Stellar Stratifications on Classifying Spaces

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Abstract

We extend Björner's characterization of the face poset of finite CW complexes to a certain class of stratified spaces, called cylindrically normal stellar complexes. As a direct consequence, we obtain a discrete analogue of cell decompositions in smooth Morse theory, by using the classifying space model introduced in [NTT]. As another application, we show that the exit-path category $\mathsf{Exit}(X)$, in the sense of [Lur], of a finite cylindrically normal CW stellar complex X is a quasi-category.

Contents

1 Introduction

In this paper, we study stratifications on classifying spaces of acyclic topological categories. In particular, the following three questions are addressed.

Question 1.1. How can we recover the original category C from its classifying space BC?

Question 1.2. For a stratified space X with a structure analogous to a cell complex, the first author [Tam18] defined an acyclic topological category $C(X)$, called the *face category* of X, whose classifying space is homotopy equivalent to X. On the other hand, there is a way to associate an

∞-category $\mathsf{Exit}(X)$, called the *exit-path category*[1] of X, to a stratified space satisfying certain conditions [Lur]. Is $C(X)$ equivalent to $\mathsf{Exit}(X)$ as ∞-categories?

Question 1.3. For a discrete Morse function f or an acyclic partial matching on a regular CW complex X, Vidit Nanda, Kohei Tanaka, and the first author [NTT] constructed a poset-enriched category $C(f)$ whose classifying space[2] is homotopy equivalent to X. Does this classifying space have a "cell decomposition" analogous to smooth Morse theory?

The original motivation for this work was Question 1.2 posed by the second author during a series of talks by the first author at the IBS Center for Geometry of Physics in Pohang. For any stratified space X, $\mathsf{Exit}(X)$ can be defined as a simplicial set. Before Question 1.2, the first question we need to address is if $\mathsf{Exit}(X)$ is a quasi-category. Lurie proved as Theorem A.6.4 (1) in [Lur] that $\mathsf{Exit}(X)$ is a quasi-category if X is conically stratified[3].

Question 1.4. When is a CW complex X conically stratified?

It turns out that Question 1.1 is closely related to this problem. The stratified spaces in Question 1.2 are called *cylindrically normal stellar stratified spaces*[4], CNSSS for short, and an answer to Question 1.1 can be given by using CNSSS.

Theorem 1.5. *Let C be an acyclic topological category with the space of objects C_0 having discrete topology. Suppose further that the space of morphisms $C(x, y)$ is compact Hausdorff for each pair $x, y \in C_0$ and the set $P(C)_{<x} = \{y \in C_0 \mid C(y, x) \neq \emptyset \text{ and } y \neq x\}$ is finite for each $x \in C_0$. Then there exists a structure of CNSSS on the classifying space BC whose face category is isomorphic to C as topological categories.*

Roughly speaking, CNSSS is a generalization of CW complex with cells replaced by "star-shaped cells". For a regular CW complex X, the stratification on the classifying space of its face poset $F(X)$ obtained by Theorem 1.5 agrees with the original cell decomposition on X under the standard homeomorphism $X \cong BF(X)$. However, the use of "star-shaped cells" is essential for acyclic categories in general. For example, consider the acyclic category C depicted in Figure 1.

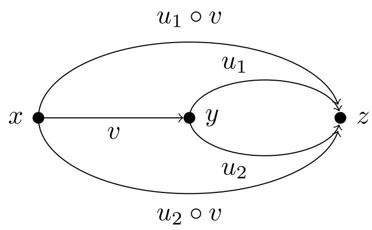

Figure 1: An acyclic category C

Its classifying space BC and the stratification of BC obtained by Theorem 1.5 is shown in Figure 2. The middle stratum in the right-hand side of the equality is the 1-cell $[x, y]$ with x removed. Similarly in the right stratum, top and bottom edges of the "hourglass" are removed. The dotted arrows indicate inclusions of strata into boundaries of closures of higher strata. By regarding the dotted arrows as morphisms and strata as objects, we recover the original category C.

[1] A precise definition is given in §4.2.

[2] See §4.3 for the choice of classifying space of 2-categories used here.

[3] Definition 4.15.

[4] Precisely speaking, CNSSS in this paper is slightly different from the one in [Tam18]. See §3.3 for a precise definition.

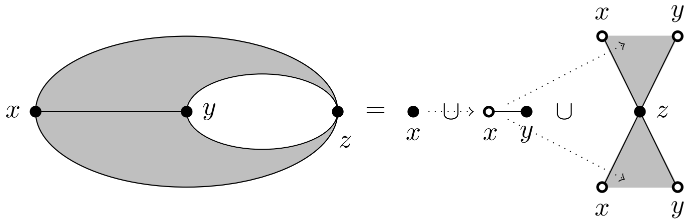

Figure 2: The classifying space BC and its unstable stratification

The stratification in Theorem 1.5 is called the *unstable stratification* on BC. There is a dual stratification called the *stable stratification*. By combining these two stratifications, we obtain the following result.

Theorem 1.6. *Let C be an acyclic topological category satisfying the conditions of Theorem 1.5. Suppose further that $P(C)_{>x} = \{y \in C_0 \mid C(x,y) \neq \emptyset \text{ and } x \neq y\}$ is finite. Then the unstable stratification on BC is conically stratified. Hence $\mathsf{Exit}(BC)$ is a quasi-category.*

It is shown in [Tam18] that, when a CNSSS X is a CW complex, $BC(X)$ is homeomorphic to X, and we obtain an answer to Question 1.4.

Corollary 1.7. *If a finite CW complex X has a structure of CNSSS, then X is conically stratified, hence $\mathsf{Exit}(X)$ is a quasi-category.*

Examples of CW complexes with such a structure are abundant. Regular CW complexes are CNSSS. Among non-regular CW complexes, real and complex projective spaces are typical examples. See Example 4.25, 4.26, and 4.27 of [Tam18]. See §4.2 of the paper for more examples. PLCW complexes introduced by Alexander Kirillov, Jr. [Kir12] also provide non-regular examples of CNSSS.

Question 1.3 is more directly related to Question 1.1. Since the classifying space in Question 1.3 is defined as the classifying space of an acyclic topological category, Theorem 1.5 can be applied.

Theorem 1.8. *For a discrete Morse function f on a finite regular CW complex X, there exists a structure of a CNSSS on the classifying space $B^2C(f)$ of the flow category $C(f)$ constructed in [NTT] satisfying the following conditions:*

1. *Strata are indexed by the set of critical cells of f.*
2. *The face category is isomorphic to the topological category $BC(f)$ associated with the 2-category $C(f)$.*

The classifying space $B^2C(f)$ has a canonical structure of a cell complex but this cell decomposition is much finer than the one obtained from smooth Morse theory. For example, consider the acyclic partial matching on the boundary of a 3-simplex $[v_0, v_1, v_2]$ in Figure 3 which corresponds to a "height function" h.

Matched pairs are indicated by arrows. For example, the 0-simplex $[v_1]$ is matched with a 1-simplex $[v_0, v_1]$ and the 1-simplex $[v_1, v_2]$ is matched with a 2-simplex $[v_0, v_1, v_2]$. The 2-category $C(h)$ has two objects corresponding to critical simplices, i.e. the top face $[v_1, v_2, v_3]$ and the bottom vertex $[v_0]$. As is shown in Example 3.17 of [NTT], the category (poset) of morphisms

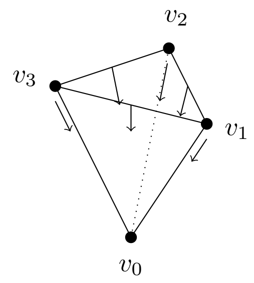

Figure 3: An acyclic partial matching on $\partial[v_0, v_1, v_2]$

from $[v_0]$ to $[v_1, v_2, v_3]$ is isomorphic to the face poset of $\partial[v_1, v_2, v_3]$, and hence its classifying space is the boundary of a hexagon. Thus the classifying space of $C(h)$ is a regular cell complex consisting of two 0-cells, six 1-cells, and six 2-cells. However the cell decomposition of S^2 we usually obtain from a height function is the minimal cell decomposition $S^2 = e^0 \cup e^2$. We should glue six 2-cells, six 1-cells, and one of the 0-cells together to obtain a single 2-cell e^2 so that we have $B^2C(h) = e^0 \cup e^2$. The motivation for Question 1.3 is to generalize this construction and Theorem 1.8 solves the problem.

From the viewpoint of topological combinatorics, Theorem 1.5 is closely related to the well-known characterization of the face poset of a regular CW complex by Björner.

Theorem 1.9 ([Bjö84])**.** *The category of finite regular CW complexes is equivalent to the category of finite CW posets via the face poset functor.*

$$F : \mathbf{RegCW}^f \xrightarrow{\simeq} \mathbf{CWPoset}^f.$$

Let $\mathbf{AcycTopCat}$ and $\mathbf{AcycTopCat}^{lf}_{\mathrm{cpt,Haus}}$ denote the category of acyclic topological categories and the full subcategory of those satisfying the conditions of Theorem 1.5, respectively. The face poset functor for regular CW complexes has been extended to the face category functor in [Tam18]

$$C : \mathbf{CNSSS} \longrightarrow \mathbf{AcycTopCat} \tag{1}$$

from the category of CNSSSs to the category of acyclic topological categories. Our construction in Theorem 1.5 is a right inverse to this face category functor when restricted to appropriate subcategories.

