

# Cellular Stratified Spaces

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September 15, 2016

Dedicated to Professor Fred Cohen on the occasion of his 70th birthday.

## Abstract

The notion of cellular stratified spaces was introduced in a joint work of the author with Basabe, González, and Rudyak with the aim of constructing a cellular model of the configuration space of a sphere. Although the original aim was not achieved in the project, the notion of cellular stratified spaces turns out to be useful, at least, in the study of configuration spaces of graphs. In particular, the notion of totally normal cellular stratified spaces was used successfully in a joint work with the former students of the author [FMT15] to study the homotopy type of configuration spaces of graphs with a small number of vertices.

Roughly speaking, totally normal cellular stratified spaces correspond to acyclic categories in the same way regular cell complexes correspond to posets.

In this paper, we extend this correspondence by replacing cells by stellar cells and acyclic categories by topological acyclic categories.

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# 1 Introduction

The author has been interested in configuration spaces since he was a graduate student at the University of Rochester under the guidance of Professor Fred Cohen. The interest arose when the author tried to understand the global structure of homotopy groups of spheres in his Ph.D. thesis [Tam93, Tam94a, Tam94b]. One of key aspects is the combinatorial structures underlying in the configuration spaces of Euclidean spaces.

Recently the author renewed his interest in configuration spaces during a joint project with Basabe, González, and Rudyak [BGRT14] on higher symmetric topological complexities. The notion of cellular stratified spaces was discovered during the discussion with them.

It turns out that cellular stratified spaces have already appeared in many areas in topology. For example, the stratification on the complement of a complexified hyperplane arrangement used in the construction of the Salvetti complex [Sal87, BZ92, DCS00] is one of motivating examples.

## 1.1 Cellular Stratified Spaces Everywhere

Before we state main results, let us take a look at examples of cellular stratified spaces. We begin with configuration spaces. The configuration space of  $n$  distinct points in a topological space  $X$  is defined by

$$\begin{aligned} \text{Conf}_n(X) &= \{(x_1, \dots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\} \\ &= X^n \setminus \Delta_n(X), \end{aligned}$$

where

$$\Delta_n(X) = \bigcup_{1 \leq i < j \leq n} \{(x_1, \dots, x_n) \in X^n \mid x_i = x_j\}.$$

The starting point of this work is the following problem:

**Problem 1.1.** Given a space  $X$ , construct a combinatorial model for the homotopy type of the configuration space  $\text{Conf}_k(X)$  of  $k$  distinct points in  $X$ . In other words, find a regular cell complex or a simplicial complex  $C_k(X)$  embedded in  $\text{Conf}_k(X)$  as a  $\Sigma_k$ -equivariant deformation retract.

Several solutions are known in special cases.

**Example 1.2.** For a finite CW-complex  $X$  of dimension 1, namely a graph, Abrams constructed a subspace  $C_k^{\text{Abrams}}(X)$  contained in  $\text{Conf}_k(X)$  in his thesis [Abr00] and proved that there is a homotopy equivalence

$$C_k^{\text{Abrams}}(X) \simeq \text{Conf}_k(X)$$

as long as the following two conditions are satisfied:

1. each path connecting vertices in  $X$  of valency more than 2 has length at least  $k + 1$ , and
2. each homotopically essential path connecting a vertex to itself has length at least  $k + 1$ .

Here a path means a 1-dimensional subcomplex homeomorphic to a closed interval. □

**Example 1.3.** Consider the case  $X = \mathbb{R}^n$ . For  $1 \leq i < j \leq k$ , the hyperplane

$$H_{i,j} = \{(x_1, \dots, x_k) \in \mathbb{R}^k \mid x_i = x_j\}$$

in  $\mathbb{R}^k$  defines a linear subspace  $H_{i,j} \otimes \mathbb{R}^n$  in  $\mathbb{R}^k \otimes \mathbb{R}^n = \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_k = X^k$  and we have

$$\text{Conf}_k(\mathbb{R}^n) = \mathbb{R}^k \otimes \mathbb{R}^n \setminus \bigcup_{1 \leq i < j \leq k} H_{i,j} \otimes \mathbb{R}^n.$$

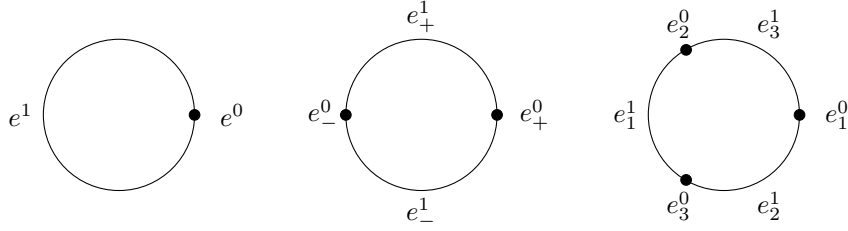
The collection  $\{H_{i,j} \mid 1 \leq i < j \leq k\}$  is called the *braid arrangement* of rank  $k - 1$  and is denoted by  $\mathcal{A}_{k-1}$ .

When  $n = 2$ , the construction due to Salvetti [Sal87] gives us a regular cell complex  $\text{Sal}(\mathcal{A}_{k-1})$  embedded in  $\text{Conf}_k(\mathbb{R}^2)$  as a  $\Sigma_k$ -equivariant deformation retract.

More generally, the construction sketched at the end of [BZ92] by Björner and Ziegler and elaborated in [DCS00] by De Concini and Salvetti gives us a regular cell complex  $\text{Sal}^{(n)}(\mathcal{A}_{k-1})$  embedded in  $\text{Conf}_k(\mathbb{R}^n)$  as a  $\Sigma_k$ -equivariant deformation retract.

This construction is a special case of the construction of a regular cell complex whose homotopy type represents the complement of the subspace arrangement associated with a real hyperplane arrangement. □

There are pros and cons in these two constructions. The conditions in Abrams' theorem require us to subdivide a given 1-dimensional CW-complex finely. For example, his construction fails to give the right homotopy type of the configuration space  $\text{Conf}_2(S^1)$  of two points in  $S^1$  when it is applied to the minimal cell decomposition;  $S^1 = e^0 \cup e^1$ . The minimal regular cell decomposition  $S^1 = e_-^0 \cup e_+^0 \cup e_-^1 \cup e_+^1$  is not fine enough, either. We need to subdivide  $S^1$  into three 1-cells to use Abrams' model.



Another problem is that his theorem is restricted to 1-dimensional CW-complexes, although the construction of the model itself works for any cell complex<sup>1</sup>.

The second construction suggests that we should consider more general stratifications than cell decompositions. The complex  $\text{Sal}^{(n)}(\mathcal{A}_{k-1})$  is constructed from the combinatorial structure of the “cell decomposition” of  $\mathbb{R}^k \otimes \mathbb{R}^n$  defined by the hyperplanes in the arrangement  $\mathcal{A}_{k-1}$  together with the standard framing in  $\mathbb{R}^n$ . “Cells” in this decomposition are unbounded regions in  $\mathbb{R}^k \otimes \mathbb{R}^n$ . Although such a decomposition is not regarded as a cell decomposition in the usual sense, we may extend the definition of face posets to such generalized cell decompositions. And the complex  $\text{Sal}^{(n)}(\mathcal{A}_{k-1})$  is constructed in terms of the combinatorial structure of the face poset of the “cell decomposition” of  $\mathbb{R}^k \otimes \mathbb{R}^n$ . The crucial deficiency of the second construction is, however, that it works only for Euclidean spaces.

One of the motivations of this paper is to find a common framework for working with configuration spaces and complements of arrangements. Although there are many interesting “parallel theories” between configuration spaces and arrangements, e.g. the Fulton-MacPherson-Axelrod-Singer compactification [FM94, AS94] and the De Concini-Procesi wonderful model [DCP95], there is no “Salvetti complex” for configuration spaces in general. A more concrete motivation is, therefore, to solve Problem 1.1 in such a way that it generalizes the Salvetti complex for the braid arrangement.

By analyzing the techniques of combinatorial algebraic topology used in the proof of Salvetti’s theorem, the notion of cellular stratified spaces was introduced in [BGRT]. It turns out that, in the case of configuration spaces of spheres, which was the main target of study in the project, it was not easy to use cellular stratified spaces to construct a combinatorial model. The section for cellular stratified spaces was removed from the published version [BGRT14].

However, the theory of cellular stratified spaces can be used to study configuration spaces of graphs, as is done in [FMT15], in which the notion of totally normal cellular stratified spaces played a central role.

**Definition 1.4** (Definition 3.6). Let  $X$  be a normal cellular stratified space.  $X$  is called *totally normal* if, for each  $n$ -cell  $e_\lambda$ ,

1. there exists a structure of regular cell complex on  $S^{n-1}$  containing  $\partial D_\lambda$  as a stratified subspace, and
2. for any cell  $e$  in  $\partial D_\lambda$ , there exists a cell  $e_\mu$  in  $\partial e_\lambda$  such that  $e_\mu$  and  $e$  share the same domain and the characteristic map of  $e_\mu$  factors through  $D_\lambda$  via the characteristic map of  $e$ :

$$\begin{array}{ccccccc}
 \bar{e} & \hookrightarrow & \partial D_\lambda & \hookrightarrow & D_\lambda & \xrightarrow{\varphi_\lambda} & X \\
 \uparrow & & & & & \nearrow \varphi_\mu & \\
 D & \xlongequal{\quad} & D_\mu & & & & 
 \end{array}$$

<sup>1</sup>Recently higher dimensional cases appeared in [AGH13].

The category  $C(X)$  consisting of cells in  $X$  as objects and above maps  $D_\mu \rightarrow \bar{e} \rightarrow D_\lambda$  as morphisms is called the *face category* of  $X$ .

The following result says that we can always recover the homotopy type of  $X$  from its face poset, if  $X$  is totally normal.

**Theorem 1.5** (Theorem 2.50 in [FMT15]). *For a totally normal cellular stratified space  $X$ , the classifying space  $BC(X)$  of the face category  $C(X)$  can be embedded in  $X$  as a strong deformation retract.*

*When  $X = \mathbb{R}^k \otimes \mathbb{R}^n$  and the stratification is given by a real hyperplane arrangement  $\mathcal{A}$  in  $\mathbb{R}^k$  by the method of Björner-Ziegler and De Concini-Salvetti [BZ92, DCS00], then  $BC(X)$  is homeomorphic to the higher order Salvetti complex  $\text{Sal}^{(n)}(\mathcal{A})$ .*

Note that, if  $X$  is a regular cell complex, it is a fundamental fact in combinatorial algebraic topology that the classifying space  $BF(X)$  of the face poset  $F(X)$  is the barycentric subdivision of  $X$  and is homeomorphic to  $X$ . The above result is a generalization of this well-known fact.

Examples of totally normal cellular stratified spaces include

- regular cell complexes,
- the Björner-Ziegler stratification [BZ92] of Euclidean spaces defined by subspace arrangements,
- graphs regarded as 1-dimensional cell complexes,
- the minimal cell decomposition of  $\mathbb{R}P^n$ ,
- Kirillov’s PLCW-complexes [KJ12] satisfying a certain regularity condition, and
- the geometric realization of  $\Delta$ -sets.

Another source of inspirations for this paper is a preprint [CJS] of R. Cohen, J.D.S. Jones, and G.B. Segal. Given a Morse-Smale function  $f : M \rightarrow \mathbb{R}$  on a smooth closed manifold  $M$ , they constructed a topological acyclic category  $C(f)$  and proved that the classifying space  $BC(f)$  is homeomorphic to  $M$ . Under a weaker assumption, they also proved that  $BC(f)$  is homotopy equivalent to  $X$ . Their results strongly suggest that a “topological acyclic category version” of Theorem 1.5 should exist.

## 1.2 Statements of Results

The aim of this paper is to extend these results and develop the theory of cellular stratified spaces. For this purpose, we first introduce the notion of cylindrical structures on cellular stratified spaces.

**Definition 1.6** (Definition 4.1). *A cylindrical structure on a normal cellular stratified space  $X$  consists of*

- a normal stratification on  $S^{n-1}$  containing  $\partial D_\lambda$  as a stratified subspace for each  $n$ -cell  $\varphi_\lambda : D_\lambda \rightarrow \bar{e}_\lambda$  in  $X$ ,
- a stratified space  $P_{\mu,\lambda}$  and a morphism of stratified spaces

$$b_{\mu,\lambda} : P_{\mu,\lambda} \times D_\mu \longrightarrow \partial D_\lambda$$

for each pair of cells  $e_\mu \subset \partial e_\lambda$ , and

- a morphism of stratified spaces

$$c_{\lambda_0, \lambda_1, \lambda_2} : P_{\lambda_1, \lambda_2} \times P_{\lambda_0, \lambda_1} \longrightarrow P_{\lambda_0, \lambda_2}$$

for each sequence  $\overline{e_{\lambda_0}} \subset \overline{e_{\lambda_1}} \subset \overline{e_{\lambda_2}}$ ,

satisfying certain compatibility and associativity conditions. A cellular stratified space equipped with a cylindrical structure is called a *cylindrically normal* cellular stratified space.

