \cos \frac{4\pi}{k} + 6 \cos \frac{8\pi}{k} + 2 \cos \frac{16\pi}{k} + \frac{2}{\cos^2 \frac{2\pi}{k}} - \frac{1}{\cos \frac{4\pi}{k}} \right) Q_w^4 + \mathcal{O}(Q_w^5), \\ V_m &= \cot \frac{\pi k}{2} \cos \frac{\pi k}{2} Q_m + (4 + 5 \cos \pi k) \cot \pi k Q_m^2 + \mathcal{O}(Q_m^3). \end{aligned} \quad (6.2)$$

Interestingly, the membrane instanton correction V_m in (6.2) is exactly the same as that of the fundamental Wilson loop. From this observation, we conjecture that V_m for $n = 2$ is again given by the NS free energy (4.23). One can show that poles in the convergence region $k > 4$ are canceled between worldsheet instantons and membrane instantons, and the remaining finite part reproduces the result in (A.3). This implies that the worldsheet instanton corrections are essentially given by the S-dual of NS free energy (4.28), up to regular terms.

6.2 Winding number $n \geq 3$

Let us consider the Wilson loops with winding number $n \geq 3$. From the convergence condition $k > 2n$, as the winding number increases the membrane instanton correction $e^{-2\mu}$ becomes highly suppressed relative to the worldsheet instanton correction

$$e^{-\frac{4\mu}{k}} \gg e^{-2\mu}. \quad (6.3)$$

Because of this, it becomes harder and harder to study membrane instanton corrections numerically as the winding number increases. We were unable to find the analytic expressions of the membrane instanton coefficients for $n \geq 3$.

On the other hand, we can determine the worldsheet instanton corrections of $\mathcal{W}_n(\mu, k)$ from numerical fitting. For $n = 3$, the worldsheet instanton corrections are given by

$$\begin{aligned} \mathcal{W}_3^{\text{WS}} &= \frac{e^{\frac{6\mu}{k}}}{k\pi \sin \frac{6\pi}{k}} \left[f_w \left(\mu - \pi \cot \frac{2\pi}{k} V_w \right) - \pi \cot \frac{4\pi}{k} - \pi \cot \frac{6\pi}{k} \right], \\ f_w &= 1 - \frac{\sin^2 \frac{6\pi}{k}}{\sin^2 \frac{2\pi}{k}} Q_w + 3 \frac{\sin^2 \frac{6\pi}{k}}{\sin^2 \frac{2\pi}{k}} Q_w^2 + \left(2 - 9 \frac{\sin^2 \frac{6\pi}{k}}{\sin^2 \frac{2\pi}{k}} \right) Q_w^3 + \mathcal{O}(Q_w^4), \\ V_w &= 1 + \frac{\cos \frac{4\pi}{k} \sin \frac{6\pi}{k}}{\cos^2 \frac{2\pi}{k} \sin \frac{2\pi}{k}} Q_w + \frac{\sin \frac{6\pi}{k}}{\cos^2 \frac{2\pi}{k} \sin \frac{2\pi}{k}} \left(\frac{3}{2} + \cos \frac{4\pi}{k} + \cos \frac{8\pi}{k} + \cos \frac{12\pi}{k} \right) Q_w^2 \\ &\quad + \mathcal{O}(Q_w^3), \end{aligned} \tag{6.4}$$

and for $n = 4$, worldsheet instanton corrections are given by

$$\begin{aligned} \mathcal{W}_4^{\text{WS}} &= \frac{e^{\frac{8\mu}{k}}}{k\pi \sin \frac{8\pi}{k}} \left[f_w \left(\mu - \pi \cot \frac{2\pi}{k} V_w \right) - \pi \cot \frac{4\pi}{k} - \pi \cot \frac{6\pi}{k} - \pi \cot \frac{8\pi}{k} \right], \\ f_w &= 1 - \frac{\sin^2 \frac{8\pi}{k}}{\sin^2 \frac{2\pi}{k}} Q_w + 4 \cos^2 \frac{4\pi}{k} \left(7 + 8 \cos \frac{4\pi}{k} + 2 \cos \frac{8\pi}{k} \right) Q_w^2 + \mathcal{O}(Q_w^3), \\ V_w &= 1 + 4 \frac{\cos \frac{4\pi}{k} \sin \frac{2\pi}{k}}{\sin \frac{6\pi}{k}} \left(\cos \frac{4\pi}{k} + \cos \frac{8\pi}{k} - \cos \frac{12\pi}{k} \right) Q_w + \mathcal{O}(Q_w^2). \end{aligned} \tag{6.5}$$

We find that the worldsheet 1-instanton correction in the coefficient of μ can be written in a closed form for general $n \geq 2$

$$f_w = 1 - \frac{\sin^2 \frac{2n\pi}{k}}{\sin^2 \frac{2\pi}{k}} Q_w + \mathcal{O}(Q_w^2). \tag{6.6}$$

It would be very interesting to understand the structure of instanton correction of $\mathcal{W}_n(\mu, k)$ and see if it is related to the quantum volume for the general winding number n .