Theorem 1.10. *Let* $\mathbf{CNCW}$ *be the full subcategory of* $\mathbf{CNSSS}$ *consisting of cylindrically normal CW stellar complexes. Then the restriction of the face category functor (1)*

$$C : \mathbf{CNCW} \longrightarrow \mathbf{AcycTopCat}^{lf}_{\mathrm{cpt,Haus}}$$

is an equivalence of categories.

This paper is organized as follows.

- §2 is preliminary. We fix notion and terminology for nerves and classifying spaces of small categories, including simplicial techniques.
- §3 collects necessary materials for stellar stratified spaces from [Tam18] with some generalizations and extensions.
- §4 is the main part. Theorems mentioned above are proved.
- We conclude this paper by a couple of remarks and comments in §5.

1.1 Acknowledgments

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2 Recollections

2.1 Simplicial Terminology

Here we fix notation and terminology for simplicial homotopy theory.

Definition 2.1.

1. The category of isomorphism classes of finite totally ordered sets and order preserving maps is denoted by Δ.

2. The wide subcategory of injective maps is denoted by Δ_{inj}.

3. Objects in these categories are represented by $[n] = \{0 < 1 < \cdots < n\}$ for nonnegative integers n.

4. A *simplicial space* is a functor $X : \Delta^{\mathrm{op}} \to \mathbf{Top}$, where $\mathbf{Top}$ is the category of topological spaces and continuous maps.

5. Dually a *cosimplicial space* is a functor $Y : \Delta \to \mathbf{Top}$.

6. A functor $X : \Delta_{\mathrm{inj}}^{\mathrm{op}} \to \mathbf{Top}$ is called a Δ*-space*.

7. For a nonnegative integer n, the *geometric* n*-simplex* is defined by
$$\underline{\Delta}^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=0}^{n} t_i = 1, t_i \geq 0 \right\}.$$

8. The geometric realization of a simplicial space X is denoted by $|X|$, while the geometric realization of a Δ-space[5] is denoted by $\|X\|$.

2.2 Nerves and Classifying Spaces

Let us first recall basic properties of the classifying space construction. For a topological category C, the spaces of objects and morphisms are denoted by C_0 and C_1, respectively. The space of morphisms from x to y is denoted by $C(x, y)$. The space $N_k(C)$ of k-chains in C is defined to be the set of all functors $[k] \to C$ topologized as a subspace of C_1^k under the identification
$$N_k(C) \cong \left\{ (u_k, u_{k-1}, \ldots, u_1) \in C_1^k \,\middle|\, x_0 \xrightarrow{u_1} x_1 \to \cdots \to x_{k-1} \xrightarrow{u_k} x_k \right\}.$$

[5] Δ-spaces are sometimes called semisimplicial spaces, e.g. in [Lur09] and [ER].

With this notation the structure of a category is given by a pair of maps

$$\circ : N_2(C) \longrightarrow C_1$$
$$\iota : C_0 \longrightarrow C_1$$

satisfying the associativity and the unit conditions.

The collection $N(C) = \{N_k(C)\}_{k\geq 0}$ can be made into a simplicial space, called the *nerve* of C. The geometric realization of the nerve is denoted by BC and is called the *classifying space* of C. The defining quotient map is denoted by

$$p_C : \coprod_{k\geq 0} N_k(C) \times \underline{\Delta}^k \longrightarrow BC. \tag{2}$$

We are mainly interested in acyclic categories.

Definition 2.2. A topological category C is called *acyclic* if the following conditions are satisfied:

1. For any pair of distinct objects $x, y \in C_0$, either $C(x, y)$ or $C(y, x)$ is empty.
2. For any object $x \in C_0$, $C(x, x)$ consists only of the identity morphism.
3. Regard C_0 as the subspace of identity morphisms in C_1. Then $C_1 = C_0 \amalg (C_1 \setminus C_0)$ as topological spaces.

For $x, y \in C_0$, define $x \leq y$ if and only if $C(x, y) \neq \emptyset$. When C is acyclic, C_0 becomes a poset under this relation. This poset is denoted by $P(C)$.

An acyclic topological category C is called a *topological poset* if it is a poset when the topology is forgotten.

The last condition in the definition of acyclicity simplifies the description of the classifying space BC.

Lemma 2.3. *For an acyclic topological category C, define*

$$\overline{N}_k(C) = N_k(C) \setminus \bigcup_i s_i N_{k-1}(C),$$

where $s_i : N_{k-1}(C) \to N_k(C)$ is the i-th degeneracy operator. Then the collection $\overline{N}(C) = \{\overline{N}_k(C)\}$ together with restrictions of the face operators in $N(C)$ forms a Δ-space and the canonical inclusion $\overline{N}(C) \hookrightarrow N(C)$ induces a homeomorphism $\|\overline{N}(C)\| \cong |N(C)| = BC$.

The Δ-space $\overline{N}(C)$ is called the *nondegenerate nerve* of C.

3 Stellar Stratified Spaces and Their Face Categories

The notion of stellar stratified spaces was introduced by the first author in §2.4 of [Tam18]. Roughly speaking, a stellar stratification on a topological space X is a stratification of X together with identifications of strata with "star-shaped cells". In [Tam18], these star-shaped cells are defined as subspaces of closed disks. Here we extend the definition by using cones on stratified spaces.

3.1 Stratifications by Posets

Before we give a definition of stellar stratified spaces, we need to clarify what we mean by a stratification, since the meaning of stratification varies in the literature. Generally, a stratification of a topological space X is a decomposition of X into a mutually disjoint union of subspaces, satisfying some boundary conditions. The boundary conditions are often described in terms of posets.

Decomposing a space X into a union of mutually disjoint subspaces

$$X = \bigcup_{\lambda \in \Lambda} e_\lambda.$$

is equivalent to giving a surjective map $\pi : X \to \Lambda$ with $\pi^{-1}(\lambda) = e_\lambda$. When can we call such a decomposition a stratification? Several conditions have been proposed. Recall that a partial order on Λ generates a topology, called the Alexandroff topology, on Λ. Here we use the following definition from [Tam18].

Definition 3.1. A *stratification* of a topological space X by a poset Λ is an open continuous map $\pi : X \to \Lambda$ such that $\pi^{-1}(\lambda)$ is connected and locally closed for each $\lambda \in \operatorname{Im} \pi$, where Λ is equipped with the Alexandroff topology. Such a pair (X, π) is called a Λ-*stratified space* when π is surjective.

The image of π is denoted by $P(X)$ and is called the *face poset* of X. It is regarded as a full subposet of Λ. For $\lambda \in P(X)$, $\pi^{-1}(\lambda)$ is called the *stratum* indexed by λ and is denoted by e_λ.

Nowadays many authors use simpler definitions. The simplest one only assumes the continuity of π. On the other hand, we need to impose some additional "niceness" conditions to have a good hold on the topological properties of stratified spaces.

In order to understand the meanings of these conditions, let us compare the following five conditions.

(1) π is continuous.

(2) π is an open map.

(3) For any $\mu, \lambda \in \Lambda$, $e_\mu \subset \overline{e_\lambda}$ if and only if $\mu \leq \lambda$.

(4) For any $\mu, \lambda \in \Lambda$, $e_\mu \cap \overline{e_\lambda} \neq \emptyset$ implies $e_\mu \subset \overline{e_\lambda}$, or equivalently, for any $\lambda \in \Lambda$, $\overline{e_\lambda} = \displaystyle\bigcup_{e_\mu \cap \overline{e_\lambda} \neq \emptyset} e_\mu$.

(5) For any closed subset $C \subset \Lambda$, $\displaystyle\bigcup_{\lambda \in C} \overline{e_\lambda}$ is closed.

The condition (1) seems to be the most popular one these days. It is used in Andrade's thesis [And10], Lurie's book [Lur], papers concerning factorization homology [AFT17b; AFT17a], and so on. The combination (1)+(2) is used by the first author [Tam18]. The condition (3) is used by the first author in a series of talks at the Center for Geometry and Physics in Pohang. Cell complexes satisfying the condition (4) are usually called *normal* [LW69]. This condition has been also known as "the axiom of the frontier" in the classical stratification theory due to Thom [Tho69] and Mather [Mat70]. The condition (5) was mentioned by the second author during the above mentioned talks in Pohang.

The following fact is stated and proved in [Tam18] as Lemma 2.3.

Proposition 3.2. *Suppose (1) is satisfied. Then (2) is equivalent to (3).*

For the convenience of the reader, we record a proof. We use the following fact.

Lemma 3.3. *A map $f : X \to Y$ between topological spaces is open if and only if $f^{-1}(\overline{B}) \subset \overline{f^{-1}(B)}$ for any $B \subset Y$. In particular, f is open and continuous if and only if $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$ for any $B \subset Y$.*

Proof. Suppose f is open. Suppose further that $f^{-1}(\overline{B}) \not\subset \overline{f^{-1}(B)}$. Then there exists $x \in f^{-1}(\overline{B}) \setminus \overline{f^{-1}(B)}$. In other words, $f(x) \in \overline{B}$ and there exists an open neighborhood U of x such that $U \cap f^{-1}(B) = \emptyset$. The first condition implies that $V \cap B \neq \emptyset$ for any open neighborhood V of $f(x)$ in Y. The second condition implies that $f(U) \cap B = \emptyset$. Since f is open, $f(U)$ is an open neighborhood of $f(x)$ and this is a contradiction.