Examples of cylindrically normal cellular stratified spaces include

- totally normal cellular stratified spaces,
- PLCW complexes,
- the minimal cell decomposition of  $\mathbb{C}P^n$ , and
- the geometric realization of simplicial sets.

The cylindrical structure on  $\mathbb{C}P^n$  can be defined by using the “moduli spaces of flows” of a Morse-Smale function, as is done in the preprint [CJS] by R. Cohen, J.D.S Jones, and G.B. Segal mentioned above. Alternatively we can construct the same cylindrical structure by identifying  $\mathbb{C}P^n$  with the Davis-Januszkiewicz construction  $M_{\lambda_n}^2$ . This observation suggests a large class of quasitoric and torus manifolds have cylindrically normal cell decompositions.

Given a cylindrically normal cellular stratified space  $X$ , we define a topological acyclic category  $C(X)$ , called the cylindrical face category of  $X$ . Objects are cells in  $X$  and the space of morphisms from  $e_\mu$  to  $e_\lambda$  is defined to be  $P_{\mu, \lambda}$ .

Our first result says that the classifying space  $BC(X)$  of  $C(X)$  can be always embedded in  $X$ .

**Theorem 1.7** (Theorem 5.16). *For any cylindrically normal cellular stratified space  $X$ , there exists an embedding*

$$i_X : BC(X) \hookrightarrow X$$

*which is natural with respect to morphisms of cellular stratified spaces. Furthermore, when all cells in  $X$  are closed,  $i_X$  is a homeomorphism.*

When  $X$  contains non-closed cells,  $i_X$  is not a homeomorphism. Our second result says that, under a reasonable condition, those non-closed cells can be collapsed into  $BC(X)$ .

**Theorem 1.8** (Theorem 5.18). *For a polyhedral<sup>3</sup> cellular stratified space  $X$ , the image of the embedding  $i_X : BC(X) \hookrightarrow X$  is a strong deformation retract of  $X$ . The deformation retraction can be taken to be natural with respect to morphisms of polyhedral cellular stratified spaces.*

It turns out that they still hold when we replace cells by “star-shaped” cells. We introduce the notion of stellar stratified spaces and show that the functor  $BC(-)$  transforms cellular stratified spaces to stellar stratified spaces, giving us a dualizing operation  $D(-)$ . With this structure, Theorem 1.7 (Theorem 5.16) can be rephrased as follows.

**Theorem 1.9** (Theorem 7.28). *For a totally normal stellar stratified space  $X$ , we have an embedding of stellar stratified spaces*

$$D(D(X)) \hookrightarrow X,$$

*which is an isomorphism when all cells in  $X$  are closed.*

<sup>2</sup>See Example 4.26 for more details.

<sup>3</sup>Definition 4.7

It should be worthwhile noting that, when  $X$  is the complement of the complexification of a real hyperplane arrangement  $\mathcal{A}$ ,  $DD(X)$  coincides with the Salvetti complex of  $\mathcal{A}$ . This fact and examples of cylindrical structures suggest that we may apply these theorems to the following problems:

1. Construct combinatorial models for configuration spaces and apply them to study the homotopy types of configuration spaces.
2. Develop a refinement and an extension of the Cohen-Jones-Segal Morse theory.
3. Reformulate Forman's discrete Morse theory [For95, For98] in terms of topological acyclic categories.

We should also be able to apply the results in this paper to other types of configurations and arrangements. We might be able to apply the results to toric topology.

### 1.3 Organization

The article is organized as follows.

- §2 is preliminary. We fix notations and terminologies for stratified spaces in §2.1. The definition of cell structures is given in §2.2. Cellular stratified spaces are introduced in §2.3. We introduce stellar stratified spaces in §2.4.
- We extend the regularity and normality for cell complexes in §3. The regularity and normality are extended to cellular stratified spaces in §3.1. In §3.2, the definition and basic properties of totally normal cellular stratified spaces are recalled from [FMT15]. §3.3 is devoted to examples of totally normal cellular stratified spaces.
- A new structure, called cylindrically normal cellular and stellar stratifications, is introduced and studied in §4. After defining cylindrically normal cellular stratified spaces in §4.1, we impose piecewise-linear structure on each cell and introduce polyhedral structures in §4.2. We review examples of cylindrically normal cellular stratified spaces in §4.3.
- Theorem 1.7 and 1.8 are proved in §5.
- For applications to configuration spaces, we need to understand basic operations on cellular stratified spaces, which is the subject of §6. We study three kinds of operations; stratified subspaces in §6.1, products in §6.2, and subdivisions in §6.3.
- As an extension of the barycentric subdivision of regular cell complexes, our framework is suitable for discussing duality, which is the subject of §7.

The paper contains three appendices for the convenience of the reader.

- Understanding the behavior of quotient maps is important in this paper, since cell structure maps are required to be quotient maps. We summarize important properties of quotient maps in Appendix A.
- In Appendix B, we recall definitions and properties of simplicial complexes, simplicial sets, and related structures.
- Appendix C is a summary on basics of topological categories, including their classifying spaces.

## 1.4 Acknowledgments

This work originates in fruitful and enjoyable discussion with Jesús González on the Google Wave system in 2010. Although Google terminated the development of the system, I would like to thank Google for providing us with such a useful collaboration tool for free. I would also like to express my appreciation to the Centro di Ricerca Matematica Ennio De Giorgi, Scuola Normale Superiore di Pisa, for supporting my participation in the research program “Configuration Spaces: Geometry, Combinatorics and Topology” in 2010, during which a part of this work was done. I would also like to thank the National University of Singapore for inviting me to the workshop “Combinatorial and Toric Topology” in 2015.

The discussion on the Google Wave has been incorporated into a joint project with Ibai Basabe and Yuli Rudyak with the aim of applying cellular stratified spaces to study the homotopy type of configuration spaces of spheres. It turns out that configuration spaces of spheres are much more complicated than we imagined and the sections for cellular stratified spaces were removed from the published paper [BGRT14]. Nonetheless the discussion during the project with them has been essential. I would like to thank Basabe, González, and Rudyak, as well as Peter Landweber, who read earlier versions of the paper and made useful comments.

Before I began the discussion with González on Google Wave, some of the ideas have been already developed during the discussion with my students. Takamitsu Jinno and Mizuki Furuse worked on Hom complexes in 2009 and configuration spaces of graphs in 2010, respectively, in their master’s theses. The possibility of finding a better combinatorial model for configuration spaces was suggested by their work. Furuse’s work was developed further by another student Takashi Mukouyama. Their work is contained in a paper [FMT15], in which theory of totally normal cellular stratified spaces has been developed. The connection with Cohen-Jones-Segal Morse theory, which resulted in the current definition of cylindrical structure, was discovered during the discussion with another former student, Kohei Tanaka. I am grateful to all these former students.

Mikiya Masuda pointed out my misunderstanding in examples concerning complex projective spaces in an early draft of this paper. Priyavrat Deshpande pointed out a mistake in [FMT15] on subdivisions of totally normal cellular stratified spaces. The mistake is corrected in this paper as Proposition 6.36. I would like to thank both of them.

This work is supported by JSPS KAKENHI Grant Number 23540082 and 15K04870.

## 2 Stratifications and Cells

This section is preliminary. We introduce the notions of stratified spaces, cell structures, and cellular stratified spaces. Although the term “stratified space” has been used in singularity theory since its beginning and is a well-established concept, we need our own definition of stratified spaces.

### 2.1 Stratified Spaces

Before we introduce *cellular* stratified spaces, let us first recall the notion of stratified spaces in general, whose theory has been developed in singularity theory. Unfortunately, however, there seems to be no standard definition of stratified spaces. There are many non-equivalent definitions in the literature. For this reason, we decided to examine several books and extract properties for our needs. As our prototypes, we use definitions in books by Kirwan [Kir88], Bridson and Haefliger [BH99], Pflaum [Pfl01], and Schürmann [Sch03].

Here is our reformulation.



**Definition 2.1.** Let  $X$  be a topological space and  $\Lambda$  be a poset. A *stratification* of  $X$  indexed by  $\Lambda$  is an open continuous map

$$\pi : X \longrightarrow \Lambda$$

satisfying the condition that, for each  $\lambda \in \text{Im } \pi$ ,  $\pi^{-1}(\lambda)$  is connected and locally closed<sup>4</sup>, where  $\Lambda$  is topologized by the Alexandroff topology<sup>5</sup>.

For simplicity, we put  $e_\lambda = \pi^{-1}(\lambda)$  and call it a *stratum* with index  $\lambda$ .

**Remark 2.2.** We may safely assume that  $\pi$  is surjective. When we define morphisms of stratified spaces and stratified subspaces, however, it is more convenient not to assume the surjectivity.

Given a map  $\pi : X \rightarrow \Lambda$ , we have a decomposition of  $X$ , i.e.

1.  $X = \bigcup_{\lambda \in \text{Im } \pi} e_\lambda$ .
2. For  $\lambda, \lambda' \in \text{Im } \pi$ ,  $e_\lambda \cap e_{\lambda'} = \emptyset$  if  $\lambda \neq \lambda'$ .

Thus the image of  $\pi$  in the indexing poset  $\Lambda$  can be identified with the set of strata. The definition of stratification can be rephrased in terms of closures of strata.

**Lemma 2.3.** Let  $\pi : X \rightarrow \Lambda$  be a continuous map from a topological space  $X$  to a poset  $\Lambda$ . Then it is open if and only if the condition  $e_\lambda \subset \overline{e_{\lambda'}}$  is equivalent to  $\lambda \leq \lambda'$  for  $\lambda, \lambda' \in \text{Im } \pi$ .

*Proof.* It is well known that a map  $\pi$  is an open continuous map if and only if  $\overline{\pi^{-1}(B)} = \pi^{-1}(\overline{B})$  for any subset  $B \subset \Lambda$ . Thus, when  $\pi$  is open continuous,  $\pi^{-1}(\lambda) \subset \overline{\pi^{-1}(\lambda')}$  if and only if  $\lambda \in \{\lambda'\}$ , which is equivalent to saying  $\lambda \leq \lambda'$ .

Conversely suppose that  $\pi^{-1}(\lambda) \subset \overline{\pi^{-1}(\lambda')}$  is equivalent to  $\lambda \leq \lambda'$ . For a subset  $B \subset \Lambda$ , we have  $\pi^{-1}(B) \subset \pi^{-1}(\overline{B})$  by the continuity of  $\pi$ . For  $x \in \pi^{-1}(\overline{B})$ ,  $\pi(x) \in \overline{B}$ . By the definition of the Alexandroff topology, there exists  $\lambda \in B$  such that  $\pi(x) \leq \lambda$ . By assumption, this is equivalent to  $\pi^{-1}(\pi(x)) \subset \overline{\pi^{-1}(\lambda)}$ . Thus  $x \in \overline{\pi^{-1}(B)}$  and we have shown that  $\pi^{-1}(\overline{B}) \subset \overline{\pi^{-1}(B)}$ .  $\square$

**Definition 2.4.** For a stratification  $\pi : X \rightarrow \Lambda$ , the image  $\text{Im } \pi$  is called the *face poset* and is denoted by  $P(X, \pi)$  or simply by  $P(X)$ .

When  $P(X)$  is finite or countable,  $(X, \pi)$  is said to be *finite* or *countable*.

**Remark 2.5.** The above structure (without the connectivity of  $\pi^{-1}(\lambda)$ ) is called a decomposition in Pflaum's book [Pfl01]. Pflaum used the notion of set germ to define a stratification from local decompositions. Furthermore, Pflaum imposed three further conditions:

- If  $e_\mu \cap \overline{e_\lambda} \neq \emptyset$ , then  $e_\mu \subset \overline{e_\lambda}$ .
- Each stratum is a smooth manifold.
- The collection  $\{e_\lambda\}_{\lambda \in \Lambda}$  is locally finite in the sense that, for any  $x \in X$ , there exists a neighborhood  $U$  of  $x$  such that  $U \cap e_\lambda \neq \emptyset$  only for a finite number of strata  $e_\lambda$ .

<sup>4</sup>A subset  $A$  of a topological space  $X$  is said to be *locally closed*, if every point  $x \in A$  has a neighborhood  $U$  in  $X$  with  $A \cap U$  closed in  $U$ . This condition is known to be equivalent to saying that  $A$  is an intersection of an open and a closed subset of  $X$ , or  $A$  is open in  $\overline{A}$ .

<sup>5</sup> $D \subset \Lambda$  is closed if and only if, for  $\lambda \in D$  and  $\mu \in \Lambda$ ,  $\mu \leq \lambda$  implies  $\mu \in D$ . Or  $U \subset \Lambda$  is open if and only if, for  $\lambda \in U$  and  $\mu \in \Lambda$ ,  $\lambda \leq \mu$  implies  $\mu \in U$ .