7 Wilson loops with two boundaries

As pointed out in [17], 1/6 BPS Wilson loops in higher rank representation involves multi-particle interaction in the Fermi gas picture, hence it is not so easy to study the 1/6 BPS Wilson loops in general representations. Here we initiate the study of Wilson loop with two boundaries $\langle (\text{Tr } U)^2 \rangle$, which is related to the second rank symmetric and anti-symmetric representations by

$$\begin{aligned} W_{\square} &= W_{S_2} = \frac{1}{2} \langle (\text{Tr } U)^2 \rangle + \frac{1}{2} \langle \text{Tr } U^2 \rangle, \\ W_{\square} &= W_{A_2} = \frac{1}{2} \langle (\text{Tr } U)^2 \rangle - \frac{1}{2} \langle \text{Tr } U^2 \rangle. \end{aligned} \tag{7.1}$$

Note that $\langle \text{Tr } U^2 \rangle$ is the 1/6 BPS Wilson loop with winding number $n = 2$ which we have studied in the previous section.

From the general formula (2.11), the generating function of 1/6 BPS Wilson loops in the anti-symmetric representations is given by

$$\sum_n t^n W_{A_n} = \frac{\det(1 + \kappa\rho(1 + tW))}{\det(1 + \kappa\rho)}, \quad (7.2)$$

while the generating function of 1/6 BPS Wilson loops in the symmetric representations is given by

$$\sum_n t^n W_{S_n} = \frac{\det(1 + \kappa\rho(1 - tW)^{-1})}{\det(1 + \kappa\rho)}. \quad (7.3)$$

Picking up the coefficient of t^2 in the above expressions, we find that W_{A_2} and W_{S_2} are written in terms of R defined in (2.14):

$$\begin{aligned} W_{A_2} &= \frac{1}{2}(\text{Tr } RW)^2 - \frac{1}{2}\text{Tr}(RW)^2, \\ W_{S_2} &= \text{Tr } RW^2 + \frac{1}{2}(\text{Tr } RW)^2 - \frac{1}{2}\text{Tr}(RW)^2. \end{aligned} \quad (7.4)$$

Note that $\text{Tr } RW$ and $\text{Tr } RW^2$ appearing in (7.4) are the grand canonical VEV of 1/6 BPS winding Wilson loops with the winding numbers $n = 1, 2$, respectively

$$\text{Tr } RW = \langle \text{Tr } U \rangle^{\text{GC}}, \quad \text{Tr } RW^2 = \langle \text{Tr } U^2 \rangle^{\text{GC}}. \quad (7.5)$$

The last term $\text{Tr}(RW)^2$ in (7.4) is the new contribution which we should compute. It turns out that the grand canonical VEV of $(\text{Tr } U)^2$, which we denote by $W_{(1,1)}$, has a simple expansion in the large μ limit

$$W_{(1,1)} = \frac{1}{2} \langle (\text{Tr } U)^2 \rangle^{\text{GC}} = \frac{1}{2}(W_{S_2} + W_{A_2}). \quad (7.6)$$

From (7.4), $W_{(1,1)}$ is written as

$$W_{(1,1)} = \frac{1}{2} \left[\text{Tr } RW^2 + (\text{Tr } RW)^2 - \text{Tr}(RW)^2 \right]. \quad (7.7)$$

7.1 Computation of $\text{Tr}(RW)^2$

To study the 1/6 BPS Wilson loop with two boundaries (7.7), we have to compute $\text{Tr}(RW)^2$. The first two terms in (7.7) have been already obtained in the previous sections. Now let us expand $\text{Tr}(RW)^2$ in κ as

$$\text{Tr}(RW)^2 = \sum_{\ell=2}^{\infty} (-1)^\ell \kappa^\ell t_\ell, \quad (7.8)$$

where t_ℓ is given by

$$t_\ell = \sum_{j=1}^{\ell-1} \text{Tr}(\rho^j W \rho^{\ell-j} W). \quad (7.9)$$

To compute t_ℓ , we rewrite the trace in (7.9) as

$$\text{Tr}(\rho^j W \rho^{\ell-j} W) = \text{Tr}(\rho^\ell W^2) + \frac{1}{2} \text{Tr}([\rho^j, W][\rho^{\ell-j}, W]). \quad (7.10)$$

Note that the first term $\text{Tr}(\rho^\ell W^2)$ in (7.10) is the trace appearing in the winding Wilson loop with winding number $n = 2$. The second term in (7.10) can be further simplified by noticing that

$$[\rho, W] = \frac{1}{2 \cosh \frac{x}{2}} \left\{ \frac{1}{2 \cosh \frac{p}{2}}, W \right\} - \left\{ \frac{1}{2 \cosh \frac{x}{2}}, W \right\} \frac{1}{2 \cosh \frac{p}{2}}, \quad (7.11)$$

and using the fact that the anti-commutators in (7.11) have a simple matrix element [18]

$$\begin{aligned} \langle x_1 | \left\{ \frac{1}{2 \cosh \frac{p}{2}}, W \right\} | x_2 \rangle &= \frac{1}{\hbar} e^{\frac{x_1+x_2}{2k}}, \\ \langle p_1 | \left\{ \frac{1}{2 \cosh \frac{x}{2}}, W \right\} | p_2 \rangle &= \frac{1}{\hbar} e^{\frac{p_1+p_2}{2k}}. \end{aligned} \quad (7.12)$$