Conversely suppose that $f^{-1}(\overline{B}) \subset \overline{f^{-1}(B)}$ for any $B \subset Y$. For an open set $U \subset X$, we have

$$f^{-1}(\overline{Y \setminus f(U)}) \subset \overline{f^{-1}(Y \setminus f(U))} \subset \overline{X \setminus U} = X \setminus U,$$

which implies that $f^{-1}(\overline{Y \setminus f(U)}) \cap U = \emptyset$ or $\overline{Y \setminus f(U)} \cap f(U) = \emptyset$. Thus $f(U)$ is an open set. □

Proof of Proposition 3.2. Suppose π is continuous. When π is open, Lemma 3.3 implies that $\pi^{-1}(\overline{\{\lambda\}}) = \overline{\pi^{-1}(\lambda)}$. Thus $e_\mu \subset \overline{e_\lambda}$ if and only if $\pi^{-1}(\mu) \subset \pi^{-1}(\overline{\{\lambda\}})$, which is equivalent to $\mu \in \overline{\{\lambda\}}$. By the definition of the Alexandroff topology, this is equivalent to $\mu \leq \lambda$. □

Proposition 3.2 suggests a close relationship between the normality condition and the openness of π. In fact, we have the following.

Proposition 3.4. *The conditions (1) and (3) imply the condition (4).*

Proof. When π is continuous, we have

$$\overline{e_\lambda} = \overline{\pi^{-1}(\{\lambda\})} \subset \pi^{-1}(\overline{\{\lambda\}}).$$

If $e_\mu \cap \overline{e_\lambda} \neq \emptyset$, then $e_\mu \cap \pi^{-1}(\overline{\{\lambda\}}) \neq \emptyset$, or $\mu \in \overline{\{\lambda\}}$, or $\mu \leq \lambda$. By (3), this is equivalent to $e_\mu \subset \overline{e_\lambda}$. □

Corollary 3.5. *Any stratification in the sense of Definition 3.1 is normal in the sense of Definition 2.6 of [Tam18].*

The condition (1) obviously implies (5). For the converse, we have the following.

Proposition 3.6. *The conditions (3), (4), and (5) imply (1).*

Proof. Suppose $C \subset \Lambda$ is closed. Then we have

$$\overline{\pi^{-1}(C)} \supset \bigcup_{\lambda \in C} \overline{\pi^{-1}(\lambda)} \supset \bigcup_{\lambda \in C} \pi^{-1}(\lambda) = \pi^{-1}(C).$$

By the condition (5), we obtain

$$\overline{\pi^{-1}(C)} = \bigcup_{\lambda \in C} \overline{\pi^{-1}(\lambda)} = \bigcup_{\lambda \in C} \overline{e_\lambda}.$$

On the other hand, the conditions (3) and (4) allow us to write each $\overline{e_\lambda}$ as $\bigcup_{\mu \leq \lambda} e_\mu$. Since C is closed, $\lambda \in C$ and $\mu \leq \lambda$ imply that $\mu \in C$ and we have

$$\bigcup_{\lambda \in C} \overline{e_\lambda} = \bigcup_{\lambda \in C} \bigcup_{\mu \leq \lambda} e_\mu = \bigcup_{\lambda \in C} e_\lambda = \pi^{-1}(C)$$

and $\pi^{-1}(C)$ is shown to be closed. □

For a cell complex X, define $P(X)$ to be the set of cells of X. Define a partial order $\leq$ on $P(X)$ by saying that $e \leq e'$ if and only if $e \subset \overline{e'}$. We have a map $\pi : X \to P(X)$ which assigns the unique cell $\pi(x)$ containing x to $x \in X$.

In general, this is not a stratification in the sense of Definition 3.1. Proposition 3.6 implies, however, that π is a stratification if X is normal.

Corollary 3.7. *For a normal CW complex X, the map $X \to P(X)$ defined by the cell decomposition is open and continuous, hence is a stratification in the sense of Definition 3.1.*

In particular, the geometric realization of a simplicial complex is a stratified space.

Example 3.8. Let K be an ordered simplicial complex. Then the geometric realization $\|K\|$ has a structure of regular CW complex whose cells are indexed by the simplices of K. Since regular CW complexes are normal, by Corollary 3.7, the cell decomposition is a stratification. This stratification can be generalized to Δ-spaces, i.e. simplicial spaces without degeneracies. For a Δ-space X, the *simplicial stratification*

$$\pi_X : \|X\| = \left(\coprod_{n\geq 0} X_n \times \underline{\Delta}^n\right)\Big/_{\sim} \longrightarrow \coprod_{n\geq 0} X_n$$

is defined by $\pi_X([x, \boldsymbol{t}]) = x$ when $\boldsymbol{t} \in \operatorname{Int}\underline{\Delta}^n$, where the topology of each X_n is forgotten and the partial order on $\coprod_{n\geq 0} X_n$ is defined by

$$x \leq y \Longleftrightarrow \exists u \in \Delta_{\text{inj}}([m],[n]) \text{ such that } X(u)(y) = x$$

for $x \in X_n$, $y \in X_m$. See Example 3.16 of [Tam18], for details. □

The definition of morphisms between stratified spaces should be obvious.

Definition 3.9. A *morphism* of stratified spaces from $\pi_X : X \to P(X)$ to $\pi_Y : Y \to P(Y)$ is a pair of a continuous map $f : X \to Y$ and a morphism of posets $P(f) : P(X) \to P(Y)$ making the obvious diagram commutative.

A morphism $f : X \to Y$ of stratified spaces is called *strict* if $f(e_\lambda) = e_{P(f)(\lambda)}$.

As is the case of cell complexes, the CW condition plays an essential role when we study topological and homotopy-theoretic properties.

Definition 3.10. A stratification π on a topological space X is said to be *CW* if it satisfies the following conditions:

1. (Closure Finite) For each stratum e_λ, the boundary ∂e_λ is covered by a finite number of strata.
2. (Weak Topology) X has the weak topology determined by the covering $\{\overline{e_\lambda}\}_{\lambda\in P(X)}$.

3.2 Joins and Cones

We need to make use of the join of stratified spaces in order to define stellar structures.

Recall that, for topological spaces X and Y, the *join* $X \star Y$ is defined to be the quotient space

$$X \star Y = X \times [0,1] \times Y/_{\sim}$$

where the equivalence relation $\sim$ is generated by the following two types of relations:

1. $(x, 0, y) \sim (x, 0, y')$ for all $x \in X$ and $y, y' \in Y$.

2. $(x, 1, y) \sim (x', 1, y)$ for all $x, x' \in X$ and $y \in Y$.

The class represented by $(x, t, y) \in X \times [0, 1] \times Y$ is denoted by $(1 - t)x + ty$.

Definition 3.11. When X and Y are stratified by maps $\pi_X : X \to P(X)$ and $\pi_Y : Y \to P(Y)$, define

$$\pi_X \star \pi_Y : X \star Y \longrightarrow P(X) \amalg P(X) \times P(Y) \amalg P(Y)$$

by

$$(\pi_X \star \pi_Y)((1-t)x + ty) = \begin{cases} \pi_X(x), & t = 0 \\ (\pi_X(x), \pi_Y(y)), & 0 < t < 1 \\ \pi_Y(y), & t = 1. \end{cases}$$

Lemma 3.12. *For stratified spaces $\pi_X : X \to P(X)$ and $\pi_Y : Y \to P(Y)$, define a partial order on $P(X) \amalg P(X) \times P(Y) \amalg P(Y)$ by the following rule.*

1. *$P(X)$ and $P(Y)$ are full subposets.*

2. *$P(X) \times P(Y)$ with the product partial order is a full subposet.*

3. *$\lambda < (\lambda, \mu)$ for all $\lambda \in P(X)$ and $\mu \in P(Y)$.*

4. *$\mu < (\lambda, \mu)$ for all $\lambda \in P(X)$ and $\mu \in P(Y)$.*

Let us denote this poset as $P(X) \star P(Y)$. Then the map $\pi_X \star \pi_Y : X \star Y \to P(X) \star P(Y)$ defines a stratification on the join $X \star Y$.

Proof. Let us denote the strata in X and Y by e^X_λ and e^Y_μ for $\lambda \in P(X)$ and $\mu \in P(Y)$, respectively. Regard X and Y as subspaces of $X \star Y$. Then we have a decomposition

$$X \star Y = \coprod_{\lambda \in P(X)} e^X_\lambda \amalg \coprod_{(\lambda,\mu) \in P(X) \times P(Y)} e^{X\star Y}_{(\lambda,\mu)} \amalg \coprod_{\mu \in P(Y)} e^Y_\mu,$$

where

$$e^{X\star Y}_{(\lambda,\mu)} = (\pi_X \star \pi_Y)^{-1}(\lambda, \mu) = e^X_\lambda \times (0, 1) \times e^Y_\mu.$$

It remains to show that $\pi_X \star \pi_Y$ is open and continuous. By Proposition 3.6 and Proposition 3.2, it suffices to show that $\pi_X \star \pi_Y$ satisfies the conditions (3), (4), and (5) in §3.1.