The first condition corresponds to the normality of cell complexes. We would like to separate the third condition as one of the conditions for CW stratifications. For the second condition, as is remarked in his book, we may replace smooth manifolds by any collection of geometric objects such as complex manifolds, real analytic sets, polytopes, and so on. In §2.3, we choose the class of spaces equipped with “cell structures” and define the notion of cellular stratified spaces. As an example of another choice, we use “star-shaped cells” and define the notion of stellar stratified spaces in §2.4. We also impose more structures on cells and introduce notions of cubical and polyhedral structures in Definition 6.20 and Definition 4.7, respectively.

Bridson and Haefliger [BH99] define stratifications by using closed strata. Their strata correspond to closures of strata in our definition. Furthermore, they also required their stratifications to be normal in the following sense.

**Definition 2.6.** We say a stratum  $e_\lambda$  in a stratified space  $(X, \pi)$  is *normal* if  $e_\mu \subset \overline{e_\lambda}$  whenever  $e_\mu \cap \overline{e_\lambda} \neq \emptyset$ . When all strata are normal, the stratification  $\pi$  is said to be *normal*.

It is immediate to verify the following.

**Lemma 2.7.** *A stratum  $e_\lambda$  is normal if and only if  $\partial e_\lambda = \overline{e_\lambda} \setminus e_\lambda$  is a union of strata.*

Another difference between our definition and the one by Bridson and Haefliger is that they considered intersections of closed strata.

**Lemma 2.8.** *Let  $(X, \pi)$  be a normal stratified space. Then, for any pair of strata  $e_\mu, e_\lambda$ , the intersection  $\overline{e_\mu} \cap \overline{e_\lambda}$  is a union of strata.*

*Proof.* This is obvious, since closures of different strata can intersect only on the boundaries, which are unions of strata by the definition of normality.  $\square$

The following is a typical example of stratifications we are interested in.

**Example 2.9.** Let  $S_1 = \{-1, 0, 1\}$  with poset structure  $0 < \pm 1$ . The sign function

$$\text{sign} : \mathbb{R} \longrightarrow S_1$$

given by

$$\text{sign}(x) = \begin{cases} +1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

defines a stratification on  $\mathbb{R}$ :

$$\mathbb{R} = (-\infty, 0) \cup \{0\} \cup (0, \infty).$$

This innocent-looking stratification turns out to be one of the most important ingredients in the theory of real hyperplane arrangements. Let  $\mathcal{A} = \{H_1, \dots, H_k\}$  be a real affine hyperplane arrangement in  $\mathbb{R}^n$  defined by affine 1-forms  $L = \{\ell_1, \dots, \ell_k\}$ . Hyperplanes cut  $\mathbb{R}^n$  into convex regions that are homeomorphic to the interior of the  $n$ -disk. Each hyperplane  $H_i$  is cut into convex regions of dimension  $n - 1$  by other hyperplanes, and so on. These cuttings can be described as a stratification defined by the sign function as follows. Define a map

$$\pi_{\mathcal{A}} : \mathbb{R}^n \longrightarrow \text{Map}(L, S_1)$$

by

$$\pi_{\mathcal{A}}(a)(\ell_i) = \text{sign}(\ell_i(a)).$$

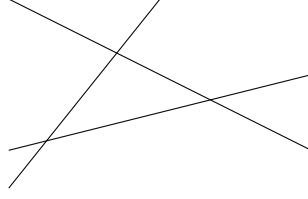


Figure 1: Hyperplanes cut the ambient Euclidean space.

The partial order in  $S_1$  induces a partial order on  $\text{Map}(L, S_1)$  by  $\varphi \leq \psi$  if and only if  $\varphi(\ell_i) \leq \psi(\ell_i)$  for all  $i$ . Then  $\pi_{\mathcal{A}}$  is the indexing map for the stratification of  $\mathbb{R}^n$  induced by  $\mathcal{A}$ .

There is a standard way to extend the above construction to a stratification on  $\mathbb{C}^n$  by the complexification  $\mathcal{A} \otimes \mathbb{C} = \{H_1 \otimes \mathbb{C}, \dots, H_k \otimes \mathbb{C}\}$  as is studied intensively in the theory of hyperplane arrangements. A good reference is the paper [BZ92] by Björner and Ziegler. As is sketched at the end of the above paper, the stratification of  $\mathbb{R}^n$  defined above can be extended to a stratification on  $\mathbb{R}^n \otimes \mathbb{R}^\ell$  as follows: Let  $\mathcal{A} = \{H_1, \dots, H_k\}$  and  $L = \{\ell_1, \dots, \ell_k\}$  be as above. Then the maps

$$\ell_1 \otimes \mathbb{R}^\ell, \dots, \ell_k \otimes \mathbb{R}^\ell : \mathbb{R}^n \otimes \mathbb{R}^\ell \longrightarrow \mathbb{R}^\ell$$

define a subspace arrangement  $\mathcal{A} \otimes \mathbb{R}^\ell = \{H_1 \otimes \mathbb{R}^\ell, \dots, H_k \otimes \mathbb{R}^\ell\}$  in  $\mathbb{R}^n \otimes \mathbb{R}^\ell$ .

Let  $S_\ell = \{0, \pm e_1, \dots, \pm e_\ell\}$  be the poset with partial ordering  $0 < \pm e_1 < \dots < \pm e_\ell$ . Define the  $\ell$ -th order sign function

$$\text{sign}_\ell : \mathbb{R}^\ell \longrightarrow S_\ell$$

by

$$\text{sign}_\ell(\mathbf{x}) = \begin{cases} \text{sign}(x_\ell)e_\ell, & x_\ell \neq 0 \\ \text{sign}(x_{\ell-1})e_{\ell-1}, & x_\ell = 0, x_{\ell-1} \neq 0 \\ \vdots \\ \text{sign}(x_1)e_1, & x_n = \dots = x_2 = 0, x_1 \neq 0 \\ 0 & \mathbf{x} = 0. \end{cases}$$

Define a stratification on  $\mathbb{R}^n \otimes \mathbb{R}^\ell$

$$\pi_{\mathcal{A} \otimes \mathbb{R}^\ell} : \mathbb{R}^n \otimes \mathbb{R}^\ell \longrightarrow \text{Map}(L, S_\ell)$$

by

$$\pi_{\mathcal{A} \otimes \mathbb{R}^\ell}(a \otimes x)(\ell_i) = \text{sign}_\ell(\ell_i(a)x).$$

This is a normal stratification on  $\mathbb{R}^n \otimes \mathbb{R}^\ell$ . □

**Example 2.10.** Consider the standard  $n$ -simplex

$$\Delta^n = \{(x_0, \dots, x_n) \in \mathbb{R}^n \mid x_0 + \dots + x_n = 1, x_i \geq 0\}.$$

Define

$$\pi_n : \Delta^n \longrightarrow 2^{[n]}$$

by

$$\pi_n(x_0, \dots, x_n) = \{i \in [n] \mid x_i \neq 0\},$$

where  $[n] = \{0, \dots, n\}$  and  $2^{[n]}$  is the power set of  $[n]$ . Under the standard poset structure on  $2^{[n]}$ ,  $\pi$  is a stratification with strata simplices in  $\Delta^n$ . This is a normal stratification.

Define another stratification  $\pi_n^{\max} : \Delta^n \rightarrow [n]$  by

$$\pi_n^{\max}(x_0, \dots, x_n) = \max \{i \mid x_i \neq 0\},$$

where  $[n]$  is equipped with the partial order  $0 < 1 < \dots < n$ . The resulting decomposition is

$$\Delta^n = (\Delta^n \setminus \Delta^{n-1}) \cup (\Delta^{n-1} \setminus \Delta^{n-2}) \cup \dots \cup (\Delta^1 \setminus \Delta^0) \cup \Delta^0.$$

This is also a normal stratification. □

**Example 2.11.** Let  $G$  be a compact Lie group acting smoothly on a smooth manifold  $M$ . M. Davis [Dav78] defined a stratification on  $M$  and on the quotient space  $M/G$  as follows. The indexing set  $I(G)$  is called the set of normal orbit types and is defined by

$$I(G) = \{(\varphi : H \rightarrow \mathrm{GL}(V)) \mid H < G \text{ a closed subgroup, } \varphi \text{ a representation, } V^H = \{0\}\} / \sim,$$

where  $(\varphi : H \rightarrow \mathrm{GL}(V)) \sim (\varphi' : H' \rightarrow \mathrm{GL}(V'))$  if and only if there exist an element  $g \in G$  and a linear isomorphism  $f : V \rightarrow V'$  such that  $H' = gHg^{-1}$  and the diagram

$$\begin{array}{ccc} H & \xrightarrow{\varphi} & \mathrm{GL}(V) \\ g(-)g^{-1} \downarrow & & \downarrow f_* \\ H' & \xrightarrow{\varphi'} & \mathrm{GL}(V') \end{array}$$

is commutative.

He defined a map

$$\pi_M : M \longrightarrow I(G)$$

by

$$\pi_M(x) = (G_x, S_x / (S_x)^{G_x}),$$

where  $G_x$  is the isotropy subgroup at  $x$  and  $S_x = T_x M / T_x(Gx)$ . There is a canonical partial order on  $I(G)$  and it is proved that  $\pi_M$  is a normal stratification (Theorem 1.6 in [Dav78]). It is also proved that the stratification descends to  $M/G$ .

In particular, for a torus manifold<sup>6</sup> or a small cover<sup>7</sup>  $M$  of dimension  $2n$  or  $n$ , the quotient  $M/T^n$  or  $M/\mathbb{Z}_2^n$  has a canonical normal stratification, respectively.

When  $M = \mathbb{C}P^n$  or  $\mathbb{R}P^n$ , the quotient is  $\Delta^n$  and the stratification by normal orbit types corresponds to the stratification  $\pi_n$  in Example 2.10. We will see the other stratification  $\pi_n^{\max}$  in Example 2.10 corresponds to the minimal cell decompositions of  $\mathbb{R}P^n$  and  $\mathbb{C}P^n$  in Example 4.26. □

**Example 2.12.** The stratification on the quotient  $M/G$  has been generalized to the notion of manifolds with faces or more generally manifolds with corners. See §6 of Davis' paper [Dav83], for example. These are also important examples of normal stratifications. □

It is easy to verify that the product of two stratifications is again a stratification.

<sup>6</sup>in the sense of Hattori-Masuda [HM03]

<sup>7</sup>in the sense of Davis-Januszkiewicz [DJ91]

**Lemma 2.13.** Let  $(X, \pi_X)$  and  $(Y, \pi_Y)$  be stratified spaces. The map

$$\pi_X \times \pi_Y : X \times Y \longrightarrow P(X) \times P(Y)$$

defines a stratification on  $X \times Y$ .

*Proof.* The product of open maps is again open. □

The following requirements for morphisms of stratified spaces should be reasonable.

**Definition 2.14.** Let  $(X, \pi_X)$  and  $(Y, \pi_Y)$  be stratified spaces.

- A *morphism of stratified spaces* is a pair  $\mathbf{f} = (f, \underline{f})$  of a continuous map  $f : X \rightarrow Y$  and a map of posets  $\underline{f} : P(X) \rightarrow P(Y)$  making the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \pi_X \downarrow & & \downarrow \pi_Y \\ P(X) & \xrightarrow{\underline{f}} & P(Y). \end{array}$$

- When  $X = Y$  and  $f$  is the identity,  $\mathbf{f} = (f, \underline{f})$  is called a *subdivision*. We also say that  $(X, \pi_X)$  is a subdivision of  $(Y, \pi_Y)$  or  $(Y, \pi_Y)$  is a *coarsening* of  $(X, \pi_X)$ .
- When  $f(e_\lambda) = e_{\underline{f}(\lambda)}$  for each  $\lambda$ , it is called a *strict morphism*.
- When  $\mathbf{f} = (f, \underline{f})$  is a strict morphism of stratified spaces and  $f$  is an embedding of topological spaces,  $\mathbf{f}$  is said to be an *embedding* of  $X$  into  $Y$ .

For a stratified space  $\pi : Y \rightarrow \Lambda$  and a continuous map  $f : X \rightarrow Y$ , the composition  $f^*(\pi) = \pi \circ f : X \rightarrow \Lambda$  may or may not be a stratification.

**Definition 2.15.** Let  $\mathbf{f} : (X, \pi_X) \rightarrow (Y, \pi_Y)$  be a morphism of stratified spaces. When  $\underline{f} : P(X) \rightarrow \pi_Y(f(X))$  is an isomorphism of posets

$$\begin{array}{ccccc} X & \xrightarrow{f} & f(X) & \hookrightarrow & Y \\ \pi_X \downarrow & & \downarrow \pi_Y & & \downarrow \pi_Y \\ P(X) & \xrightarrow{\cong} & \pi_Y(f(X)) & \hookrightarrow & P(Y), \end{array}$$

we say  $\pi_X$  is *induced* from  $\pi_Y$  via  $f$ . We sometimes denote it by  $f^*(\pi_Y)$ .