Then one can show that t_ℓ is written as

$$t_\ell = (\ell - 1) \text{Tr}(\rho^\ell W^2) + \sum_{j=1}^{\ell-1} \sum_{a=0}^{j-1} \sum_{b=0}^{\ell-j-1} (I_{1/2}^{(a+b)} I_{1/2}^{(\ell-2-a-b)} - \bar{I}_{1/6}^{(a+b+1)} I_{1/6}^{(\ell-2-a-b)}), \quad (7.13)$$

where

$$\begin{aligned} I_{1/2}^{(\ell)} &= \int \frac{dx dy}{\hbar} e^{\frac{x}{2k}} \langle x | \rho^\ell | y \rangle \frac{e^{\frac{y}{2k}}}{2 \cosh \frac{y}{2}}, \\ I_{1/6}^{(\ell)} &= \int \frac{dx dp}{\hbar} e^{\frac{x}{2k}} \langle x | \rho^\ell | p \rangle e^{\frac{p}{2k}} = \int dx \frac{e^{\frac{x}{2k}}}{\sqrt{k}} \langle x | \rho^\ell | \pi i \rangle. \end{aligned} \quad (7.14)$$

The integral $I_{1/2}^{(\ell)}$ has appeared in the computation of 1/2 BPS Wilson loops in a hook representation [18]. As shown in [18], $I_{1/2}^{(\ell)}$ can be computed systematically by constructing a sequence of functions. Using the algorithm in section 2.2, we can numerically evaluate $I_{1/2}^{(\ell)}$ with high precision. Similarly, $I_{1/6}^{(\ell)}$ in (7.14) can be computed by constructing a sequence of functions $\psi_\ell(x)$

$$\begin{aligned} I_{1/6}^{(0)} &= \frac{1}{\sqrt{k}} e^{\frac{\pi i}{2k}}, \\ I_{1/6}^{(\ell)} &= \int dx e^{\frac{x}{2k}} \psi_\ell(x), \quad (\ell \geq 1), \end{aligned} \quad (7.15)$$

where $\psi_\ell(x)$ is defined recursively

$$\psi_\ell(x) = \frac{1}{2 \cosh \frac{x}{2}} \int \frac{dy}{2\pi k} \frac{1}{2 \cosh \frac{x-y}{2k}} \psi_{\ell-1}(y), \quad \psi_1(x) = \frac{1}{\sqrt{k}} \frac{1}{2 \cosh \frac{x}{2}} \frac{1}{2 \cosh \frac{x-\pi i}{2k}}. \quad (7.16)$$

Again, we can compute $I_{1/6}^{(\ell)}$ numerically using the algorithm in section 2.2.

For some values of integer k , we can compute the trace t_ℓ exactly by closing the contour and picking up the residue of poles. For $k = 6$ we find

$$\begin{aligned} t_2 &= \frac{e^{\frac{2\pi i}{6}}}{648} \left(5 - \frac{6\sqrt{3}}{\pi} \right), \\ t_3 &= \frac{-972 + 216i\sqrt{3} + 180i\pi - 864\sqrt{3}\pi + 585\pi^2 - 46i\sqrt{3}\pi^2}{93312\pi^2}. \end{aligned} \quad (7.17)$$

For $k = 8$ we find

$$t_2 = \frac{e^{\frac{2\pi i}{8}} (2\sqrt{2} + (\sqrt{2} - 2)\pi)}{256\pi}, \tag{7.18}$$

$$t_3 = \frac{-4 + (16 + 4i)\pi + (32 - 16i)\sqrt{2}\pi - (13 - 12i)\pi^2 - (4 + 4i)\sqrt{2}\pi^2}{8192\pi^2}.$$

For $k = 12$ we find

$$t_2 = e^{\frac{2\pi i}{12}} \frac{18 - 27\pi + 13\sqrt{3}\pi}{2592\pi}, \tag{7.19}$$

$$t_3 = \frac{-54 + ((486 + 54i) + (120 - 216i)\sqrt{3})\pi + ((-209 - 279i) + 222i\sqrt{3})\pi^2}{373248\pi^2}. \tag{7.20}$$

We have checked that the numerical computation using the method in section 2.2 correctly reproduces the exact values of t_ℓ for $k = 6, 8, 12$ with high precision. We should stress that we can compute t_ℓ numerically at arbitrary k in the convergence region $k > 4$.

7.2 Imaginary part of $W_{(1,1)}$

Let us first consider the imaginary part of $W_{(1,1)}$, which we denote by $W_{(1,1)}^{\text{Im}}$. From the numerical fitting, we find that the perturbative part of $W_{(1,1)}^{\text{Im}}$ is given by

$$W_{(1,1)}^{\text{Im,pert}} = \frac{e^{\frac{4\mu}{k}}}{\pi k (2 \sin \frac{2\pi}{k})^2} \left(\mu - 2\pi \cot \frac{4\pi}{k} \right). \tag{7.21}$$