By Proposition 3.2 and Proposition 3.4, π_X and π_Y satisfy (3), (4), and (5). By the definition of the partial order on $P(X) \star P(Y)$, $\pi_X \star \pi_Y$ satisfies (3). The condition (4) follows from the fact that $\overline{e^X_\lambda \star e^Y_\mu} = \overline{e^X_\lambda} \star \overline{e^Y_\mu}$. Let C be a closed subset of $P(X) \star P(Y)$. It decomposes as $C = C_X \cup C_{X,Y} \cup C_Y$ with $C_X \subset P(X)$, $C(Y) \subset P(Y)$, and $C_{X,Y} \subset P(X) \times P(Y)$. We have

$$\begin{aligned} \bigcup_{\nu \in C} \overline{(\pi_X \star \pi_Y)^{-1}(\nu)} &= \bigcup_{\lambda \in C_X} \overline{e^X_\lambda} \cup \bigcup_{(\lambda,\mu) \in C_{X,Y}} \overline{e^X_\lambda \times (0,1) \times e^Y_\mu} \cup \bigcup_{\mu \in C_Y} \overline{e^Y_\mu} \\ &= \bigcup_{\lambda \in C_X} \overline{e^X_\lambda} \cup \bigcup_{(\lambda,\mu) \in C_{X,Y}} \overline{e^X_\lambda} \star \overline{e^Y_\mu} \cup \bigcup_{\mu \in C_Y} \overline{e^Y_\mu}. \end{aligned}$$

Let $p_{X\star Y} : X \times [0,1] \times Y \to X \star Y$ be the defining quotient map. Then we have

$$\begin{aligned}
&p_{X\star Y}^{-1}\left(\bigcup_{\lambda\in C_X} \overline{e_\lambda^X} \cup \bigcup_{(\lambda,\mu)\in C_{X,Y}} \overline{e_\lambda^X} \star \overline{e_\mu^Y} \cup \bigcup_{\mu\in C_Y} \overline{e_\mu^Y}\right) \\
&= \bigcup_{\lambda\in C_X} p_{X\star Y}^{-1}\left(\overline{e_\lambda^X}\right) \cup \bigcup_{(\lambda,\mu)\in C_{X,Y}} p_{X\star Y}^{-1}\left(\overline{e_\lambda^X} \star \overline{e_\mu^Y}\right) \cup \bigcup_{\mu\in C_Y} p_{X\star Y}^{-1}\left(\overline{e_\mu^Y}\right) \\
&= \bigcup_{\lambda\in C_X} \overline{e_\lambda^X} \times \{0\} \times Y \cup \bigcup_{(\lambda,\mu)\in C_{X,Y}} \overline{e_\lambda^X} \times [0,1] \times \overline{e_\mu^Y} \cup \bigcup_{\mu\in C_Y} X \times \{1\} \times \overline{e_\mu^Y}.
\end{aligned}$$

Since C_X, $C_{X,Y}$, and C_Y are closed in $P(X)$, $P(X)\times P(Y)$, and $P(Y)$, respectively, this is closed in $X \times [0,1] \times Y$. Hence $\bigcup_{\nu\in C} \overline{(\pi_X \star \pi_Y)^{-1}(\nu)}$ is closed in $X \star Y$. □

In particular, we have the (closed) cone construction on stratified spaces.

Definition 3.13. For a stratified space $\pi_X : X \to \Lambda$, the *cone on* X is defined by

$$\mathsf{cone}(X) = \{*\} \star X.$$

The complement

$$\mathsf{cone}(X) \setminus X \times \{1\} = \{(1-t)*+tx \in \mathsf{cone}(X) \mid 0 \le t < 1\}$$

is called the *open cone* and is denoted by $\mathsf{cone}^\circ(X)$.

Remark 3.14. Make a set $\{b,i\}$ into a poset by $b < i$. Then the face poset of $\mathsf{cone}(X)$ can be identified with $\Lambda \times \{b,i\} \amalg \{*\}$. The element $*$ is the unique minimal element in $\Lambda \times \{i\} \amalg \{*\}$ and is unrelated to elements in $\Lambda \times \{b\}$.

With this identification, the stratification on $\mathsf{cone}(X)$

$$\pi_{\mathsf{cone}(X)} : \mathsf{cone}(X) \longrightarrow \Lambda \times \{b,i\} \amalg \{*\}$$

is given by

$$\pi_{\mathsf{cone}(X)}((1-t)*+tx) = \begin{cases} *, & t = 0 \\ (\pi_X(x), i), & 0 < t < 1 \\ (\pi_X(x), b), & t = 1. \end{cases}$$

The face poset of the open cone $\mathsf{cone}^\circ(X)$, which is $\Lambda \times \{i\} \amalg \{*\}$, is denoted by $\Lambda^{\triangleleft}$ in Lurie's book [Lur].

3.3 Stellar Stratified Spaces

Recall that a characteristic map of a cell e in a cell complex X is a surjective continuous map $\varphi : D^{\dim e} \to \overline{e}$ which is a homeomorphism onto e when restricted to the interior of $D^{\dim e}$. The notion of stellar structure was introduced in [Tam18] by replacing disks by stellar cells, in which a stellar cell was defined as a subspace of a closed disk D^N obtained by taking the join of the center and a subspace S of the boundary. In other words, it is a cone $\mathsf{cone}(S)$ on S embedded in D^N. Here we do not require this embeddability condition.

Definition 3.15. Let S be a stratified space. A subset $D \subset \mathsf{cone}(S)$ is called an *aster* if, for any $x \in D$ with $x = (1-t)*+ty$, $(1-t')*+t'y \in D$ for any $0 \le t' \le t$. The subset $S \cap D$ is called the boundary of D and is denoted by ∂D. The complement $D \setminus \partial D$ is called the *interior* and is denoted by $\mathrm{Int}(D)$. An aster D is called *thin* if $D = \{*\} \star \partial D$.

Definition 3.16. Let $\pi : X \to \Lambda$ be a stratified space. A *stellar structure* on a stratum e_λ is a morphism of stratified spaces $\varphi_\lambda : D_\lambda \longrightarrow \overline{e_\lambda}$, for an aster $D_\lambda \subset \mathsf{cone}(S_\lambda)$, which is a quotient map and whose restriction $\varphi_\lambda|_{\mathrm{Int}(D_\lambda)} : \mathrm{Int}(D_\lambda) \to e_\lambda$ is a homeomorphism. If S_λ is a stratified subspace of a stratification of a sphere S^{n-1} and $\mathrm{Int}(D_\lambda) = \mathrm{Int}(D^n)$, then the stellar structure is called a *cell structure*.

A *stellar stratified space* is a triple (X, π_X, Φ_X) of a topological space X, a stratification $\pi_X : X \to P(X)$, and a collection of stellar structures $\Phi_X = \{\varphi_\lambda\}_{\lambda \in P(X)}$ on strata such that, for each stratum e_λ, $\partial e_\lambda = \overline{e_\lambda} \setminus e_\lambda$ is covered by strata indexed by $P(X)_{<\lambda} = \{\mu \in P(X) \mid \mu < \lambda\}$. When all stellar structures are cell structures, it is called a *cellular stratified space*.

A stratum e_λ in a stellar stratified space is called *thin* if the domain D_λ of the stellar structure $\varphi_\lambda : D_\lambda \to \overline{e_\lambda}$ is a thin aster. A stellar stratified space is called a *stellar complex* if all strata are thin.

Remark 3.17. We do not require the restriction $\varphi|_{\mathrm{Int}(D_\lambda)}$ to be an isomorphism of stratified spaces.

Example 3.18. Consider a cell e_λ in a normal CW complex X. The characteristic map $\varphi_\lambda : D^{\dim e_\lambda} \to \overline{e_\lambda}$ defines a stellar structure on e_λ by setting $S_\lambda = \partial D^{\dim e_\lambda} = S^{\dim e_\lambda - 1}$. The stratification on S_λ is defined by taking connected components of strata obtained by pulling back the stratification on ∂e_λ. Thus any normal CW complex can be regarded as a stellar stratified space. □

Example 3.19. Let Y be the geometric realization of a 1-dimensional simplicial complex of the shape of Y as is shown in Figure 4. Let e^0_0, e^0_1, e^0_2 be the three outer vertices. Denote the

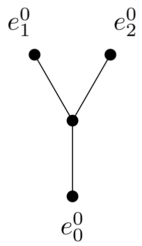

Figure 4: A stellar stratification on “Y”

complement $Y \setminus \{e^0_0, e^0_1, e^0_2\}$ by e^1. Then the decomposition

$$Y = e^0_0 \cup e^0_1 \cup e^0_2 \cup e^1$$

is a stellar stratification. The stellar structure on e^1 is given by the identity map $Y \to \overline{e^1}$. □

Definition 3.20. Let (X, π_X, Φ_X) and (Y, π_Y, Φ_Y) be stellar stratified spaces. A *morphism of stellar stratified spaces* from (X, π_X, Φ_X) to (Y, π_Y, Φ_Y) consists of

- a morphism $\boldsymbol{f} : (X, \pi_X) \to (Y, \pi_Y)$ of stratified spaces, and

- a family of maps $f_\lambda : D^X_\lambda \to D^Y_{P(f)(\lambda)}$ indexed by $P(X)$ making the diagrams
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \varphi_\lambda}\uparrow & & \uparrow{\scriptstyle \psi_{P(f)(\lambda)}} \\ D^X_\lambda & \xrightarrow[f_\lambda]{} & D^Y_{P(f)(\lambda)} \end{array}$$
commutative, where φ_λ and $\psi_{\underline{f}(\lambda)}$ are stellar structures in X and Y, respectively.