**Example 2.16.** Consider the double covering map

$$2 : S^1 \longrightarrow S^1.$$

The minimal cell decomposition  $\pi_{\min} : S^1 = e^0 \cup e^1$  on  $S^1$  in the range does not induce a stratification on  $S^1$  in the domain, since the inverse images of strata are not connected. But we have a strict morphism of stratified spaces

$$\begin{array}{ccc} S^1 & \xrightarrow{2} & S^1 \\ \downarrow & & \downarrow \\ \{0_+, 0_-, 1_+, 1_-\} & \longrightarrow & \{0, 1\} \end{array}$$

if  $S^1$  in the domain is equipped with the minimal  $\Sigma_2$ -equivariant cell decomposition  $S^1 = e_-^0 \cup e_+^0 \cup e_-^1 \cup e_+^1$ .

Consider the complex analogue of the double covering on  $S^1$ , i.e. the Hopf bundle

$$\eta : S^3 \longrightarrow S^2.$$

This is a principal fiber bundle with fiber  $S^1$ . The minimal cell decomposition on  $S^2$ ,  $S^2 = e^0 \cup e^2$ , is a stratification. This stratification induces a stratification on  $S^3$

$$S^3 = \eta^{-1}(e^0) \amalg \eta^{-1}(e^2).$$

Note that we have

$$\begin{aligned} \eta^{-1}(e^0) &\cong e^0 \times S^1 \\ \eta^{-1}(e^2) &\cong e^2 \times S^1 \end{aligned}$$

The face posets of these stratifications are isomorphic to the poset  $\{0 < 2\}$  and we have a commutative diagram

$$\begin{array}{ccc} S^3 & \xrightarrow{\eta} & S^2 \\ \downarrow & & \downarrow \\ \{0, 2\} & \cong & \{0, 2\}. \end{array}$$

Note that these two examples can be described as

$$\begin{aligned} S^1 &= e^0 \times S^0 \amalg e^1 \times S^0 \\ S^3 &= e^0 \times S^1 \amalg e^2 \times S^1, \end{aligned}$$

suggesting the existence of a common framework for handling them simultaneously. We propose the notion of cylindrical structures as such in §4.1.  $\square$

As a special case of embeddings of stratified spaces, we have the notion of stratified subspaces.

**Definition 2.17.** Let  $(X, \pi)$  be a stratified space and  $A$  be a subspace of  $X$ . If the restriction  $\pi|_A$  is a stratification,  $(A, \pi|_A)$  is called a *stratified subspace* of  $(X, \pi)$ .

When the inclusion  $i : A \hookrightarrow X$  is a strict morphism,  $A$  is called a *strict stratified subspace* of  $X$ .

**Remark 2.18.** We study stratified subspaces in detail in §6.1.

As is the case of cell complexes, the CW condition is useful.

**Definition 2.19.** A stratification  $\pi$  on  $X$  is said to be *CW* if it satisfies the following two conditions:

1. (Closure Finite) For each stratum  $e_\lambda$ ,  $\partial e_\lambda$  is covered by a finite number of strata.
2. (Weak Topology)  $X$  has the weak topology determined by the covering  $\{\overline{e_\lambda} \mid \lambda \in P(X)\}$ .

For example, weak topologies are useful when we glue quotient maps.

**Lemma 2.20.** *Let  $\pi_X : X \rightarrow \Lambda$  and  $\pi_Y : Y \rightarrow \Lambda$  be normal CW stratified spaces with  $P(X) = P(Y) = \Lambda$  and*

$$f : X \longrightarrow Y$$

*be a strict morphism of stratified spaces with  $\underline{f} = 1_\Lambda$ . Let us denote the strata in  $X$  and  $Y$  indexed by  $\lambda$  by  $e_\lambda^X$  and  $e_\lambda^Y$ , respectively. Suppose*

$$f_\lambda = f|_{\overline{e_\lambda^Y}} : \overline{e_\lambda^Y} \rightarrow \overline{e_\lambda^X}$$

*is a quotient map for all  $\lambda \in \Lambda$ . Then  $f : X \rightarrow Y$  is a quotient map.*

*Proof.* For a subset  $U \subset Y$ , suppose  $f^{-1}(U)$  is open in  $X$ . By the weak topology condition,  $f^{-1}(U) \cap \overline{e_\lambda^X}$  is open in  $\overline{e_\lambda^X}$  for each  $\lambda \in \Lambda$ . We have

$$f\left(f^{-1}(U) \cap \overline{e_\lambda^X}\right) = U \cap f\left(\overline{e_\lambda^X}\right) = U \cap \overline{e_\lambda^Y},$$

since  $f$  is a strict morphism and both  $X$  and  $Y$  are normal. Thus

$$f_\lambda^{-1}\left(U \cap \overline{e_\lambda^Y}\right) = f^{-1}\left(f\left(f^{-1}(U) \cap \overline{e_\lambda^X}\right)\right) = f^{-1}(U) \cap \overline{e_\lambda^X}$$

is open in  $\overline{e_\lambda^X}$  for each  $\lambda$ . By assumption,  $f_\lambda$  is a quotient map and  $Y$  is CW. And  $U$  is open in  $Y$ .  $\square$

It is straightforward to verify the following by using the fact that any topological space has the weak topology with respect to a locally finite closed covering.

**Proposition 2.21.** *Any locally finite stratified space is CW.*

**Corollary 2.22.** *Let  $(X, \pi)$  be a CW stratified space and  $(X, \pi')$  be a subdivision. If each stratum  $e_\lambda$  in  $(X, \pi)$  is subdivided into a finite number of strata in  $(X, \pi')$ , then  $(X, \pi')$  is CW.*

*Proof.* For each cell  $e_\lambda$  in  $(X, \pi)$ ,  $\overline{e_\lambda}$  has the weak topology with respect to the covering

$$\overline{e_\lambda} = \bigcup_{\lambda' \in P(X, \pi'), e_{\lambda'} \subset e_\lambda} \overline{e_{\lambda'}}$$

because of the finiteness assumption. Thus  $X$  has the weak topology with respect to the covering

$$X = \bigcup_{\lambda' \in P(X, \pi')} \overline{e_{\lambda'}} = \bigcup_{\lambda \in P(X, \pi)} \left( \bigcup_{\lambda' \in P(X, \pi'), e_{\lambda'} \subset e_\lambda} \overline{e_{\lambda'}} \right).$$

The closure finiteness condition also follows from the finiteness of the subdivision of each stratum.  $\square$

As is the case of CW complexes, metrizable implies local finiteness.

**Lemma 2.23.** *Any metrizable CW stratified space is locally finite.*

*Proof.* This fact is well known for CW complexes. The same argument can be used to prove this generalized statement. We give a proof for the convenience of the reader.

If  $X$  is not locally finite, there exists a point  $x \in X$  such that, for any open neighborhood  $U$  of  $x$ ,  $U$  intersects with infinitely many strata. For each  $n$ , let  $U_n$  be the  $\frac{1}{n}$ -neighborhood of  $x$  and choose a stratum  $e_n$  with  $U_n \cap e_n \neq \emptyset$  and  $x \notin e_n$ . Choose  $x_n \in U_n \cap e_n$ . Then the set  $A = \{x_n\}_{n=1,2,\dots}$  is closed by the CW conditions. This contradicts to the fact that  $x \notin A$  and  $x \in \overline{A}$ . Thus  $X$  is locally finite.  $\square$

## 2.2 Cells

We would like to define a cellular stratification on a topological space  $X$  as a stratification on  $X$  whose strata are “cells”. As we have seen in Example 2.9, we would like to regard chambers and faces of a real hyperplane arrangement as “cells”, suggesting the need of non-closed cells.

**Definition 2.24.** A *globular  $n$ -cell* is a subset  $D$  of  $D^n$  containing  $\text{Int}(D^n)$ . We call  $D \cap \partial D^n$  the *boundary* of  $D$  and denote it by  $\partial D$ . The number  $n$  is called the *globular dimension* of  $D$ .

**Remark 2.25.** We introduce another dimension, called the stellar dimension, for a more general class of subsets of  $D^n$  in §2.4.

We use the following definition of cell structures.

**Definition 2.26.** Let  $X$  be a topological space. For a non-negative integer  $n$ , an  *$n$ -cell structure* on a subspace  $e \subset X$  is a pair  $(D, \varphi)$  of a globular  $n$ -cell  $D$  and a continuous map

$$\varphi : D \longrightarrow X$$

satisfying the following conditions:

1.  $\varphi(D) = \bar{e}$  and  $\varphi : D \rightarrow \bar{e}$  is a quotient map.
2. The restriction  $\varphi|_{\text{Int}(D^n)} : \text{Int}(D^n) \rightarrow e$  is a homeomorphism.

For simplicity, we denote an  $n$ -cell structure  $(D, \varphi)$  on  $e$  by  $e$  when there is no risk of confusion. The map  $\varphi$  is called the *cell structure map* or the *characteristic map* of  $e$  and  $D$  is called the *domain* of  $e$ . The number  $n$  is called the (globular) *dimension* of  $e$ .

**Example 2.27.** The open  $n$ -disk  $\text{Int}(D^n)$  is a globular  $n$ -cell. The standard homeomorphism

$$\text{Int}(D^n) \xrightarrow{\cong} \mathbb{R}^n$$

defines an  $n$ -cell structure on  $\mathbb{R}^n$ . The domain is  $\text{Int}(D^n)$ . □

**Example 2.28.** Let  $X = \text{Int}(D^2) \cup \{(1, 0)\}$ . The identity map defines a 2-cell structure on  $X$ .

There is another choice. Let  $D = \text{Int}(D^2) \cup S_+^1$ . The deformation retraction of  $S_+^1$  onto  $(1, 0)$  can be extended to a continuous map

$$\varphi : D \longrightarrow X$$

whose restriction to  $\text{Int}(D^2)$  is a homeomorphism. For example,  $\varphi$  is given in polar coordinates by

$$\varphi(re^{i\theta}) = \begin{cases} re^{i(1-r)\theta}, & |\theta| \leq \frac{\pi}{2} \\ re^{i\{\theta - (\pi - \theta)r\}}, & \frac{\pi}{2} \leq \theta \leq \pi \\ re^{i\{\theta + (\pi + \theta)r\}}, & -\pi \leq \theta \leq -\frac{\pi}{2}. \end{cases}$$

Note that  $\varphi$  is not a quotient map. For example, the image of  $\{(x, y) \in D \mid x > 0\}$  under  $\varphi$  is open under the quotient topology, but it is not open under the relative topology on  $X$ . Thus this is not a 2-cell structure on  $X$ . □

In other words, we need to put the quotient topology on  $X$  in order for the map  $\varphi$  in the above example to be a cell structure. Fortunately, the quotient topology is a popular choice in many practical examples.



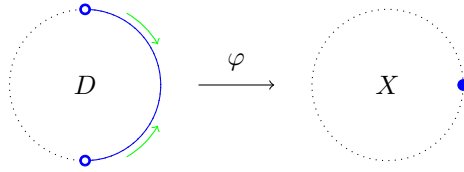


Figure 2: An exotic cell structure on  $\text{Int}(D^2) \cup \{(1, 0)\}$

**Example 2.29.** For any simplicial set  $X$ , the geometric realization  $|X|$  is known to be a CW complex, whose cells are in one-to-one correspondence with nondegenerate simplices in  $X$ .

Consider the simplicial set  $X = s(\Delta^2)/s(\Delta^1)$ , where  $\Delta^1$  and  $\Delta^2$  are regarded as ordered simplicial complexes and  $s(-)$  is the functor in Example B.12 which transforms ordered simplicial complexes to simplicial sets. The geometric realization  $|X|$  is a cell complex consisting of two 0-cells  $[0] = [1]$ ,  $[2]$ , two 1-cells  $[0, 2]$ ,  $[1, 2]$ , and a 2-cell  $[0, 1, 2]$ . The characteristic map for the 2-cell is given by the composition

$$\psi : D^2 \cong \Delta^2 \times \{[0, 1, 2]\} \hookrightarrow \prod_{i=0}^{\infty} \Delta^n \times X_n \longrightarrow |X|.$$

Thus it is given by collapsing the blue arc in Figure 3. By definition,  $|X|$  is equipped with the

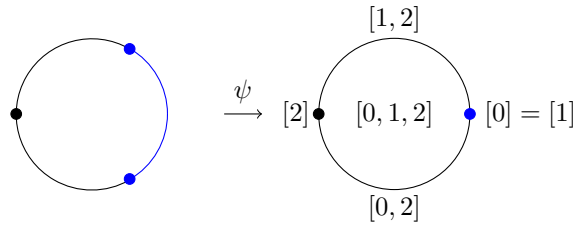


Figure 3: A 2-cell structure on  $s(\Delta^2)/s(\Delta^1)$ .

quotient topology and  $\psi$  defines a 2-cell structure. □

Cells satisfying one (or both) of the following conditions appear frequently.

**Definition 2.30.** Let  $X$  be a topological space and  $e \subset X$  a subspace. An  $n$ -cell structure  $(D, \varphi)$  on  $e$ , or simply an  $n$ -cell  $e$ , is said to be

- *closed* if  $D = D^n$ ,
- *regular* if  $\varphi : D \rightarrow \bar{e}$  is a homeomorphism.