We conjecture that the instanton corrections have the following structure

$$W_{(1,1)}^{\text{Im}} = \frac{e^{\frac{4\mu_{\text{eff}}}{k}}}{\pi k (2 \sin \frac{2\pi}{k})^2} f_w \frac{\partial \mu_{\text{eff}}}{\partial \mu} \left(\mu_{\text{eff}} - 2\pi \cot \frac{4\pi}{k} \frac{V_4}{f_w} + 2\pi \cot \frac{2\pi}{k} V_2 - \pi k V_m \right), \tag{7.22}$$

where the membrane instanton correction V_m is given by the NS free energy (4.23), and the worldsheet instanton corrections are given by

$$f_w = 1 + 4 \sin^2 \frac{2\pi}{k} Q_w - 2 \left(1 + 4 \sin^2 \frac{2\pi}{k} \right) Q_w^2$$

$$+ 16 \sin^2 \frac{2\pi}{k} Q_w^3 + \left(15 - 96 \sin^2 \frac{2\pi}{k} + 48 \sin^4 \frac{2\pi}{k} \right) Q_w^4, \tag{7.23}$$

$$V_4 = 1 - Q_w^2 + 4Q_w^3,$$

$$V_2 = Q_w - 4 \sin^2 \frac{2\pi}{k} Q_w^2 + \frac{2 \sin \frac{2\pi}{k}}{\sin \frac{6\pi}{k}} \left(-3 + 4 \cos \frac{4\pi}{k} - 6 \cos \frac{8\pi}{k} + \cos \frac{12\pi}{k} \right) Q_w^3.$$

We have checked that the membrane 1-instanton and (1,1)-boundstate are correctly reproduced from (7.22). Also, one can check that the pole cancellation between worldsheet instantons and membrane instantons works for $k = 6, 8, 12$, and the remaining finite terms correctly reproduce the result in appendix A.5.

7.3 Real part of $W_{(1,1)}$

Next consider the real part of $W_{(1,1)}$, which we denote by $W_{(1,1)}^{\text{Re}}$. Again, from the numerical fitting, we find that the perturbative part of $W_{(1,1)}^{\text{Re}}$ is given by

$$W_{(1,1)}^{\text{Re,pert}} = \frac{e^{\frac{4\mu}{k}}}{(2 \sin \frac{2\pi}{k})^2} \left[-\frac{2}{\pi^2 k^2} \left(\mu - 2\pi \cot \frac{4\pi}{k} \right)^2 + \frac{1}{8} - \frac{\tan \frac{2\pi}{k}}{\pi k} \right], \quad (7.24)$$

and the worldsheet instanton corrections are given by

$$\begin{aligned} e^{-\frac{4\mu_{\text{eff}}}{k}} W_{(1,1)}^{\text{Re,WS}} &= \frac{Q_w}{\sin^2 \frac{2\pi}{k}} \left[-\frac{2\mu_{\text{eff}}^2}{\pi^2 k^2} \left(1 + \sin^2 \frac{2\pi}{k} \right) + \frac{2\mu_{\text{eff}}}{\pi k^2} \cot \frac{2\pi}{k} - \frac{2}{k^2} + \frac{\sin \frac{4\pi}{k}}{2\pi k} \right] \\ &+ \frac{Q_w^2}{\sin^2 \frac{2\pi}{k}} \left[\frac{\mu_{\text{eff}}^2}{\pi^2 k^2} \left(-1 + 4 \sin^2 \frac{2\pi}{k} \right) + \frac{2\mu_{\text{eff}}}{\pi k^2} \left(-4 \cot \frac{2\pi}{k} + 3 \cot \frac{4\pi}{k} \right) \right. \\ &\quad \left. + \frac{2}{k^2} + \frac{4}{k^2 \sin^2 \frac{2\pi}{k}} + \frac{3}{16} + \frac{\cos \frac{4\pi}{k}}{8} - \frac{\sin \frac{6\pi}{k}}{2\pi k \cos \frac{2\pi}{k}} \right] \quad (7.25) \\ &+ \frac{Q_w^3}{\sin^2 \frac{2\pi}{k}} \left[-\frac{32\mu_{\text{eff}}^2}{\pi^2 k^2} \sin^2 \frac{2\pi}{k} + \frac{\mu_{\text{eff}}}{\pi k^2} \frac{13 + 13 \cos \frac{4\pi}{k} + 5 \cos \frac{8\pi}{k} - 3 \cos \frac{12\pi}{k}}{\cos \frac{2\pi}{k} \sin \frac{6\pi}{k}} \right. \\ &\quad \left. + \frac{2 \sin \frac{4\pi}{k}}{\pi k} + \mathcal{O}(k^{-2}) + \mathcal{O}(k^0) \right]. \end{aligned}$$

We find that the membrane 1-instanton has the form

$$W_{(1,1)}^{\text{Re,M2-1}} = \frac{e^{\frac{4\mu_{\text{eff}}}{k}}}{\sin^2 \frac{2\pi}{k}} \left[-\frac{4 \cos \frac{\pi k}{2}}{\pi^2 k^2} \mu_{\text{eff}}^2 + b_1(k) \mu_{\text{eff}} + c_1(k) \right] e^{-2\mu_{\text{eff}}}. \quad (7.26)$$