The category of stellar stratified spaces is denoted by **SSS**.

The aim of [Tam18] was to find a structure on a stellar stratified space from which the homotopy type can be recovered. The notion of cylindrical normality was introduced for this purpose. Let us recall the definition.

Definition 3.21. Let $\pi : X \to P(X)$ be a normal stellar stratified space whose stellar structure is given by $\{\varphi_\lambda : D_\lambda \to \overline{e_\lambda}\}$. A *cylindrical structure* on this stellar stratified space consists of

- a space $P_{\mu,\lambda}$ and a strict morphism of stratified spaces
$$b_{\mu,\lambda} : P_{\mu,\lambda} \times D_\mu \longrightarrow \partial D_\lambda$$
for each pair of strata $e_\mu \subset \partial e_\lambda$, where each $P_{\mu,\lambda}$ is regarded as a stratified space with a single stratum, and
- a map
$$c_{\lambda_0,\lambda_1,\lambda_2} : P_{\lambda_1,\lambda_2} \times P_{\lambda_0,\lambda_1} \longrightarrow P_{\lambda_0,\lambda_2}$$
for each sequence $\overline{e_{\lambda_0}} \subset \overline{e_{\lambda_1}} \subset \overline{e_{\lambda_2}}$

satisfying the following conditions:

1. The restriction of $b_{\mu,\lambda}$ to $P_{\mu,\lambda} \times \mathrm{Int}(D_\mu)$ is a homeomorphism onto its image.
2. The following three types of diagrams are commutative.
$$\begin{array}{ccc} D_\lambda & \xrightarrow{\varphi_\lambda} & X \\ {\scriptstyle b_{\mu,\lambda}}\uparrow & & \uparrow{\scriptstyle \varphi_\mu} \\ P_{\mu,\lambda} \times D_\mu & \xrightarrow{\mathrm{pr}_2} & D_\mu . \end{array}$$
$$\begin{array}{ccc} P_{\lambda_1,\lambda_2} \times P_{\lambda_0,\lambda_1} \times D_{\lambda_0} & \xrightarrow{1\times b_{\lambda_0,\lambda_1}} & P_{\lambda_1,\lambda_2} \times D_{\lambda_1} \\ {\scriptstyle c_{\lambda_0,\lambda_1,\lambda_2}\times 1}\downarrow & & \downarrow{\scriptstyle b_{\lambda_1,\lambda_2}} \\ P_{\lambda_0,\lambda_2} \times D_{\lambda_0} & \xrightarrow[b_{\lambda_0,\lambda_2}]{} & D_{\lambda_2} \end{array}$$
$$\begin{array}{ccc} P_{\lambda_2,\lambda_3} \times P_{\lambda_1,\lambda_2} \times P_{\lambda_0,\lambda_1} & \xrightarrow{c_{\lambda_1,\lambda_2,\lambda_3}\times 1} & P_{\lambda_1,\lambda_3} \times P_{\lambda_0,\lambda_1} \\ {\scriptstyle 1\times c_{\lambda_0,\lambda_1,\lambda_2}}\downarrow & & \downarrow{\scriptstyle c_{\lambda_0,\lambda_1,\lambda_3}} \\ P_{\lambda_2,\lambda_3} \times P_{\lambda_0,\lambda_2} & \xrightarrow[c_{\lambda_0,\lambda_2,\lambda_3}]{} & P_{\lambda_0,\lambda_3} . \end{array}$$

3. We have
$$D_\lambda = \bigcup_{e_\mu \subset \partial e_\lambda} b_{\mu,\lambda}(P_{\mu,\lambda} \times \mathrm{Int}(D_\mu))$$
as a stratified space.

The space $P_{\mu,\lambda}$ is called the *parameter space* for the inclusion $e_\mu \subset \overline{e_\lambda}$. When $\mu = \lambda$, we define $P_{\lambda,\lambda}$ to be a single point. A stellar stratified space equipped with a cylindrical structure is called a *cylindrically normal stellar stratified space* (CNSSS, for short).

Definition 3.22. For a CNSSS X, define a category $C(X)$ as follows. Objects are strata of X. For each pair $e_\mu \subset \overline{e_\lambda}$, define
$$C(X)(e_\mu, e_\lambda) = P_{\mu,\lambda}.$$
The composition of morphisms is given by
$$c_{\lambda_0,\lambda_1,\lambda_2} : P_{\lambda_1,\lambda_2} \times P_{\lambda_0,\lambda_1} \longrightarrow P_{\lambda_0,\lambda_2}.$$
The category $C(X)$ is called the *face category* of X.

The following fact is obvious from the definition.

Lemma 3.23. *For any CNSSS X, its face category $C(X)$ is an acyclic topological category whose underlying poset is $P(X)$.*

Definition 3.24. Let (X, π_X, Φ_X) and (Y, π_Y, Φ_Y) be CNSSSs with cylindrical structures given by $\{b^X_{\mu,\lambda} : P^X_{\mu,\lambda} \times D^X_\mu \to D^X_\lambda\}$ and $\{b^Y_{\alpha,\beta} : P^Y_{\alpha,\beta} \times D^Y_\alpha \to D^Y_\beta\}$, respectively.

A *morphism of CNSSSs* from (X, π_X, Φ_X) to (Y, π_Y, Φ_Y) is a morphism of stellar stratified spaces
$$\boldsymbol{f} : (X, \pi_X, \Phi_X) \longrightarrow (Y, \pi_Y, \Phi_Y)$$
together with maps $f_{\mu,\lambda} : P^X_{\mu,\lambda} \to P^Y_{P(f)(\mu),P(f)(\lambda)}$ making the diagram
$$\begin{array}{ccc} P^X_{\mu,\lambda} \times D_\mu & \xrightarrow{f_{\mu,\lambda} \times f_\mu} & P^Y_{P(f)(\mu),P(f)(\lambda)} \times D_{P(f)(\mu)} \\ \downarrow{\scriptstyle b^X_{\mu,\lambda}} & & \downarrow{\scriptstyle b^Y_{\underline{f}(\mu),P(f)(\lambda)}} \\ D_\lambda & \xrightarrow[f_\lambda]{} & D_{P(f)(\lambda)} \end{array}$$
commutative.

The category of CNSSSs is denoted by **CNSSS**. The full subcategories of cylindrically normal CW stellar complexes and of cylindrically normal cellular stratified spaces are denoted by **CNCW** and **CNCSS**, respectively.

One of main results of [Tam18] is the following.

Theorem 3.25 (Theorem 5.16 of [Tam18])**.** *For a CW cylindrically normal cellular stratified space X, there exists a natural embedding $i_X : BC(X) \hookrightarrow X$ which is a homeomorphism when X is a CW complex.*

The embedding i_X is, in fact, constructed for "cylindrically normal stellar stratified spaces" in the sense of [Tam18]. They differ from CNSSS in this paper by the requirement that the domain D_λ of the stellar structure $\varphi_\lambda : D_\lambda \to \overline{e_\lambda}$ of each stratum is embedded in a disk. This embeddability condition is not used in the construction of i_X.

Corollary 3.26. *The embedding in Theorem 3.25 can be extended to CW CNSSSs. When X is a cylindrically normal stellar complex, the embedding $i_X : BC(X) \to X$ is a homeomorphism.*

4 Stellar Stratifications on Classifying Spaces of Acyclic Categories

Note that the space of objects in the face category $C(X)$ of a CNSSS X has the discrete topology. Let us call such a topological category a *top-enriched category*. In the rest of this paper, we restrict our attention to acyclic top-enriched categories. We introduce two stellar stratifications on BC for such a category C.

4.1 Stable and Unstable Stratifications

Let C be an acyclic top-enriched category. We first need to define stratifications on BC. An obvious choice is the simplicial stratification

$$\pi_{\overline{N}(C)} : BC \longrightarrow \coprod_{n\geq 0} \overline{N}_n(C)$$

in Example 3.8, since BC is homeomorphic to the geometric realization of a Δ-space $\overline{N}(C)$. Unfortunately this stratification is too fine for our purpose.

Definition 4.1. The composition

$$BC \stackrel{\pi_{\overline{N}(C)}}{\longrightarrow} \coprod_{n\geq 0} \overline{N}_n(C) \stackrel{t}{\longrightarrow} C_0$$

is denoted by π_C. It is easy to see that this is a stratification when C_0 is regarded as the poset $P(C)$ associated with C. This is called the *unstable stratification* on BC.

We also have a dual stratification

$$\pi_{C^{\mathrm{op}}} : BC \stackrel{\pi_{\overline{N}(C)}}{\longrightarrow} \coprod_{n\geq 0} \overline{N}_n(C) \stackrel{s}{\longrightarrow} C_0,$$

which should be called the *stable stratification*.

Remark 4.2. We regard BC as a stratified space by the unstable stratification. The stable stratification will be used in §4.2. These terminologies are borrowed from the "classifying space approach" to Morse theory [CJS; NTT].