**Example 2.31.** Given a cell complex  $X$  and its  $n$ -cell  $e$ , the characteristic map

$$\varphi : D^n \longrightarrow X$$

defines a closed  $n$ -cell structure on  $e$ . When  $X$  is a regular cell complex, it is a regular  $n$ -cell structure on  $e$ . □

**Example 2.32.** Let  $\mathcal{A}$  be an arrangement of a finite number of hyperplanes in  $\mathbb{R}^n$ . Bounded strata in the stratification  $\pi_{\mathcal{A}}$  are convex polytopes and they are closed and regular cells.

We may also define non-closed but regular cell structures on unbounded strata as follows. Suppose  $\mathcal{A}$  is essential, namely normal vectors to the hyperplanes span  $\mathbb{R}^n$ . Then we may choose a closed ball  $B$  with center at the origin in  $\mathbb{R}^n$  which contains all bounded strata. Hyperplanes

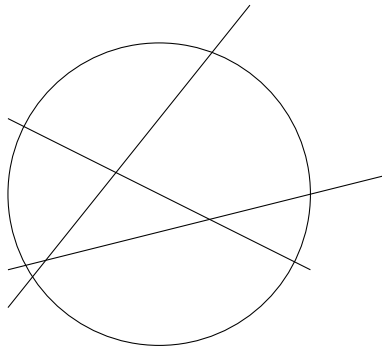


Figure 4: A bounding sphere for bounded strata.

also cut the boundary sphere and define a stratification  $\pi_{\mathcal{A},B}$  on  $B$  whose strata are all closed cells. The inclusion  $\text{Int}(B) \hookrightarrow \mathbb{R}^n$  is a morphism of stratified spaces under which face posets can be identified

$$\begin{array}{ccc} \text{Int}(B) & \hookrightarrow & \mathbb{R}^n \\ \pi_{\mathcal{A},B} \downarrow & & \downarrow \pi_{\mathcal{A}} \\ P(\pi_{\mathcal{A},B}|_{\text{Int}(B)}) & \equiv & P(\pi_{\mathcal{A}}). \end{array}$$

For each unbounded stratum  $e$  in  $\pi_{\mathcal{A}}$ , the intersection  $e \cap \text{Int}(B)$  is a cell in  $B$ . Let  $\varphi : D^k \rightarrow \overline{e \cap B}$  be a characteristic map for  $e \cap \text{Int}(B)$  and define  $D = \varphi^{-1}(\overline{e \cap \text{Int}(B)})$ . We can compress the outside of  $B$  into  $e \cap \text{Int}(B)$  via a homeomorphism  $\psi : \bar{e} \rightarrow \bar{e} \cap \text{Int}(B)$ . The composition

$$D \xrightarrow{\varphi|_D} \bar{e} \cap \text{Int}(B) \xrightarrow{\psi^{-1}} \bar{e}$$

defines a regular cell structure on  $e$ . □

Non-closed cells might have a bad topology.

**Example 2.33.** Let

$$p : D^2 \setminus \{(0,1)\} \longrightarrow \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$$

be the homeomorphism given by extending the stereographic projection  $S^1 \setminus \{(0,1)\} \rightarrow \mathbb{R}$ . Let

$$X = \{(x,y) \in \mathbb{R}^2 \mid y > 0\} \cup \{(x,0) \mid x \in \mathbb{Q}\}$$

and define  $D = p^{-1}(X)$ . Then the restriction

$$p|_D : D \longrightarrow X$$

defines a 2-cell structure on  $X$ .  $X$  and  $D$  are not locally compact. This example suggests

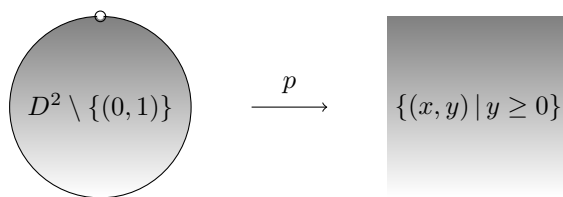


Figure 5: A 2-cell structure for the closed upper half-plane.

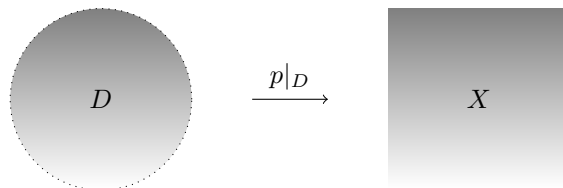


Figure 6: A 2-cell structure for  $X$ .

that taking a product of cell structures might not be easy because of our requirement for a characteristic map to be a quotient map.

For example, let  $Y$  be the quotient of  $X$  under the relation  $(x, 0) \sim (x', 0)$  if and only if  $x - x' \in \mathbb{Z}$ . The composition of  $p|_D$  with the canonical projection

$$\varphi_Y : D \xrightarrow{p|_D} X \longrightarrow Y$$

defines a 2-cell structure on  $Y$ . But the product with  $p|_D$

$$p|_D \times \varphi_Y : D \times D \longrightarrow X \times Y$$

is not a quotient map, since the product  $\mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{Q} \times (\mathbb{Q}/\mathbb{Z})$  of the identity map and the quotient map is not a quotient map.

We need to impose certain conditions to take products of cell structures freely. Our solution is to require cellular structures on the boundaries of the domains of cells. See §4.1 for more details.  $\square$

### 2.3 Cellular Stratifications

So far we have defined the notions of stratified spaces and cell structures. Now we are ready to define cellular stratified spaces by combining these two structures.

**Definition 2.34.** Let  $X$  be a Hausdorff space. A *cellular stratification* on  $X$  is a pair  $(\pi, \Phi)$  of a stratification  $\pi : X \rightarrow P(X)$  on  $X$  and a collection of cell structures  $\Phi = \{\varphi_\lambda : D_\lambda \rightarrow \bar{e}_\lambda\}_{\lambda \in P(X)}$  satisfying the condition that, for each  $n$ -cell  $e_\lambda$ ,  $\partial e_\lambda$  is covered by cells of dimension less than or equal to  $n - 1$ .

A *cellular stratified space* is a triple  $(X, \pi, \Phi)$  where  $(\pi, \Phi)$  is a cellular stratification on  $X$ . As usual, we abbreviate it by  $(X, \pi)$  or  $X$ , if there is no danger of confusion.

**Remark 2.35.** The term “cellular stratified space” has been already used in the study of singularities. See, for example, Schürmann’s book [Sch03]. We found, however, that his definition is too restrictive for our purposes.

**Example 2.36.** Consider the “topologist’s sine curve”

$$S = \{(x, \sin \frac{1}{x}) \mid 0 < x \leq 1\}.$$

Its closure in  $\mathbb{R}^2$  is given by

$$\bar{S} = S \cup \{(0, t) \mid -1 \leq t \leq 1\}.$$

Since  $S$  is homeomorphic to the half interval  $(0, 1]$  via the function  $\sin \frac{1}{x}$ , the decomposition

$$\bar{S} = \{(1, \sin 1)\} \cup \{(0, 1)\} \cup \{(0, -1)\} \cup \{(x, \sin \frac{1}{x}) \mid 0 < x < 1\} \cup \{(0, t) \mid -1 < t < 1\}$$

is a stratification of  $\bar{S}$  consisting of five strata. Although the stratum  $\{(x, \sin \frac{1}{x}) \mid 0 < x < 1\}$  is homeomorphic to  $\text{Int}(D^1)$ , there is no 1-cell structure on this stratum, since there is no way to extend a homeomorphism  $(0, 1] \cong S$  to a continuous map  $[0, 1] \rightarrow \bar{S}$ .

Furthermore this stratification does not satisfy the dimension condition in Definition 2.34, since  $\partial S = \{(0, t) \mid -1 \leq t \leq 1\}$ .  $\square$

**Remark 2.37.** The above example is borrowed from Pflaum’s book [Pfl01]. He describes an even more pathological example. See 1.1.12 on page 18 of his book.

Here is another non-example.

**Example 2.38.** Consider the 2-cell structure on  $X = \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \cup \{(x, 0) \mid x \in \mathbb{Q}\}$  defined in Example 2.33. Let  $e^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$  and  $e_x^0 = \{(x, 0)\}$  for  $x \in \mathbb{Q}$ . We have a decomposition

$$X = \bigcup_{x \in \mathbb{Q}} e_x^0 \cup e^2$$

and, for each  $x \in \mathbb{Q}$ , the identification

$$\psi_x : D^0 \cong \{(x, 0)\} \hookrightarrow X$$

defines a 0-cell structure on  $\{(x, 0)\}$ . But this is not a cellular stratified space, since  $e_x^0$  is not locally closed.

On the other hand, let  $Y = \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \cup \mathbb{Z} \times \{0\}$  and consider the decomposition

$$Y = \bigcup_{n \in \mathbb{Z}} e_n^0 \cup e^2.$$

Each  $e_n^0$  is locally closed in  $Y$  and this is a cellular stratification on  $Y$ . Note, however, this is not a CW stratification, since we need infinitely many 0-cells to cover the boundary of the 2-cell  $\partial e^2$ .  $\square$

**Definition 2.39.** We say a cellular stratification is *CW* if its underlying stratification is CW (Definition 2.19).

**Lemma 2.40.** A *CW* cellular stratification  $(\pi, \Phi)$  on a space  $X$  defines and is defined by the following structure

- a filtration  $X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n \subset \cdots$  on  $X$ ,
- an  $n$ -cell structure on each connected component of  $X_n \setminus X_{n-1}$

satisfying the following conditions:

1.  $X = \bigcup_{n=0}^{\infty} X_n$ .
2. For each stratum  $e$  in  $X_n$ ,  $\partial e$  is covered by a finite number of strata in  $X_{n-1}$ .
3.  $X$  has the weak topology determined by the covering  $\{\bar{e} \mid e \in P(X)\}$ .

*Proof.* Suppose  $(\pi, \Phi)$  is a CW cellular stratification. Define

$$X_n = \bigcup_{e \in P(X), \dim e \leq n} e$$

then we obtain a filtration on  $X$  with  $X = \bigcup_{n=0}^{\infty} X_n$ . By definition, the boundary  $\partial e$  of each  $n$ -cell  $e$  is covered by a finite number of cells of dimension less than or equal to  $n-1$ . Also by definition,  $X$  has the weak topology by the covering  $\{\bar{e} \mid e \in P(X)\}$ .

It remains to show that the  $n$ -cells are connected components of the difference  $X_n \setminus X_{n-1}$ , i.e. each  $n$ -cell  $e$  is open and closed in  $X_n \setminus X_{n-1}$ . By the local closedness,  $e$  is open in  $\bar{e}$ . Since  $\partial e \subset X_{n-1}$ ,  $e$  is open in  $X_n \setminus X_{n-1}$ . For any  $n$ -cell  $e'$ , the intersection  $e \cap \bar{e}'$  in  $X_n \setminus X_{n-1}$  is  $e \cap e'$  and is  $\emptyset$  or  $e = e'$ . And thus it is closed in  $\bar{e}'$ .

It is left to the reader to check the converse.  $\square$

**Example 2.41.** The stratification  $\pi_{\mathcal{A}}$  of  $\mathbb{R}^n$  defined by a hyperplane arrangement  $\mathcal{A}$  (Example 2.9) is a CW cellular stratification.

The Björner-Ziegler stratification  $\pi_{\mathcal{A} \otimes \mathbb{R}^\ell}$  on  $\mathbb{R}^n \otimes \mathbb{R}^\ell$  is also a CW cellular stratification.  $\square$

In Definition 2.14, we defined morphisms of stratified spaces as “stratification-preserving maps”. Note that, in our definition of cellular stratified spaces, we include characteristic maps as defining data. When we consider maps between cellular stratified spaces, we require them to be compatible with characteristic maps.

**Definition 2.42.** Let  $(X, \pi_X, \Phi_X)$  and  $(Y, \pi_Y, \Phi_Y)$  be cellular stratified spaces. A *morphism of cellular stratified spaces* from  $(X, \pi_X, \Phi_X)$  to  $(Y, \pi_Y, \Phi_Y)$  consists of

- a morphism  $\mathbf{f} : (X, \pi_X) \rightarrow (Y, \pi_Y)$  of stratified spaces, and
- a family of maps

$$f_\lambda : D_\lambda \longrightarrow D_{\underline{f}(\lambda)}$$

indexed by cells  $\varphi_\lambda : D_\lambda \rightarrow \bar{e}_\lambda$  in  $X$  making the diagrams

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \varphi_\lambda \uparrow & & \uparrow \psi_{\underline{f}(\lambda)} \\ D_\lambda & \xrightarrow{f_\lambda} & D_{\underline{f}(\lambda)} \end{array}$$

commutative, where  $\psi_{\underline{f}(\lambda)} : D_{\underline{f}(\lambda)} \rightarrow \bar{e}_{\underline{f}(\lambda)}$  is the characteristic map for  $e_{\underline{f}(\lambda)}$ .

The category of cellular stratified spaces is denoted by **CSSpaces**.

**Remark 2.43.** When  $\mathbf{f} : (X, \pi_X, \Phi_X) \rightarrow (Y, \pi_Y, \Phi_Y)$  is a morphism of cellular stratified spaces, the compatibility of  $f_\lambda$  with characteristic maps implies  $f_\lambda(\text{Int}(D_\lambda)) \subset \text{Int}(D_{\underline{f}(\lambda)})$ .