In the limit $k \rightarrow 6$, we find that the membrane 1-instanton (7.26) and the worldsheet 3-instanton in (7.25) reproduce the coefficient of μ_{eff}^2 term for $k = 6$ in (A.7)

$$\lim_{k \rightarrow 6} e^{-\frac{4\mu_{\text{eff}}}{k}} \left[W_{(1,1)}^{\text{Re,M2-1}} + W_{(1,1)}^{\text{Re,WS-3}} \right] = \left[-\frac{32\mu_{\text{eff}}^2}{52\pi^2} + \mathcal{O}(\mu_{\text{eff}}) \right] e^{-2\mu_{\text{eff}}}. \quad (7.27)$$

The $\mathcal{O}(\mu_{\text{eff}})$ term in the worldsheet 3-instanton has a pole at $k = 6$ which should be canceled by the membrane 1-instanton. From this pole cancellation condition, we conjecture that the membrane 1-instanton coefficient of μ_{eff} in (7.26) has the structure

$$b_1(k) = \frac{\cos \frac{\pi k}{2} \cot \frac{\pi k}{2}}{\pi k} + (\text{regular}). \quad (7.28)$$

We leave the further study of membrane instanton corrections of $W_{(1,1)}^{\text{Re}}$ as an interesting future problem.

7.4 Connected part of $W_{(1,1)}$

We expect that the connected part of $\langle (\text{Tr } U)^2 \rangle$ has a simpler behavior

$$\frac{1}{2} \langle (\text{Tr } U)^2 \rangle_{\text{conn}} = \frac{1}{2} \langle (\text{Tr } U)^2 \rangle - \frac{1}{2} \langle \text{Tr } U \rangle^2. \quad (7.29)$$

Let us consider the perturbative part of $\langle \text{Tr } U \rangle$ and $\langle (\text{Tr } U)^2 \rangle$

$$\begin{aligned} \langle \text{Tr } U \rangle^{\text{pert}} &= \frac{e^{\frac{2\mu}{k}}}{2 \sin \frac{2\pi}{k}} \left[\frac{1}{2} + \frac{2i}{\pi k} \left(\mu - \pi \cot \frac{2\pi}{k} \right) \right], \\ \frac{1}{2} \langle (\text{Tr } U)^2 \rangle^{\text{pert}} &= \frac{e^{\frac{4\mu}{k}}}{(2 \sin \frac{2\pi}{k})^2} \left[-\frac{2}{\pi^2 k^2} \left(\mu - 2\pi \cot \frac{4\pi}{k} \right)^2 + \frac{1}{8} - \frac{\tan \frac{2\pi}{k}}{\pi k} + \frac{i}{\pi k} \left(\mu - 2\pi \cot \frac{4\pi}{k} \right) \right]. \end{aligned} \tag{7.30}$$

From this expression, we find that the μ^2 term is canceled in the connected part of $\langle (\text{Tr } U)^2 \rangle$

$$\frac{1}{2} \langle (\text{Tr } U)^2 \rangle_{\text{conn}}^{\text{pert}} = \frac{e^{\frac{4\mu}{k}}}{2\pi k \sin \frac{4\pi}{k}} \left[-\frac{2}{k} \left(2\mu - 2\pi \cot \frac{4\pi}{k} - \pi \cot \frac{2\pi}{k} \right) - 1 + \pi i \right]. \tag{7.31}$$

However, this cancellation of μ^2 term does not hold for the worldsheet instanton corrections. It would be interesting to study the structure of the connected part $\langle (\text{Tr } U)^2 \rangle_{\text{conn}}$ further.

8 Conclusions

In this paper we have studied the instanton corrections to the VEV of 1/6 BPS Wilson loops in ABJM theory from Fermi gas approach. For the fundamental representation, we find that the grand canonical VEV of the imaginary part of Wilson loop has a remarkably simple form (4.33), and it is closely related to the quantum volume given by the NS free energy on local $\mathbb{P}^1 \times \mathbb{P}^1$. The poles at rational value of k in the membrane instanton corrections are canceled by the worldsheet instanton corrections and the singular part of worldsheet instantons is essentially given by the ‘‘S-dual’’ of membrane instantons. This is reminiscent of the exact quantization condition of the spectrum. It would be very interesting to understand the physical meaning of this relation between the quantum volume and the 1/6 BPS Wilson loops.

It is curious that the NS free energy is determined by the BPS invariant of *closed* topological string, while we naively expect that the Wilson loops are related to some *open* string amplitudes. This type of *open-closed duality* was observed for the 1/2 BPS Wilson loops in ABJ(M) theory [20]. We speculate that the relation (4.33) between 1/6 BPS Wilson loops and the quantum volume is a ‘‘1/6 BPS version’’ of the open-closed duality. It would be interesting to study 1/6 BPS Wilson loops in ABJ theory along the lines of [20, 42, 43], and see if the relation to quantum volume also appears in Wilson loops in ABJ theory.

We have also studied 1/6 BPS winding Wilson loops and initiated the study of 1/6 BPS Wilson loops with two boundaries. For both of the imaginary part of winding Wilson loop \mathcal{W}_n and the imaginary part of Wilson loop with two boundaries $W_{(1,1)}^{\text{Im}}$, we find that the membrane instanton correction V_m is again given by the NS free energy (4.23). We also find that the perturbative part of winding Wilson loop is different from the expression in [17]. It would be interesting to derive our result of $\mathcal{W}_n^{\text{pert}}$ in (3.11) analytically.