Example 4.3. Consider the acyclic top-enriched category C with two objects x and y and $C(x,y) = S^{n-1}$. The classifying space BC is a quotient of

$$C_0 \times \underline{\Delta}^0 \amalg (C_1 \setminus \{1_x, 1_y\}) \times \underline{\Delta}^1 \cong C_0 \amalg C(x,y) \times [0,1].$$

The relation is defined by identifying $C(x,y) \times \{0\}$ and $C(x,y) \times \{1\}$ with x and y, respectively. Thus it is a suspension of S^{n-1}.

The simplicial stratification $\pi_{\overline{N}(C)} : BC = \Sigma(S^{n-1}) \to C_0 = \{x, y\}$ is given by

$$\pi_{\overline{N}(C)}([u,t]) = \begin{cases} x, & t = 0, \\ u, & 0 < t < 1, \\ y, & t = 1, \end{cases}$$

while the unstable stratification π_C is given by

$$\pi_C([u,t]) = \begin{cases} x, & t < 1 \\ y, & t = 1. \end{cases}$$

The stratum indexed by x can be identified with the space of flows going out of x. This example justify the terminology. □

Example 4.4. Let $P = [2] = \{0 < 1 < 2\}$. The classifying space BP is homeomorphic to the standard 2-simplex $\underline{\Delta}^2$. Under this identification, the unstable stratification $\pi_{[2]} : \underline{\Delta}^2 \to [n]$ is given by

$$\pi_{[2]}(t_0, t_1, t_2) = \max\{i \mid t_i \neq 0\}.$$

Let $e_i = \pi_{[2]}^{-1}(i)$ for $i \in [2]$. This stratification on $\underline{\Delta}^2$ is given as in Figure 5.

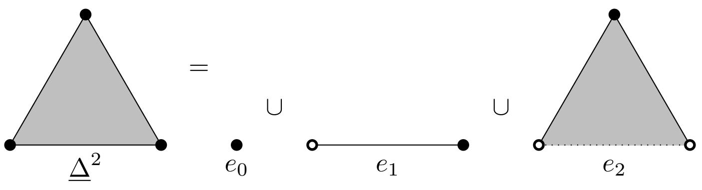

Figure 5: The unstable stratification on $B[2]$

The inclusions $e_0 \subset \overline{e_1}$ and $e_1 \subset \overline{e_2}$ imply that the poset [2] can be recovered from this stratification.

More generally, the unstable stratification on $B[n] = \underline{\Delta}^n$ is given by

$$\pi_{[n]}(t_0, \dots, t_n) = \max\{i \mid t_i \neq 0\}$$

This is the stratification appeared as Example 2.10 in [Tam18]. This stratification is used in the definition of the exit-path ∞-category. See §4.2. □

Our next task is to define a stellar structure on the unstable stratification. This is done by using comma categories.

Definition 4.5. Let C be an acyclic top-enriched category and x an object of C. The nondegenerate nerve $\overline{N}(C \downarrow x)$ of the comma category $C \downarrow x$ is denoted by $\mathrm{St}_{\leq x}(C)$ and is called the *lower star* of x in C.

The full subcategory of $C \downarrow x$ consisting of $(C \downarrow x)_0 \setminus \{1_x\}$ is denoted by $C_{<x}$. The nondegenerate nerve $\overline{N}(C_{<x})$ is denoted by $\mathrm{Lk}_{<x}(C)$ and is called the *lower link* of x in C.

The functor induced by the source map in C is denoted by

$$s_x : C_{<x} \subset C \downarrow x \longrightarrow C.$$

The induced map of Δ-spaces is also denoted by

$$s_x : \mathrm{Lk}_{<x}(C) \subset \mathrm{St}_{\leq x}(C) \longrightarrow \overline{N}(C).$$

Dually, we define $\mathrm{St}_{\geq x}(C)$, $C_{>x}$, $\mathrm{Lk}_{>x}(C)$ and $t_x : C_{>x} \subset x \downarrow C \to C$ by using $x \downarrow C$. The map induced by the functor t_x is denoted by $t_x : \mathrm{Lk}_{>x}(C) \subset \mathrm{St}_{\geq x}(C) \to \overline{N}(C)$.

It is straightforward to verify that $C \downarrow x$ and $x \downarrow C$ are acyclic when C is.

We have the following description of the lower link.

Lemma 4.6. *For an acyclic top-enriched category C and an object $x \in C_0$, we have*

$$\mathrm{Lk}_{<x}(C)_k \cong \begin{cases} \coprod_{x \neq y} C(y, x), & k = 0 \\ \left\{ \boldsymbol{u} \in \overline{N}_{k+1}(C) \,\middle|\, t(\boldsymbol{u}) = x \right\}, & k > 0. \end{cases}$$

Proof. By inspection. □

Definition 4.7. For $x \in C_0$, define

$$\begin{aligned} D_x &= \|\mathrm{St}_{\leq x}(C)\| = B(C \downarrow x) \\ \partial D_x &= \|\mathrm{Lk}_{<x}(C)\| = BC_{<x} \\ D_x^{\mathrm{op}} &= \|\mathrm{St}_{\geq x}(C)\| = B(x \downarrow C) \\ \partial D_x^{\mathrm{op}} &= \|\mathrm{Lk}_{>x}(C)\| = BC_{>x}. \end{aligned}$$

The complements $D_x \setminus \partial D_x$ and $D_x^{\mathrm{op}} \setminus \partial D_x^{\mathrm{op}}$ are denoted by D_x° and $D_x^{\mathrm{op},\circ}$, respectively. The maps induced by the source and the target maps are denoted by $s_x : D_x \to BC$ and $t_x : D_x^{\mathrm{op}} \to BC$, respectively.

The geometric realization $\|\mathrm{Lk}_{<x}(C)\|$ has a stratification based on the Δ-space structure. Lemma 7.12 of [Tam18] says that we have a homeomorphism

$$D_x^{\mathrm{op}} = \|\mathrm{St}_{\geq x}(C)\| \cong \mathsf{cone}(\|\mathrm{Lk}_{>x}(C)\|) = \{1_x\} \star \partial D_x^{\mathrm{op}}.$$

The following is a dual.

Lemma 4.8. *For an acyclic top-enriched category C and an object $x \in C_0$, we have a homeomorphism*

$$D_x = \|\mathrm{St}_{\leq x}(C)\| \cong \mathsf{cone}(\|\mathrm{Lk}_{<x}(C)\|) = \partial D_x \star \{1_x\}.$$

In particular, D_x° is an open cone on ∂D_x.

In order to use explicit descriptions of these homeomorphisms, let us sketch a proof.

Proof. For $x \in C_0$, defined a map $h_x : D_x \to BC_{<x} \star \{1_x\}$ as follows. For $[(\boldsymbol{u}, \boldsymbol{a})] \in D_x = \|\overline{N}(x \downarrow C)\|$, choose a representative $(\boldsymbol{u}, \boldsymbol{a}) \in \overline{N}_k(C \downarrow x) \times \underline{\Delta}^k$ such that $\boldsymbol{a}$ is in the interior of $\underline{\Delta}^k$. The k-chain $\boldsymbol{u}$ in $C \downarrow x$ can be regarded as a $(k+1)$-chain in C of the following form

$$\boldsymbol{u} : x_0 \xrightarrow{u_1} x_1 \to \cdots \to x_{k-1} \xrightarrow{u_k} x_k \xrightarrow{u_{k+1}} x.$$

Note that $u_1, \ldots, u_k$ are not identity morphisms, but u_{k+1} can be. When u_{k+1} is not an identity morphism, $[\boldsymbol{u}, \boldsymbol{a}]$ defines an element of $BC_{<x}$. Define

$$h_x([\boldsymbol{u}, \boldsymbol{a}]) = 1[\boldsymbol{u}, \boldsymbol{a}] + 01_x.$$

When $u_{k+1} = 1_x$, use the standard identification $\underline{\Delta}^k \cong \underline{\Delta}^{k-1} \star \{\boldsymbol{e}_{k+1}\}$ to denote $\boldsymbol{a} = (1-t)\boldsymbol{a}' + t\boldsymbol{e}_{k+1}$, where $\boldsymbol{e}_i$ is the i-th vertex of $\underline{\Delta}^k$. And define

$$h_x([\boldsymbol{u}, \boldsymbol{t}]) = (1-t)[(u_1, u_2, \ldots, u_{k+1} \circ u_k), \boldsymbol{a}'] + t1_x.$$

Here we regard $(u_1, u_2, \ldots, u_{k+1} \circ u_k)$ as an element of $\overline{N}_{k-1}(C_{<x})$ under the identification of Lemma 4.6.

A map $h_x^{\mathrm{op}} : D_x^{\mathrm{op}} \to \{1_x\} \star BC_{>x}$ is defined by reversing arrows. It is straightforward to verify that these maps are homeomorphisms. □

In order to describe the image of D_x° under s_x, we need the following generalization of the dual of Lemma 7.15 of [Tam18].

Lemma 4.9. *Let C be an acyclic top-enriched category. Then, for each $x \in C_0$, we have*

$$\pi_C^{-1}(x) = s_x(D_x^\circ) \text{ and } \overline{\pi_C^{-1}(x)} = s_x(D_x).$$

Furthermore

$$s_x(D_x^\circ) = p_C\left(\coprod_k \coprod_{x_0<\cdots<x_{k-1}<x} C(x_{k-1},x)\times\cdots\times C(x_0,x_1)\times\left(\underline{\Delta}^k \setminus d^k(\underline{\Delta}^{k-1})\right)\right), \qquad (3)$$

where $p_C : \coprod_k \overline{N}_k(C)\times\underline{\Delta}^k \to BC$ is the projection.