**Remark 2.44.** In algebraic topology, the requirement for maps between cell complexes is much weaker. A map  $f : X \rightarrow Y$  between cell complexes is said to be *cellular*, if  $f(X_n) \subset Y_n$  for each  $n$ . The author thinks, however, this terminology is misleading.

**Definition 2.45.** A morphism  $(\mathbf{f}, \{f_\lambda\}) : (X, \pi_X, \Phi_X) \rightarrow (Y, \pi_Y, \Phi_Y)$  of cellular stratified spaces is said to be *strict* if  $\mathbf{f} : (X, \pi_X) \rightarrow (Y, \pi_Y)$  is a strict morphisms of stratified spaces and  $f_\lambda(0) = 0$  for each  $\lambda \in P(X, \pi_X)$ .

Once we have morphisms between cellular stratified spaces, we have a notion of equivariant stratification.

**Definition 2.46.** Let  $G$  be a group. A cellular stratified space  $X$  equipped with a monoid morphism<sup>8</sup>  $G \rightarrow \mathbf{CSSpaces}(X, X)$  is called a  $G$ -cellular stratified space.

In order to study configuration spaces, products and subdivisions are important. However, the product of two cell structures may not be a cell structure as we have seen in Example 2.33. Subdivisions of cell structures are not easy to handle, either. We discuss products and subdivisions of cellular stratified spaces in §6.2 and §6.3, respectively.

## 2.4 Stellar Stratified Spaces

In Definition 2.26, we required the domain  $D$  of an  $n$ -cell to contain  $\text{Int}(D^n)$ . As we will see in the proof of Theorem 5.16, this condition is requiring too much. Furthermore, the definition of the globular dimension of a cell is not appropriate when the cell is not closed. In this section, we introduce stellar cells and study stratified spaces whose strata are stellar cells. Stellar structures also play an essential role in our description of the classifying space of the face category of cellular stratified spaces.

Let us first define “star-shaped cells”.

**Definition 2.47.** A subset  $S$  of  $D^N$  is said to be an *aster* if  $\{0\} * \{x\} \subset S$  for any  $x \in S$ , where  $*$  is the join operation defined by connecting points by line segments<sup>9</sup>. The subset  $S \cap \partial D^N$  is called the *boundary* of  $S$  and is denoted by  $\partial S$ . The complement  $S \setminus \partial S$  of the boundary is called the *interior* of  $S$  and is denoted by  $\text{Int}(S)$ .

We say  $S$  is *thin* if  $S = \{0\} * \partial S$ .

We require the existence of a cellular stratification on the boundary in order to define the dimension.

**Definition 2.48.** A *stellar cell* is an aster  $S$  in  $D^N$  for some  $N$  such that there exists a cellular stratification on  $\partial D^N$  containing  $\partial S$  as a cellular stratified subspace.

When the (globular) dimension of  $\partial S$  is  $n - 1$ , we define the *stellar dimension* of  $S$  to be  $n$  and call  $S$  a *stellar  $n$ -cell*.

An  $n$ -cell in the sense of §2.2 is stellar if its boundary has a structure of cellular stratified space. However, the dimension as a stellar cell might be smaller than  $n$ .

**Example 2.49.** Consider the globular  $n$ -cell  $\text{Int}(D^n)$  in Example 2.27. It is a stellar cell with empty boundary. Thus its stellar dimension is 0.

By adding three points to the boundary, for example, we obtain a globular  $n$ -cell  $D$  whose stellar dimension is 1.

These two stellar cells are not thin. The first example contains  $\{0\}$  as a thin stellar cell. The second example contains a graph of the shape of  $Y$  as a thin stellar cell. See Figure 8. □

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<sup>8</sup>Or a functor  $G \rightarrow \mathbf{CSSpaces}$ .

<sup>9</sup>Definition B.20.

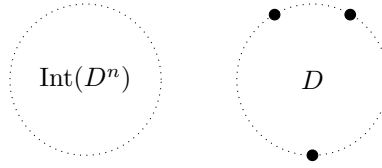


Figure 7: An open disk and an open disk with three points added.

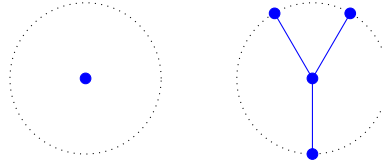


Figure 8: Thin stellar cells.

**Definition 2.50.** An  $n$ -stellar structure on a subset  $e$  of a topological space  $X$  is a pair  $(S, \varphi)$  of a stellar  $n$ -cell  $S$  and a quotient map

$$\varphi : S \longrightarrow X$$

satisfying the following conditions:

1.  $\varphi(S) = \bar{e}$ .
2. The restriction of  $\varphi$  to  $\text{Int}(S)$  is a homeomorphism onto  $e$ .

By replacing cell structures by stellar structures in the definition of cellular stratifications, we obtain the notion of stellar stratifications.

**Definition 2.51.** A *stellar stratification* on a topological space  $X$  consists of a stratification  $(X, \pi)$ , and a stellar structure on each  $e_\lambda = \pi^{-1}(\lambda)$  for  $\lambda \in P(X)$  satisfying the condition that for each stellar  $n$ -cell  $e_\lambda$ ,  $\partial e_\lambda$  is covered by stellar cells of stellar dimension less than or equal to  $n - 1$ .

A space equipped with a stellar stratification is called a *stellar stratified space*.

The following is a typical example.

**Example 2.52.** Consider a finite graph  $X$  regarded as a 1-dimensional cell complex. Suppose  $X$  is regular as a cell complex. Then we may define the barycentric subdivision  $\text{Sd}(X)$  of  $X$ . This cell complex can be expressed as a union of open stars of vertices of  $X$  and the barycenters of 1-cells in  $X$ . And we have a structure of stellar stratified space on  $\text{Sd}(X)$ .  $\square$

**Remark 2.53.** We will extend the definition of the barycentric subdivision to cellular and stellar stratified spaces in §5.2 and investigate more precise relations between stellar stratifications and their barycentric subdivisions.

The definition of morphisms of stellar stratified spaces should be obvious.

**Definition 2.54.** The category of stellar stratified spaces is denoted by **SSSpaces**.

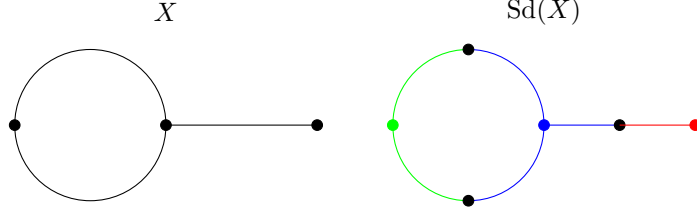


Figure 9: A stellar stratification on  $\text{Sd}(X)$ .

If a stellar stratified space  $X$  is CW, we may describe  $X$  as a quotient space of the collection of domains of cell structures.

**Lemma 2.55.** *For a CW stellar stratified space  $X$  with stellar structure  $\{\varphi_\lambda : D_\lambda \rightarrow \overline{e_\lambda}\}_{\lambda \in \Lambda}$ , define*

$$D(X) = \coprod_{\lambda \in \Lambda} D_\lambda$$

and let  $\Phi : D(X) \rightarrow X$  be the map defined by the stellar structure maps. Then  $\Phi$  is a quotient map. More explicitly, we have a homeomorphism

$$X \cong D(X) / \sim_\Phi,$$

where the relation  $\sim_\Phi$  is defined by  $x \sim_\Phi y$  if and only if  $\varphi_\mu(x) = \varphi_\lambda(y)$  for  $x \in D_\mu$  and  $y \in D_\lambda$ .

This fact is proved for CW cellular stratified spaces as Lemma 2.28 in [FMT15]. The proof can be applied to stellar stratified spaces without a change and is omitted.

### 3 Totally Normal Cellular Stratified Spaces

In order to study their homotopy types, we would like to impose appropriate “niceness conditions” on cellular and stellar stratified spaces.

#### 3.1 Regularity and Normality

Regularity and normality are frequently used conditions on CW complexes. We have already defined normality for stratified spaces (Definition 2.6) and regularity for cells (Definition 2.30).

**Definition 3.1.** Let  $X$  be a cellular or stellar stratified space. We say  $X$  is *normal*, if it is normal as a stratified space. We say  $X$  is *regular* if all cells in  $X$  are regular.

**Example 3.2.** The cellular stratification  $\pi_{\mathcal{A} \otimes \mathbb{R}^\ell}$  on  $\mathbb{R}^n \otimes \mathbb{R}^\ell$  in Example 2.9 defined by a real arrangement  $\mathcal{A}$  in  $\mathbb{R}^n$  is regular and normal.  $\square$

**Example 3.3.** Consider the following cellular stratified space obtained by gluing  $\text{Int}(D^2)$  to the boundary of a 2-simplex at the middle point of an edge. The domain of the characteristic map of the 2-cell is  $\text{Int}(D^2) \cup \{(1, 0)\}$ , whose boundary is mapped into the 1-skeleton. Thus this is a regular cellular stratified space. However, this is not normal.

If we regard the globular 2-cell  $\text{Int}(D^2) \cup \{(1, 0)\}$  as a stellar cell, its stellar dimension is 1 and this stellar structure does not satisfy the dimensional requirement. Thus this is not a stellar stratified space.  $\square$



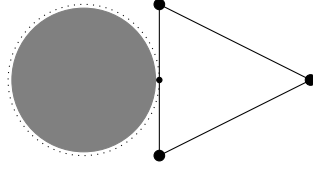


Figure 10: A non-normal regular cellular stratified space.

In the case of CW complexes, regularity always implies normality. See Theorem 2.1 in Chapter III of the book [LW69] by Lundell and Weingram. The above example suggests that the failure of this fact for cellular stratified spaces is partly due to the “wrong definition” of dimensions of globular cells. The right notion of dimension is the stellar one.

Even for stellar stratified spaces, however, regularity does not necessarily imply normality.

**Example 3.4.** By adding an arc to the previous example, we obtain a stratified space as follows. The globular 2-cell  $\text{Int}(D^2) \cup \{(1, 0)\} \cup \{(x, y) \in S^1 \mid x \leq 0\}$  is a stellar 2-cell. And this is a

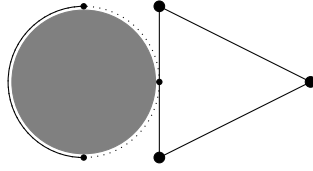


Figure 11: A non-normal regular stellar stratified space.

regular stellar stratified space. But this is not normal. □

**Remark 3.5.** It seems that, if  $X$  is a stellar stratified space in which the boundary of each stellar  $n$ -cell is “pure of dimension  $n - 1$ ” and if  $X$  is regular, then  $X$  is normal.

### 3.2 Total Normality

In the case of cellular or stellar stratified spaces, relations among cells are not as rigid as those in cell complexes.

**Definition 3.6.** Let  $X$  be a normal cellular stratified space.  $X$  is called *totally normal* if, for each  $n$ -cell  $e_\lambda$ ,

1. there exists a structure of regular cell complex on  $S^{n-1}$  containing  $\partial D_\lambda$  as a strict stratified subspace of  $S^{n-1}$  and
2. for any cell  $e$  in  $\partial D_\lambda$ , there exists a cell  $e_\mu$  in  $\partial e_\lambda$  such that  $e_\mu$  and  $e$  share the same domain and the characteristic map  $\varphi_\mu$  of  $e_\mu$  factors through  $D_\lambda$  via the characteristic map of  $e$ :

$$\begin{array}{ccccccc}
 \bar{e} & \hookrightarrow & \partial D_\lambda & \hookrightarrow & D_\lambda & \xrightarrow{\varphi_\lambda} & \bar{e}_\lambda & \hookrightarrow & X \\
 \uparrow & & & & & & & & \nearrow \\
 D & \xlongequal{\quad} & D_\mu & \xrightarrow{\varphi_\mu} & \bar{e}_\mu & & & & 
 \end{array}$$

The total normality for stellar stratified spaces is defined analogously. But we need to use cellular subdivisions.

**Definition 3.7.** Let  $X$  be a normal stellar stratified space. We say  $X$  is *totally normal* if, for each stellar  $n$ -cell  $e_\lambda$  with domain  $D_\lambda \subset D^N$ ,

1. there exist a structure of regular cell complex on  $S^{N-1}$  and a structure of stellar stratified space on  $\partial D_\lambda$  such that each stellar cell in  $\partial D_\lambda$  is a strict stratified subspace of  $S^{N-1}$ , and
2. for any stellar cell  $e$  in  $\partial D_\lambda$ , there exists a stellar cell  $e_\mu$  in  $\partial e_\lambda$  such that  $e_\mu$  and  $e$  share the same domain and the characteristic map of  $e_\mu$  factors through  $D_\lambda$  via the characteristic map of  $e$ .

**Lemma 3.8.** *Any totally normal cellular stratified space has a structure of a totally normal stellar stratified space.*

The characteristic maps of totally normal stellar stratified spaces preserve cells.