Our method in section 7.1 can in principle be generalized to arbitrary number of boundaries, but the computation becomes cumbersome as the number of boundaries increases. It would be important to develop a more efficient method to compute the multi-boundary Wilson loops both numerically and analytically. From our result of $W_{(1,1)}$, it is natural to

expect that as the number of boundaries increases, the order of polynomial of μ_{eff} in the grand canonical VEV also increases. It would be interesting to find the general structure of the grand canonical VEV of 1/6 BPS Wilson loops with h boundaries.

Acknowledgments

I would like to thank Yasuyuki Hatsuda for a collaboration during the initial stage of this work. I would also like to thank Marcos Mariño for correspondence. This work was supported in part by JSPS KAKENHI Grant Number 16K05316, and JSPS Japan-Hungary and Japan-Russia bilateral joint research projects.

A Instanton corrections at integer k

In this appendix, we summarize the instanton corrections at some integer k .

A.1 Fundamental representation

For the fundamental representation with $k = 3, 4, 6, 8, 12$ we find

$$\begin{aligned}
 e^{-\frac{2\mu}{3}} \mathcal{W}_1(k=3) &= \left(\frac{2\mu}{3\sqrt{3}\pi} + \frac{2}{9} \right) + \frac{4\mu}{3\sqrt{3}\pi} e^{-\frac{4\mu}{3}} + \left(-\frac{2\mu}{3\sqrt{3}\pi} - \frac{14}{9} \right) e^{-\frac{8\mu}{3}} + \left(\frac{-96\mu + 18}{27\sqrt{3}\pi} + \frac{40}{27} \right) e^{-4\mu} \\
 &\quad + \left(\frac{-146\mu + 12}{9\sqrt{3}\pi} - \frac{70}{9} \right) e^{-\frac{16\mu}{3}} + \left(\frac{48\mu - 2}{3\sqrt{3}\pi} + \frac{320}{9} \right) e^{-\frac{20\mu}{3}} + \left(\frac{652\mu - 177}{27\sqrt{3}\pi} - \frac{4892}{81} \right) e^{-8\mu}, \\
 e^{-\frac{\mu}{3}} \mathcal{W}_1(k=6) &= \left(\frac{\mu}{3\sqrt{3}\pi} - \frac{1}{9} \right) + \frac{2\mu}{3\sqrt{3}\pi} e^{-\frac{2\mu}{3}} + \left(-\frac{\mu}{3\sqrt{3}\pi} + \frac{7}{9} \right) e^{-\frac{4\mu}{3}} + \left(\frac{-48\mu + 18}{27\sqrt{3}\pi} - \frac{20}{27} \right) e^{-2\mu} \\
 &\quad + \left(\frac{-73\mu + 12}{9\sqrt{3}\pi} + \frac{35}{9} \right) e^{-\frac{8\mu}{3}} + \left(\frac{24\mu - 2}{3\sqrt{3}\pi} - \frac{160}{9} \right) e^{-\frac{10\mu}{3}} + \left(\frac{326\mu - 177}{27\sqrt{3}\pi} + \frac{2446}{81} \right) e^{-4\mu}, \\
 e^{-\frac{\mu}{2}} \mathcal{W}_1(k=4) &= \frac{\mu}{4\pi} + \frac{\mu}{2\pi} e^{-\mu} + \frac{3\mu - 1}{2\pi} e^{-2\mu} + \frac{5\mu - 1}{\pi} e^{-3\mu} + \frac{74\mu - 21}{4\pi} e^{-4\mu} \\
 &\quad + \frac{130\mu - 29}{2\pi} e^{-5\mu} + \frac{1518\mu - 403}{6\pi} e^{-6\mu}, \\
 e^{-\frac{\mu}{4}} \mathcal{W}_1(k=8) &= \left(\frac{\mu}{4\sqrt{2}\pi} - \frac{1}{4\sqrt{2}} \right) + \frac{\mu}{2\sqrt{2}\pi} e^{-\frac{\mu}{2}} + \left(-\frac{\mu}{4\sqrt{2}\pi} + \frac{5}{4\sqrt{2}} \right) e^{-\mu} \\
 &\quad + \left(\frac{\mu}{2\sqrt{2}\pi} - 2\sqrt{2} \right) e^{-\frac{3\mu}{2}} + \left(\frac{\mu - 4}{8\sqrt{2}\pi} + \frac{79}{8\sqrt{2}} \right) e^{-2\mu} \\
 &\quad + \left(\frac{41\mu - 4}{4\sqrt{2}\pi} - 16\sqrt{2} \right) e^{-\frac{5\mu}{2}} + \left(\frac{-139\mu + 4}{8\sqrt{2}\pi} + \frac{791}{8\sqrt{2}} \right) e^{-3\mu}, \\
 e^{-\frac{\mu}{6}} \mathcal{W}_1(k=12) &= \frac{\mu}{6\pi} \left(1 + 2e^{-\frac{\mu}{3}} - e^{-\frac{2\mu}{3}} + 2e^{-\mu} - 7e^{-\frac{4\mu}{3}} + 30e^{-\frac{5\mu}{3}} \right) \\
 &\quad - \frac{1}{2\sqrt{3}} \left(1 - \frac{13}{3} e^{-\frac{2\mu}{3}} + \frac{46}{3} e^{-\mu} - \frac{185}{3} e^{-\frac{4\mu}{3}} + \frac{730}{3} e^{-\frac{5\mu}{3}} \right). \tag{A.1}
 \end{aligned}$$