Proof. Let us first show (3). An element of D_x° can be represented by a pair $(\boldsymbol{u},\boldsymbol{t})$ of $\boldsymbol{u}\in\overline{N}_k(C_{<x})$ and $\boldsymbol{t}\in\underline{\Delta}^k$, where $\boldsymbol{u}$ is a sequence of morphisms $v_0 \xrightarrow{u_1} v_1 \to \cdots \to v_{k-1} \xrightarrow{u_k} v_k$ in $C_{<x}$ with $v_i \neq 1_x$ for all i and $u_k = 1_x$. The only possibility for such an element to be equivalent to an element of $\coprod_k \overline{N}_k(C_{<x})\times\underline{\Delta}^k$ is that $\boldsymbol{t}\in d^k(\underline{\Delta}^{k-1})$. And we obtain (3).

By definition, the composition $\pi_C\circ s_x$ is a constant map onto x when restricted to D_x° and thus $s_x(D_x^\circ)\subset\pi_C^{-1}(x)$.

Suppose $\pi_C([\boldsymbol{u},\boldsymbol{t}]) = x$ for $\boldsymbol{u} = (u_k,\ldots,u_1)\in\overline{N}_k(C)$ and $\boldsymbol{t}\in\operatorname{Int}\underline{\Delta}^k$. Then $t(u_k) = x$ and the sequence $(u_k,\ldots,u_1)$ can be regarded as an element of $\overline{N}_k(C\downarrow x) = \mathrm{St}_{\leq x}(C)_k$ as follows

$$\begin{array}{ccccc} x_0 & \xrightarrow{u_1} & \cdots & \xrightarrow{u_k} & x \\ & \searrow^{u_k\circ\cdots\circ u_1} & & & \downarrow 1_x \\ & & & & x. \end{array} \qquad (4)$$

Since $\boldsymbol{t}\in\operatorname{Int}\Delta^k$, the pair $(\boldsymbol{u},\boldsymbol{t})$ represents and element of D_x° whose image under s_x is $[\boldsymbol{u},\boldsymbol{t}]$. Thus we have $s_x(D_x^\circ) = \pi_C^{-1}(x)$.

By taking the closure, we have $\overline{s_x(D_x^\circ)} = \overline{\pi_C^{-1}(x)}$. Since C_0 is discrete, the topology of BC is given by the weak topology defined by the covering

$$\{p_C(C(x_{k-1},x_k)\times\cdots\times C(x_0,x_1)\times\Delta^k)\}_{k\geq 0, x_0\leq\cdots\leq x_k}.$$

Under the description of (3), the closure of $s_x(D_x^\circ)$ is given by adding $C(x_{k-1},x)\times\cdots\times C(x_0,x_1)\times d^k(\underline{\Delta}^k)$. And we have $\overline{\pi_C^{-1}(x)} = s_x(D_x)$. □

Note that we used the fact that C_0 has the discrete topology.

In order for s_x to be a stellar structure, we need to impose a finiteness condition on C.

Definition 4.10. An acyclic top-enriched category C is called *locally finite* if $P(C)_{<x} = \{y\in C_0 \mid y < x\}$ is finite.

Lemma 4.11. *Let C be an acyclic top-enriched category. If C is locally finite and the morphism space $C(x,y)$ is compact Hausdorff for each pair $x,y\in C_0$, then $s_x : D_x \to BC$ is a stellar structure on $\pi_C^{-1}(x)$ for each $x\in C_0$.*

Proof. By Lemma 4.8, D_x is a cone on ∂D_x. By Lemma 4.9, the image of the map s_x is $\overline{\pi_C^{-1}(x)}$. By the compactness of each $C(x,y)$ and the finiteness of $P(C)(< x)$, D_x is compact. By a result of de Seguins Pazzis [Seg13], BC is Hausdorff. Hence $s_x : D_x \to \overline{\pi_C^{-1}(x)}$ is a quotient map.

The fact that the restriction of s_x to D_x° is a homeomorphism onto $\pi_C^{-1}(x)$ follows from the description (3). In fact, the inverse to $s_x|_{D_x^\circ}$ is given by assigning (4) to $(u_k,\ldots,u_1)\in\overline{N}_k(C)$ with $t(u_k) = x$. □

The stratification we have defined on BC fits well into the face category of CNSSS.

Proposition 4.12. *For a CW CNSSS X, the embedding*

$$i_X : BC(X) \hookrightarrow X$$

in Theorem 3.25 is a morphism of stratified spaces when $BC(X)$ is equipped with the unstable stratification. If each parameter space of X is compact Hausdorff, i_X is a morphism of stellar stratified spaces.

In particular, if X is a CW stellar complex, i_X is an isomorphism of stellar stratified spaces.

Proof. This follows immediately from the explicit definitions of i_X and the unstable stratification on $BC(X)$. Note that the closure finiteness implies that $C(X)$ is locally finite. □

The next task is to find a cylindrical structure, in the sense of Definition 3.21, on the stellar stratification on BC we have constructed.

Definition 4.13. Let C be an acyclic topological category. For $x, y \in C_0$ with $x < y$, a morphism $u : x \to y$ induces a functor

$$u \circ (-) : C \downarrow x \longrightarrow C \downarrow y.$$

The induced map on the classifying spaces is denoted by

$$b_{x,y} : C(x, y) \times D_x \longrightarrow D_y.$$

We are ready to prove Theorem 1.5.

Proof of Theorem 1.5. It remains to show that maps $b_{x,y} : C(x, y) \times D_x^\circ \to D_y$ define a cylindrical structure on the unstable stratification on BC.

The fact that each $b_{x,y} : C(x, y) \times D_x^\circ \to D_y$ is an embedding follows from the acyclicity of C. The associativity of compositions of morphisms in C implies that the commutativity of three diagrams in Definition 3.21. By definition, $e_x \subset \partial e_y$ if and only if $x \neq y$ and $C(x, y) \neq \emptyset$ and we have

$$D_y = \bigcup_{e_x \subset \partial e_y} b_{x,y}(C(x, y) \times D_x^\circ).$$

□

Theorem 1.10 is a corollary to the above argument.

Proof of Theorem 1.10. By Lemma 4.8, the structure of CNSSS on BC constructed in Theorem 1.5 is actually a stellar complex. It remains to verify that it is CW. The closure finiteness follows from the locally finiteness of C. For the weak topology, consider the commutative diagram

$$\begin{array}{ccc}
\coprod_{k \geq 0} \overline{N}_k(C) \times \underline{\Delta}^k & = & \coprod_{x \in C_0} \coprod_k t_k^{-1}(x) \times \underline{\Delta}^k \\
\Big\downarrow p_C & & \Big\downarrow \coprod_{x \in C_0} p_{D_x} \\
BC & \xleftarrow{\bigcup_{x \in C_0} s_x} & \coprod_{x \in C_0} D_x,
\end{array}$$

where $t_k : \overline{N}_k(C) \to C_0$ is given by the target map. Since both p_C and $\coprod_{x \in C_0} p_{D_x}$ are quotient maps, it follows that $\bigcup_{x \in C_0} s_x$ is a quotient map. □

4.2 The Exit-Path Category

In this section, we prove Theorem 1.6. Let us first recall the definition of the exit-path category.

Definition 4.14. For a stratified space $\pi : X \to P(X)$, define

$$\begin{aligned}\mathsf{Exit}(X) &= \{\sigma \in \mathrm{Sing}(X) \mid \sigma : \underline{\Delta}^n \to X \text{ stratification preserving}\} \\ &= \left\{ \sigma : \underline{\Delta}^n \to X \;\middle|\; \begin{array}{ccc} \underline{\Delta}^n & \xrightarrow{\sigma} & X \\ {\scriptstyle \pi_{[n]}}\downarrow & & \downarrow{\scriptstyle \pi_X} \\ [n] & \xrightarrow[\exists]{} & P(X), \end{array} \right\}\end{aligned}$$

where $\mathrm{Sing}(X)$ is the singular simplicial set of X and $\pi_{[n]}$ is the stratification in Example 4.4. When $\mathsf{Exit}(X)$ is a quasi-category, it is called the *exit-path ∞-category of X*.

The following useful criterion can be found in Lurie's book [Lur].

Definition 4.15 (Definition A.5.5 of [Lur]). A stratified space $\pi : X \to P(X)$ is called *conically stratified* if, for each $x \in X$, there exists a $P(X)_{>\pi(x)}$-stratified space Y, a topological space Z, and an open embedding $Z \times \mathsf{cone}^\circ(Y) \to X$ of $P(X)$-stratified space whose image contains x, where $P(X)_{>\pi(x)}$ is the full subposet of $P(X)$ consisting of elements λ with $\lambda > \pi(x)$.