**Lemma 3.9.** *For a cell  $e_\lambda$  in a totally normal stellar stratified space  $X$ , let  $\varphi_\lambda : D_\lambda \rightarrow \overline{e_\lambda} \subset X$  be the characteristic map. Then there exists a structure of stellar stratified space on  $D_\lambda$  under which  $\varphi_\lambda$  is a strict morphism of stellar stratified spaces.*

*Proof.* Let  $e_\lambda$  be a cell in a totally normal stellar stratified space  $X$  and  $\varphi_\lambda : D_\lambda \rightarrow \overline{e_\lambda}$  the characteristic map. Let

$$\partial D_\lambda = \bigcup_{\nu} e'_\nu$$

be the stellar stratification in the definition of total normality. We have

$$\partial e_\lambda = \varphi_\lambda(\partial D_\lambda) = \bigcup_{\nu} \varphi_\lambda(e'_\nu).$$

By the definition of total normality, for each  $\nu$ , there exists a cell  $e_{\mu'}$  in  $\partial e_\lambda$  whose characteristic map makes the following diagram commutative

$$\begin{array}{ccccc} \overline{e'_\nu} & \hookrightarrow & \partial D_\lambda & \xrightarrow{\varphi_\lambda} & \partial e_\lambda \\ \uparrow \psi_\nu & & & & \uparrow \\ D_\nu & \xlongequal{\quad} & D_{\mu'} & \xrightarrow{\varphi_{\mu'}} & \overline{e_{\mu'}}, \end{array}$$

where  $\psi_\nu$  is the characteristic map for  $e'_\nu$ . This implies that each  $\varphi_\lambda(e'_\nu)$  is a cell in  $X$  and thus  $\varphi_\lambda$  is a strict morphism of stellar stratified spaces.  $\square$

**Corollary 3.10.** *Let  $(\pi, \Phi)$  be a stellar stratification on  $X$  satisfying the first condition of total normality.*

*Then it is totally normal if and only if the following condition is satisfied: For each pair  $e_\mu < e_\lambda$ , let  $F(X)(e_\mu, e_\lambda)$  be the set of all maps*

$$b : D_\mu \longrightarrow D_\lambda$$

making the diagram

$$\begin{array}{ccc} D_\lambda & \xrightarrow{\varphi_\lambda} & \bar{e}_\lambda \\ \uparrow b & & \uparrow \\ D_\mu & \xrightarrow{\varphi_\mu} & \bar{e}_\mu \end{array}$$

commutative, where  $\varphi_\lambda$  and  $\varphi_\mu$  are the characteristic maps of  $e_\lambda$  and  $e_\mu$ , respectively. Then

$$\partial D_\lambda = \bigcup_{e_\mu < e_\lambda} \bigcup_{b \in F(X)(e_\mu, e_\lambda)} b(\text{Int}(D_\mu)). \quad (1)$$

*Proof.* Suppose  $(\pi, \Phi)$  is totally normal. For a pair of cells  $e_\mu < e_\lambda$ , by Lemma 3.9, there exists a stellar cell  $e$  in  $D_\lambda$  such that  $\varphi_\lambda(e) = e_\mu$ . By the assumption of total normality, the characteristic map  $\psi : D \rightarrow \bar{e}$  of  $e$  makes the following diagram commutative

$$\begin{array}{ccccc} \bar{e} & \hookrightarrow & \partial D_\lambda & \xrightarrow{\varphi_\lambda} & \partial e_\lambda \\ \uparrow \psi & & & & \uparrow \\ D & \xlongequal{\quad} & D_\mu & \xrightarrow{\varphi_\mu} & \bar{e}_\mu \end{array}$$

The collection of all such characteristic maps  $\psi$  is  $F(X)(e_\mu, e_\lambda)$  and we have

$$\partial D_\lambda = \bigcup_{e_\mu < e_\lambda} \bigcup_{b \in F(X)(e_\mu, e_\lambda)} b(\text{Int}(D_\mu)).$$

Conversely, the assumption (1) implies that, for any stellar cell  $e$  in  $\partial D_\lambda$ , there is a corresponding cell  $e_\mu$  in  $\partial e_\lambda$  whose characteristic map makes the required diagram commutative.  $\square$

The lifts  $b : D_\mu \rightarrow D_\lambda$  of characteristic maps appeared in the above Corollary play an essential role when we analyze totally normal stellar stratified spaces. It is easy to see that each  $b$  is an embedding.

**Lemma 3.11.** *Let  $X$  be a totally normal stellar stratified space. Then each  $b \in F(X)(e_\mu, e_\lambda)$  is an embedding of stellar stratified spaces for each pair  $e_\mu < e_\lambda$ .*

*Proof.* This follows from the assumption that the cellular stratification on  $D_\lambda$  is regular.  $\square$

### 3.3 Examples of Totally Normal Cellular Stratified Spaces

Let us take a look at some examples. The first example is borrowed from Kirillov's paper [KJ12].

**Example 3.12.** Consider  $D = \text{Int}(D^2) \cup S_+^1 = e^1 \cup e^2$ . Define  $X$  by folding the blue part of  $e^1$  according to the directions indicated by the blue arrows. Note that  $\varphi(e^1)$  is homeomorphic to  $\text{Int}(D^1)$ . Let

$$\psi : \text{Int}(D^1) \longrightarrow e^1 \subset X$$

a homeomorphism. Identifications only occur on  $e^1$  and the quotient map

$$\varphi : D \longrightarrow X$$

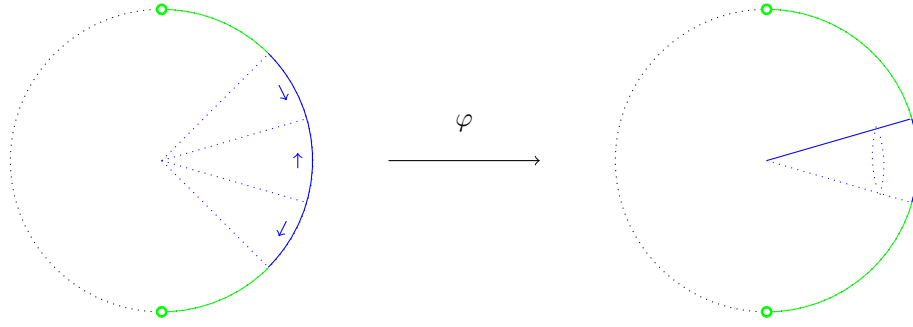


Figure 12: An example from Kirillov's paper.

is a homeomorphism onto its image when restricted to  $\text{Int}(D^2)$  and thus defines a characteristic map for the 2-cell.

These maps  $\psi$  and  $\varphi$  define a cellular stratification on  $X$ . However, there is no way to obtain a map  $h : \text{Int}(D^1) \rightarrow D$  making the diagram commutative

$$\begin{array}{ccc}
 D & \xrightarrow{\varphi} & X \\
 \uparrow h & \nearrow \psi & \\
 \text{Int}(D^1) & & 
 \end{array}$$

and thus this cellular stratification is not totally normal. Note, however, that we obtain a totally normal cellular stratification by an appropriate subdivision of  $e^1$  and  $\varphi(e^1)$ . □

**Example 3.13.** Consider the minimal cell decomposition

$$S^1 = e^0 \cup e^1 = \{(1, 0)\} \cup (S^1 \setminus \{(1, 0)\}).$$

The characteristic map for the 1-cell

$$\varphi_1 : D^1 = [-1, 1] \longrightarrow S^1$$

is given by  $\varphi(t) = (\cos(2\pi t), \sin(2\pi t))$ . There are two lifts  $b_{-1}$  and  $b_1$  of the characteristic map  $\varphi_0$  for the 0-cell.

Since  $\partial D^1 = \{-1, 1\} = b_{-1}(D^0) \cup b_1(D^0)$ , this is totally normal. □

**Example 3.14.** More generally, any 1-dimensional CW-complex is totally normal. In fact, 0-cells are always regular and there are only two types of 1-cells, i.e. regular cells and cells whose characteristic maps are given by collapsing the boundary of  $D^1$  to a point. By the above example, all 1-cells are totally normal. This fact allows us to apply results of this paper to configuration spaces of graphs [FMT15]. □

**Example 3.15.** Consider the punctured torus

$$X = S^1 \times S^1 \setminus e^0 \times e^0 = e^1 \times e^0 \cup e^0 \times e^1 \cup e^1 \times e^1$$

with the stratification induced from the product stratification of the minimal cell decomposition of  $S^1$ .

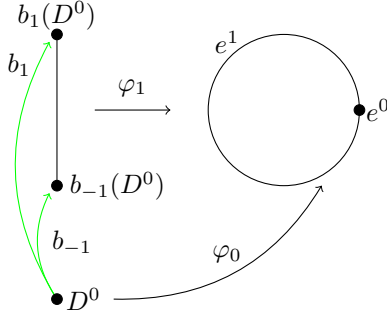


Figure 13: The minimal cell decomposition of  $S^1$  is totally normal.

A characteristic map for the 2-cell can be obtained by removing four corners from the domain of the product

$$\varphi_1 \times \varphi_1 : D^1 \times D^1 \setminus \{(1, 1), (1, -1), (-1, 1), (-1, -1)\} \longrightarrow X$$

of  $\varphi_1$  in Example 3.13. Let us denote the characteristic maps for  $e^1 \times e^0$  and  $e^0 \times e^1$  by  $\varphi_{1,0}$  and  $\varphi_{0,1}$ , respectively. The characteristic map of each 1-cell has two lifts and the images cover the

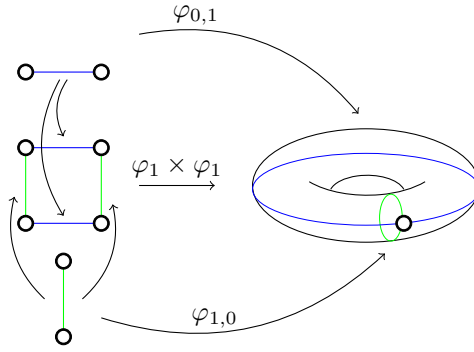


Figure 14: A totally normal punctured torus.

boundary of the domain of the 2-cell. Thus this is totally normal.  $\square$

**Example 3.16.** Let  $X$  be a  $\Delta$ -set<sup>10</sup>. Define

$$\pi_X : \|X\| \longrightarrow \prod_{n=0}^{\infty} X_n$$

by  $\pi_X(x) = \sigma$  if  $x$  is represented by  $(t, \sigma) \in \Delta^n \times X_n$  with  $t_i \neq 0$  for all  $i \in \{0, \dots, n\}$ . Note that  $P(\|X\|) = \prod_{n=0}^{\infty} X_n$  can be made into a poset by defining  $\tau \leq \sigma$  if and only if there exists a morphism  $u : [m] \rightarrow [n]$  in  $\Delta_{\text{inj}}$  with  $X(u)(\sigma) = \tau$ .

<sup>10</sup>Definition B.14.

Then the map  $\pi_X$  is a cellular stratification and we have

$$\|X\| = \bigcup_{n=0}^{\infty} \bigcup_{\sigma \in X_n} \text{Int}(\Delta^n) \times \{\sigma\}.$$

Let us denote the  $n$ -cell  $\text{Int}(\Delta^n) \times \{\sigma\}$  corresponding to  $\sigma \in X_n$  by  $e_\sigma$ . The characteristic map  $\varphi_\sigma$  for  $e_\sigma$  is defined by the composition

$$\varphi_\sigma : D^n \cong \Delta^n \times \{\sigma\} \hookrightarrow \prod_{n=0}^{\infty} \Delta^n \times X_n \longrightarrow \|X\|.$$

Suppose  $\tau < \sigma$  for  $\tau \in X_m$  and  $\sigma \in X_n$ . There exists a morphism  $u : [m] \rightarrow [n]$  with  $X(u)(\sigma) = \tau$ . With this morphism, we have the following commutative diagram

$$\begin{array}{ccc} \Delta^n \times \{\sigma\} & \xrightarrow{\varphi_\sigma} & \overline{e_\sigma} \\ \uparrow u_*^{\tau, \sigma} & & \uparrow \\ \Delta^m \times \{\tau\} & \xrightarrow{\varphi_\tau} & \overline{e_\tau} \end{array}$$

where  $u_*^{\tau, \sigma} : \Delta^m \times \{\tau\} \rightarrow \Delta^n \times \{\sigma\}$  is a copy of the affine map  $u_* : \Delta^m \rightarrow \Delta^n$  induced by the inclusion of vertices  $u : [m] \rightarrow [n]$ . Hence this cellular stratification on  $\|X\|$  is totally normal.

In particular, for any acyclic category  $C$  (with discrete topology), the classifying space  $BC$  is a totally normal CW complex by Lemma C.16.  $\square$

**Example 3.17.** A. Kirillov, Jr. introduced a structure called PLCW-complex in [KJ12]. A PLCW-complex is defined by attaching PL-disks. The attaching map of an  $n$ -cell is required to be a strict morphism of cellular stratified spaces under a suitable PLCW decomposition of the boundary sphere.