By rewriting the above expansions in terms of μ_{eff} , we find

$$\begin{aligned}
 e^{-\frac{2\mu_{\text{eff}}}{3}} \mathcal{W}_1(k=3) &= \frac{2\mu_{\text{eff}}}{3\sqrt{3}\pi} \left(1 + 2Q_w - Q_w^2 - 6Q_w^3 - 23Q_w^4 + 22Q_w^5 + 19Q_w^6 \right) \\
 &\quad + \frac{2}{9} \left(1 - 7Q_w^2 + 6Q_w^3 - 35Q_w^4 + 146Q_w^5 - 249Q_w^6 \right), \\
 e^{-\frac{\mu_{\text{eff}}}{3}} \mathcal{W}_1(k=6) &= \frac{\mu_{\text{eff}}}{3\sqrt{3}\pi} \left(1 + 2Q_w - Q_w^2 - 6Q_w^3 - 23Q_w^4 + 22Q_w^5 + 19Q_w^6 \right) \\
 &\quad - \frac{1}{9} \left(1 - 7Q_w^2 + 6Q_w^3 - 35Q_w^4 + 146Q_w^5 - 249Q_w^6 \right),
 \end{aligned}$$

$$\begin{aligned}
 e^{-\frac{\mu_{\text{eff}}}{6}} \mathcal{W}_1(k=12) &= \frac{\mu_{\text{eff}}}{6\pi} \left(1 + 2Q_w - Q_w^2 + 2Q_w^3 - 7Q_w^4 + 30Q_w^5 - 117Q_w^6 + 476Q_w^7 \right) \\
 &\quad - \frac{1}{2\sqrt{3}} \left(1 - \frac{13}{3}Q_w^2 + \frac{46}{3}Q_w^3 - \frac{185}{3}Q_w^4 + \frac{730}{3}Q_w^5 - \frac{2651}{3}Q_w^6 + 3146Q_w^7 \right), \\
 e^{-\frac{\mu_{\text{eff}}}{4}} \mathcal{W}_1(k=8) &= \frac{\mu_{\text{eff}}}{4\sqrt{2}\pi} \left(1 + 2Q_w - Q_w^2 + 2Q_w^3 + Q_w^4 + 40Q_w^5 - 68Q_w^6 \right) \\
 &\quad - \frac{1}{4\sqrt{2}} \left(1 - 5Q_w^2 + 16Q_w^3 - 39Q_w^4 + 128Q_w^5 - 388Q_w^6 \right), \\
 e^{-\frac{\mu_{\text{eff}}}{2}} \mathcal{W}_1(k=4) &= \frac{\mu_{\text{eff}}}{4\pi} \left(1 + 2Q_w + 7Q_w^2 + 18Q_w^3 + 57Q_w^4 + 160Q_w^5 + 516Q_w^6 \right). \tag{A.2}
 \end{aligned}$$

A.2 Winding number $n = 2$

For the winding number $n = 2$ with $k = 6, 8, 12$ we find

$$\begin{aligned}
 e^{-\frac{4\mu_{\text{eff}}}{k}} \mathcal{W}_2(k=6) &= \frac{\mu_{\text{eff}}}{3\sqrt{3}\pi} (1 - Q_w - 12Q_w^3 + 18Q_w^4 - 20Q_w^5 + 156Q_w^6 - 292Q_w^7) \\
 &\quad + \frac{2}{9} (Q_w - Q_w^2 + 6Q_w^3 - 23Q_w^4 + 40Q_w^5 - 133Q_w^6 + 408Q_w^7), \\
 e^{-\frac{4\mu_{\text{eff}}}{k}} \mathcal{W}_2(k=8) &= \frac{\mu_{\text{eff}}}{8\pi} (1 - 2Q_w + 2Q_w^2 - 8Q_w^3 + 35Q_w^4 - 88Q_w^5) \\
 &\quad - \frac{1}{8} (1 - 2Q_w + 2Q_w^2 - 8Q_w^3 + 39Q_w^4 - 96Q_w^5), \\
 e^{-\frac{4\mu_{\text{eff}}}{k}} \mathcal{W}_2(k=12) &= \frac{\mu_{\text{eff}}}{6\sqrt{3}\pi} (1 - 3Q_w + 4Q_w^2 - 12Q_w^3 + 50Q_w^4 - 204Q_w^5) \\
 &\quad - \frac{2}{9} \left(1 - \frac{3}{2}Q_w + \frac{3}{2}Q_w^2 - 6Q_w^3 + \frac{55}{2}Q_w^4 - 120Q_w^5 \right). \tag{A.3}
 \end{aligned}$$