Theorem 4.16 (Theorem A.6.4 (1) of [Lur]). *If a stratified space $\pi : X \to P(X)$ is conically stratified, the the map $p_\pi : \mathsf{Exit}(X) \to N(P(X))$ induced by π is an inner fibration. In particular, $\mathsf{Exit}(X)$ is a quasi-category.*

Thanks to this theorem, it suffices to show that the unstable stratification on BC is conically stratified in order to prove Theorem 1.6. This is done by combining with the dual stratification, i.e. the stable stratification on BC. Namely there exists a map

$$c_x : D_x \times D_x^{\mathrm{op}} \longrightarrow BC \tag{5}$$

for each object x in C such that the restrictions $c_x|_{D_x \times \{1_x\}}$ and $c_x|_{\{1_x\} \times D_x^{\mathrm{op}}}$ coincide with s_x and t_x, respectively. The map c_x is defined by the composition

$$D_x \times D_x^{\mathrm{op}} \xrightarrow{1_{D_x} \times h_x^{\mathrm{op}}} D_x \times (\{1_x\} \star \partial D_x^{\mathrm{op}}) \xrightarrow{j_x} D_x \star \partial D_x^{\mathrm{op}} \xrightarrow{n_x} BC,$$

where h_x^{op} is the map defined in the proof of Lemma 4.8 and the map j_x is given by $j_x(\boldsymbol{a}, (1-t)1_x + t\boldsymbol{b}) = (1-t)\boldsymbol{a} + t\boldsymbol{b}$. Let us define n_x.

Definition 4.17. For $[\boldsymbol{u}, \boldsymbol{s}] \in D_x$, choose a representative with $\boldsymbol{u} \in \overline{N}_p(C \downarrow x)$ and $\boldsymbol{s} \in \mathrm{Int}\,\underline{\Delta}^p$. The nondegenerate p-chain

$$\boldsymbol{u} : y_0 \xrightarrow{u_1} y_1 \xrightarrow{u_2} \cdots \xrightarrow{u_p} y_p \xrightarrow{u_{p+1}} x$$

in $C \downarrow x$ is regarded as a $(p+1)$-chain in C. Note that this may degenerate in $N_{p+1}(C)$. For $[\boldsymbol{v}, \boldsymbol{t}] \in \partial D_x^{\mathrm{op}}$, choose a representative with $\boldsymbol{v} \in \overline{N}_q(C_{>x})$ and $\boldsymbol{t} \in \mathrm{Int}\,\underline{\Delta}^q$. The nondegenerate q-chain

$$\boldsymbol{v} : x \xrightarrow{v_0} z_0 \xrightarrow{v_1} \cdots \xrightarrow{v_q} z_q$$

in $x \downarrow C$ is also nondegenerate as an element of $N_{q+1}(C)$, since $v_0 \neq 1_x$.

Now define

$$n_x((1-r)[\boldsymbol{u},\boldsymbol{s}]+r[\boldsymbol{v},\boldsymbol{t}]) = \begin{cases} [(v_q,\ldots,v_1,v_0\circ u_{p+1},u_p,\ldots,u_1),(1-r)\boldsymbol{s}+r\boldsymbol{t}], & r>0 \\ [(u_p,\ldots,u_1),\boldsymbol{s}] = s_x([\boldsymbol{u},\boldsymbol{s}]), & r=0 \end{cases}$$

under the identification of $\underline{\Delta}^p \star \underline{\Delta}^q \cong \underline{\Delta}^{p+q+1}$.

Lemma 4.18. *The map n_x is an embedding onto its image.*

Proof. Consider the simplicial stratification on BC. It is a cell decomposition of BC indexed by $\overline{N}(C)$, when the topology of C is discrete. Even when the topology of C is not discrete, we call a stratum of the simplicial stratification a cell in BC.

By definition, the image of n_x is the union of cells whose boundary contains x as a vertex. On the other hand, the cells in $D_x \times C^\circ(\partial D_x^{\mathrm{op}})$ are in bijective correspondence with

$$P(D_x)\cup P(D_x)\times P(\partial D_x^{\mathrm{op}})\cup P(\partial D_x^{\mathrm{op}}) = \overline{N}(C\downarrow x)\cup \overline{N}(C\downarrow x)\times \overline{N}(C_{>x})\cup \overline{N}(C_{>x}).$$

The set on the right hand side is the set of nondegenerate chains in C which contains x or factors through x.

Since n_x maps a simplex to a simplex homeomorphically, n_x is a bijection onto the stratified subspace of BC consisting of cells which contain x as a vertex. By assumption, $D_x \star \partial D_x^{\mathrm{op}}$ is compact and BC is Hausdorff. And n_x is an embedding onto its image. □

Lemma 4.19. *The restrictions of c_x to $D_x \times \{1_x\}$ and $\{1_x\}\times D_x^{\mathrm{op}}$ coincides with s_x and t_x, respectively.*

Proof. It suffices to show that the restrictions of n_x to D_x and $\{1_x\}\star \partial D_x^{\mathrm{op}}$ can be identified with s_x and t_x, respectively.

By the very definition, n_x agrees with s_x when restricted to D_x. It remains to verify the commutativity of the following diagram

$$\begin{array}{ccc} D_x \star \partial D_x^{\mathrm{op}} & \xrightarrow{n_x} & BC \\ \uparrow & & \uparrow{\scriptstyle t_x} \\ \{1_x\}\star \partial D_x^{\mathrm{op}} & \xleftarrow[h_x^{\mathrm{op}}]{} & D_x^{\mathrm{op}}. \end{array}$$

For $[\boldsymbol{v},\boldsymbol{t}]\in \partial D_x^{\mathrm{op}}$, we have

$$n_x((1-r)1_x + r[\boldsymbol{v},\boldsymbol{t}]) = \begin{cases} [(v_0\circ 1_x, v_1,\ldots,v_q),[(1-r)1+r\boldsymbol{t}]], & r>0 \\ [(1_x,1)], & r=0, \end{cases}$$

which agrees with t_x under h_x^{op}. □

Corollary 4.20. *The restriction of c_x to $D_x^\circ \times D_x^{\mathrm{op},\circ}$ is an open embedding. Hence BC is conically stratified.*

Proof. The image of $D_x^\circ \times D_x^{\mathrm{op},\circ}$ under $j_x\circ(1_{D_x}\times h_x)$ is

$$D_x^\circ \star \partial D_x^{\mathrm{op}} \setminus \partial D_x^{\mathrm{op}} \cong D_x^\circ \times \mathsf{cone}^\circ(\partial D_x^{\mathrm{op}})$$

and restriction of $j_x\circ(1_{D_x}\times h_x)$ to $D_x^\circ\times D_x^{\mathrm{op},\circ}$ is a homeomorphism onto its image. These neighborhoods cover BC and hence BC is conically stratified. □

By Proposition 4.12, we may replace X by $BC(X)$ when X is a CW stellar complex.

Corollary 4.21. *For any cylindrically normal CW stellar complex X, $\mathsf{Exit}(X)$ is a quasi-category.*

4.3 Discrete Morse Theory

Robin Forman [For95; For98] formulated an analogue of Morse theory for regular cell complexes. For a discrete Morse function $f : F(X) \to \mathbb{R}$ on the face poset of a finite regular cell complex X, Forman constructed a CW complex X_f whose cells are indexed by critical cells of f and showed that X_f is homotopy equivalent to X.

Although Forman's discrete Morse theory has been shown to be useful, the construction of X_f is ad hoc. An explicit and functorial construction would be more useful. Such a construction was proposed in a joint work of the first author with Vidit Nanda, Kohei Tanaka [NTT], in which a poset-enriched category $C(f)$ was constructed from a discrete Morse function $f : F(X) \to \mathbb{R}$.

For critical cells c and d, the set of morphisms $C(f)(c, d)$ has a structure of poset. By taking the classifying space $B(C(f)(c, d))$ of each morphism poset, we obtain an acyclic topological category $BC(f)$ whose set of objects is $C(f)_0$.

Theorem 4.22 ([NTT])**.** *For a discrete Morse function f on a regular CW complex X, The classifying space $B^2C(f) = B(BC(f))$ is homotopy equivalent to X.*

As a version of Morse theory, we would like to have a "cell decomposition" of $B^2C(f)$ whose cells are in one-to-one correspondence with critical cells of f. Theorem 1.5 and Proposition 4.12 tell us that the correct way of decomposing $B^2C(f)$ is a stellar stratification, not a cell decomposition.

Theorem 1.8 can be proved by using the unstable stratification on $B^2C(f)$.

Proof of Theorem 1.8. Let X be a finite CW complex. Given a discrete Morse function f on X, the topological category $BC(f)$ is an acyclic top-enriched category. The finiteness of X and a result of de Seguins Pazzis [Seg13] guarantee that the category $BC(f)$ satisfies the conditions of Theorem 1.5. □

5 Concluding Remarks

- In [Tam18], it is proved that if a CNSSS X has a "polyhedral structure", $BC(X)$ is a strong deformation retract of X. It is very likely that the deformation retraction can be used to extend Corollary 4.21 to CW polyhedral stellar stratified spaces.
- The original motivation of this paper was to study relations between $C(X)$ and $\mathsf{Exit}(X)$ for a stellar stratified space X. We anticipate that $C(X)$ and $\mathsf{Exit}(X)$ are equivalent as ∞-categories if X is a cylindrically normal CW stellar complex. This problem will be studied in the sequel to this paper.
- Besides the assumptions of Theorem 1.5, an extra finiteness condition is added in Theorem 1.6. This condition is introduced only for proving the map n_x to be a quotient map in Lemma 4.18. Probably this condition is not necessary or can be replaced by a weaker condition.

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