A.3 Winding number $n = 3$

For the winding number $n = 3$ with $k = 8, 12$ we find

$$\begin{aligned}
 e^{-\frac{6\mu_{\text{eff}}}{k}} \mathcal{W}_3(k=8) &= \frac{\mu_{\text{eff}}}{4\sqrt{2}\pi} (1 - Q_w + 3Q_w^2 - 7Q_w^3 + 32Q_w^4 - 73Q_w^5) \\
 &\quad + \frac{1}{4\sqrt{2}} (Q_w - 4Q_w^2 + 11Q_w^3 - 36Q_w^4 + 105Q_w^5), \\
 e^{-\frac{6\mu_{\text{eff}}}{k}} \mathcal{W}_3(k=12) &= \frac{\mu_{\text{eff}}}{12\pi} (1 - 4Q_w + 12Q_w^2 - 34Q_w^3 + 120Q_w^4 - 460Q_w^5) \\
 &\quad - \frac{1}{3\sqrt{3}} (1 - 2Q_w + 6Q_w^2 - 20Q_w^3 + 80Q_w^4 - 310Q_w^5). \tag{A.4}
 \end{aligned}$$

A.4 Winding number $n = 4$

For the winding number $n = 4$ with $k = 12$ we find

$$\begin{aligned}
 e^{-\frac{8\mu_{\text{eff}}}{k}} \mathcal{W}_4(k=12) &= \frac{\mu_{\text{eff}}}{6\sqrt{3}\pi} (1 - 3Q_w + 10Q_w^2 - 36Q_w^3 + 129Q_w^4) \\
 &\quad - \frac{1}{6} \left(1 - 2Q_w + \frac{23}{3}Q_w^2 - 30Q_w^3 + \frac{335}{3}Q_w^4 \right). \tag{A.5}
 \end{aligned}$$

A.5 Imaginary part of $W_{(1,1)}$

For the imaginary part of $W_{(1,1)}$ with $k = 6, 8, 12$ we find

$$\begin{aligned}
 e^{-\frac{2\mu_{\text{eff}}}{3}} W_{(1,1)}^{\text{Im}}(k=6) &= \frac{\mu_{\text{eff}}}{18\pi} (1 + 3Q_w - 8Q_w^2 + 4Q_w^3 - 52Q_w^4 + 140Q_w^5) \\
 &\quad + \frac{1}{9\sqrt{3}} (1 + Q_w - Q_w^2 - 2Q_w^3 - 25Q_w^4 + 52Q_w^5), \\
 e^{-\frac{\mu_{\text{eff}}}{2}} W_{(1,1)}^{\text{Im}}(k=8) &= \frac{\mu_{\text{eff}}}{16\pi} (1 + 2Q_w - 6Q_w^2 + 8Q_w^3 - 13Q_w^4 + 88Q_w^5) \\
 &\quad + \frac{1}{8} (0 + Q_w + 0 \cdot Q_w^2 - 4Q_w^3 + 0 \cdot Q_w^4 + 16Q_w^5), \\
 e^{-\frac{\mu_{\text{eff}}}{3}} W_{(1,1)}^{\text{Im}}(k=12) &= \frac{\mu_{\text{eff}}}{12\pi} (1 + Q_w - 4Q_w^2 + 4Q_w^3 - 6Q_w^4 + 20Q_w^5) \\
 &\quad - \frac{1}{6\sqrt{3}} (1 - 3Q_w - Q_w^2 + 16Q_w^3 - 55Q_w^4 + 172Q_w^5).
 \end{aligned} \tag{A.6}$$

A.6 Real part of $W_{(1,1)}$

For the real part of $W_{(1,1)}$ with $k = 6, 8$ we find

$$\begin{aligned}
 e^{-\frac{2\mu_{\text{eff}}}{3}} W_{(1,1)}^{\text{Re}}(k=6) &= -\frac{\mu_{\text{eff}}^2}{54\pi^2} (1 + 7Q_w - 4Q_w^2 + 32Q_w^3 - 222Q_w^4 + 300Q_w^5 - 1196Q_w^6) \\
 &\quad - \frac{2\mu_{\text{eff}}}{27\sqrt{3}\pi} (1 - Q_w + 7Q_w^2 + 22Q_w^3 - 9Q_w^4 - 160Q_w^5 - 681Q_w^6) \\
 &\quad + \frac{1}{648} (11 - 75Q_w + 284Q_w^2 - 696Q_w^3 + 1302Q_w^4 - 6076Q_w^5 + 12236Q_w^6) \\
 &\quad - \frac{1}{6\sqrt{3}\pi} (1 - Q_w - 16Q_w^3 + 22Q_w^4 - 20Q_w^5 + 224Q_w^6), \\
 e^{-\frac{\mu_{\text{eff}}}{2}} W_{(1,1)}^{\text{Re}}(k=8) &= -\frac{\mu_{\text{eff}}^2}{64\pi^2} (1 + 6Q_w - 2Q_w^2 + 32Q_w^3 - 89Q_w^4 + 368Q_w^5) \\
 &\quad + \frac{\mu_{\text{eff}}}{16\pi} (Q_w - 4Q_w^2 + 8Q_w^3 - 8Q_w^4 + 44Q_w^5) \\
 &\quad + \frac{1}{16} (1 - 3Q_w + 11Q_w^2 - 38Q_w^3 + 105Q_w^4 - 312Q_w^5) \\
 &\quad - \frac{1}{16\pi} (1 - 2Q_w + 2Q_w^2 - 8Q_w^3 + 39Q_w^4 - 96Q_w^5).
 \end{aligned} \tag{A.7}$$

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