Besides the PL requirement in Kirillov's definition, the only difference between totally normal CW complexes and PLCW complexes is that, in totally normal CW complexes, the boundary sphere of each characteristic map is required to have a regular cell decomposition. For example, the cell decomposition of  $D^3$  in Figure 15 is a PLCW complex but is not totally normal.  $\square$

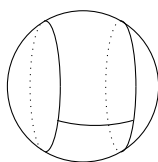


Figure 15: A PLCW complex which is not totally normal.

**Example 3.18.** Let  $X = \mathbb{R} \times \mathbb{R}_{\geq 0}$  with 0-cells  $e_n^0 = \{(n, 0)\}$  for  $n \in \mathbb{Z}$ , 1-cells  $e_n^1 = (n, n+1) \times \{0\}$  for  $n \in \mathbb{Z}$ , and a 2-cell  $e^2 = \mathbb{R} \times \mathbb{R}_{> 0}$ . The characteristic map  $\varphi$  of the 2-cell is given by extending the stereographic projection  $S^1 \setminus \{(0, 1)\} \rightarrow \mathbb{R}$ . The domain is  $D = D^2 \setminus \{(0, 1)\}$ . This is regular and normal. It even satisfies the second condition in Definition 3.6. But it is not totally normal, since the corresponding stratification on  $\partial D^2 \setminus \{(0, 1)\}$  cannot be a strict stratified subspace of a regular cell decomposition of  $\partial D^2$ .

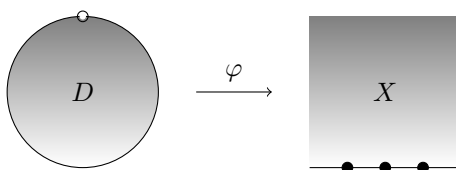


Figure 16: A cellular stratification on the closed upper half-plane with 0-cells  $\mathbb{Z}$ .

Note that this example does not satisfy the closure finiteness condition. Hence it is not CW.  $\square$

**Example 3.19.** Consider the minimal cell decomposition

$$S^2 = e^0 \cup e^2.$$

We may choose a regular cell decomposition of  $\partial D^2$  and make it a stellar stratification. For example, we may use the minimal regular cell decomposition  $S^1 = e_+^0 \cup e_-^0 \cup e_+^1 \cup e_-^1$ .

In order to make  $S^2$  into totally normal, however, this stratification is not fine enough. There are infinitely many lifts of the characteristic map of the 0-cell parametrized by points in  $S^1$ . See Figure 17. Although  $\partial D^2$  is covered by the images of  $b_z$ 's

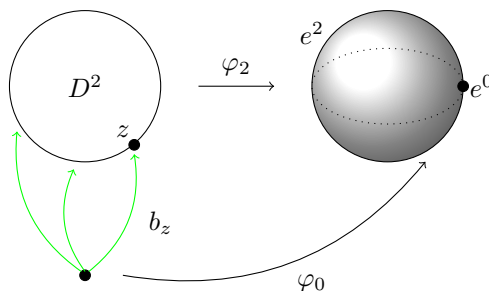


Figure 17: A minimal cell decomposition of  $S^2$ .

$$\partial D^2 = \bigcup_{z \in S^1} b_z(D^0),$$

this is not a cell decomposition of  $\partial D^2$ . Hence this is not totally normal.  $\square$

## 4 Cylindrically Normal Cellular Stratified Spaces

### 4.1 Cylindrical Structures

In Example 3.19, lifts of  $\varphi_0$  are parametrized by points in  $S^1 = \partial D^2$ . This example suggests that we need to topologize the set of all lifts. Inspired by the work of Cohen, Jones, and Segal on Morse theory [CJS] and this example, we introduce the following definition.

**Definition 4.1.** A *cylindrical structure* on a normal cellular stratified space  $(X, \pi)$  consists of

- a normal stratification on  $\partial D^n$  containing  $D_\lambda$  as a strict stratified subspace for each  $n$ -cell  $\varphi_\lambda : D_\lambda \rightarrow \overline{e_\lambda}$ ,
- a stratified space  $P_{\mu,\lambda}$  and a strict morphism of stratified spaces

$$b_{\mu,\lambda} : P_{\mu,\lambda} \times D_\mu \longrightarrow \partial D_\lambda$$

for each pair of cells  $e_\mu \subset \partial e_\lambda$ , and

- a strict morphism of stratified spaces

$$c_{\lambda_0,\lambda_1,\lambda_2} : P_{\lambda_1,\lambda_2} \times P_{\lambda_0,\lambda_1} \longrightarrow P_{\lambda_0,\lambda_2}$$

for each sequence  $\overline{e_{\lambda_0}} \subset \overline{e_{\lambda_1}} \subset \overline{e_{\lambda_2}}$

satisfying the following conditions:

1. The restriction of  $b_{\mu,\lambda}$  to  $P_{\mu,\lambda} \times \text{Int}(D_\mu)$  is a homeomorphism onto its image.
2. The following three types of diagrams are commutative.

$$\begin{array}{ccc} D_\lambda & \xrightarrow{\varphi_\lambda} & X \\ b_{\mu,\lambda} \uparrow & & \uparrow \varphi_\mu \\ P_{\mu,\lambda} \times D_\mu & \xrightarrow{\text{pr}_2} & D_\mu \end{array}$$

$$\begin{array}{ccc} P_{\lambda_1,\lambda_2} \times P_{\lambda_0,\lambda_1} \times D_{\lambda_0} & \xrightarrow{1 \times b_{\lambda_0,\lambda_1}} & P_{\lambda_1,\lambda_2} \times D_{\lambda_1} \\ c_{\lambda_0,\lambda_1,\lambda_2} \times 1 \downarrow & & \downarrow b_{\lambda_1,\lambda_2} \\ P_{\lambda_0,\lambda_2} \times D_{\lambda_0} & \xrightarrow{b_{\lambda_0,\lambda_2}} & D_{\lambda_2} \\ P_{\lambda_2,\lambda_3} \times P_{\lambda_1,\lambda_2} \times P_{\lambda_0,\lambda_1} & \xrightarrow{c \times 1} & P_{\lambda_1,\lambda_3} \times P_{\lambda_0,\lambda_1} \\ 1 \times c \downarrow & & \downarrow c \\ P_{\lambda_2,\lambda_3} \times P_{\lambda_0,\lambda_2} & \xrightarrow{c} & P_{\lambda_0,\lambda_3} \end{array}$$

3. We have

$$\partial D_\lambda = \bigcup_{e_\mu \subset \partial e_\lambda} b_{\mu,\lambda}(P_{\mu,\lambda} \times \text{Int}(D_\mu))$$

as a stratified space.

The space  $P_{\mu,\lambda}$  is called the *parameter space* for the inclusion  $e_\mu \subset \overline{e_\lambda}$ . When  $\mu = \lambda$ , we define  $P_{\lambda,\lambda}$  to be a single point. A stellar stratified space equipped with a cylindrical structure is called a *cylindrically normal stellar stratified space*.

When the map  $b_{\lambda,\mu}$  is an embedding for each pair  $e_\mu \subset \overline{e_\lambda}$ , the stratification is said to be *strictly cylindrical*.

When  $X$  is a stellar stratified space, we require that the normal stratification on each  $\partial D_\lambda$  is a coarsening<sup>11</sup> of the stratification in the definition of stellar stratified space.

<sup>11</sup>Definition 2.14



**Remark 4.2.** The author first intended to call such a structure as “locally product-like” or “locally trivial” cellular stratification. But it turns out the term “locally trivial stratification” is already used in [Pfl01] in a different sense.

We require morphisms between cylindrically normal cellular or stellar stratified spaces to preserve cylindrical structures.

**Definition 4.3.** Let  $(X, \pi_X, \Phi_X)$  and  $(Y, \pi_Y, \Phi_Y)$  be cylindrically normal cellular stratified spaces with cylindrical structures given by  $\{b_{\mu,\lambda}^X : P_{\mu,\lambda}^X \times D_\mu \rightarrow D_\lambda\}$  and  $\{b_{\alpha,\beta}^Y : P_{\alpha,\beta}^Y \times D_\alpha \rightarrow D_\beta\}$ , respectively.

A *morphism of cylindrically normal cellular stratified spaces* from  $(X, \pi_X, \Phi_X)$  to  $(Y, \pi_Y, \Phi_Y)$  is a morphism of stellar stratified spaces

$$\mathbf{f} = (f, \underline{f}) : (X, \pi_X, \Phi_X) \longrightarrow (Y, \pi_Y, \Phi_Y)$$

together with maps  $f_{\mu,\lambda} : P_{\mu,\lambda}^X \rightarrow P_{\underline{f}(\mu), \underline{f}(\lambda)}^Y$  making the diagram

$$\begin{array}{ccc} P_{\mu,\lambda}^X \times D_\mu & \xrightarrow{f_{\mu,\lambda} \times f_\mu} & P_{\underline{f}(\mu), \underline{f}(\lambda)}^Y \times D_{\underline{f}(\mu)} \\ \downarrow b_{\mu,\lambda}^X & & \downarrow b_{\underline{f}(\mu), \underline{f}(\lambda)}^Y \\ D_\lambda & \xrightarrow{f_\lambda} & D_{\underline{f}(\lambda)}. \end{array}$$

Morphisms of cylindrically normal stellar stratified spaces are defined analogously.

The categories of cylindrically normal cellular and stellar stratified spaces are denoted by  $\mathbf{CSSpaces}^{\text{cy1}}$  and  $\mathbf{SSSpaces}^{\text{cy1}}$ , respectively.

For cylindrically normal cellular/stellar stratified spaces, the relative-compactness of closures of cells can be easily verified.

**Lemma 4.4.** *Let  $X$  be a cylindrically normal cellular (stellar) stratified space with parameter spaces  $\{P_{\mu,\lambda}\}_{\mu \leq \lambda}$ . A cell  $\varphi_\lambda : D_\lambda \rightarrow \bar{e}_\lambda$  is relatively compact if and only if  $P_{\mu,\lambda}$  is compact for each  $\mu \leq \lambda$ .*

*Proof.* For  $y \in \partial e_\lambda$ , there exists  $\mu \leq \lambda$  with  $y \in e_\mu \subset \partial e_\lambda$ . In the commutative diagram

$$\begin{array}{ccc} D_\lambda & \xrightarrow{\varphi_\lambda} & \bar{e}_\lambda \\ \uparrow b_{\mu,\lambda}|_{P_{\mu,\lambda} \times \text{Int}(D_\mu)} & & \uparrow \\ P_{\mu,\lambda} \times \text{Int}(D_\mu) & \longrightarrow & \text{Int}(D_\mu) \xrightarrow{\varphi_\mu|_{\text{Int}(D_\mu)}} e_\mu \end{array}$$

the restriction  $b_{\mu,\lambda}|_{P_{\mu,\lambda} \times \text{Int}(D_\mu)}$  is an embedding and  $\partial D_\lambda$  is covered by the disjoint union of such images. Thus we have  $\varphi_\lambda^{-1}(y) \cong P_{\mu,\lambda} \times \{y\}$  and the result follows from Corollary A.15 and the assumption.  $\square$

**Corollary 4.5.** *Let  $X$  be a CW cylindrically normal cellular (stellar) stratified space. If all parameter spaces are compact,  $X$  is paracompact.*

## 4.2 Polyhedral Cellular Stratified Spaces

Although total normality and cylindrical normality play central roles in our study of cellular and stellar stratified spaces, it is not easy to prove that a given cellular stratification is totally normal or cylindrically normal.

In order to address this problem, we found polyhedral complexes and PL maps are useful. It turns out that PL maps also play an important role in our proof of Theorem 5.18. In this section, we generalize polyhedral complexes and introduce the notion of polyhedral cellular stratified spaces.

Let us first recall the definition of polyhedral complexes. See §2.2.4 of Kozlov's book [Koz08] for details.

**Definition 4.6.** A *Euclidean polyhedral complex* is a subspace  $K$  of  $\mathbb{R}^N$  for some  $N$  equipped with a finite family of maps

$$\{\varphi_i : F_i \longrightarrow K \mid i = 1, \dots, n\}$$

satisfying the conditions that

1. each  $F_i$  is a convex polytope;
2. each  $\varphi_i$  is an affine equivalence onto its image;
3.  $K = \bigcup_{i=1}^n \varphi_i(\text{Int}(F_i))$ , where  $\text{Int}(F_i)$  is the relative interior<sup>12</sup> of  $F_i$ ;
4. for  $i \neq j$ ,  $\varphi_i(F_i) \cap \varphi_j(F_j)$  is a proper face of  $\varphi_i(F_i)$  and  $\varphi_j(F_j)$ .

The polytopes  $F_i$ 's are called *generating polytopes*.

Obviously a polyhedral complex is a regular cell complex. By replacing affine cell structure maps  $\varphi_i$  by continuous maps, Kozlov defined a more general kind of polyhedral complexes in his book. The requirement of being a subspace of  $\mathbb{R}^N$  can be removed if we assume cylindrical normality. We may also remove the condition that all cells are closed. Here is our definition of “polyhedral” structure.

**Definition 4.7.** A *polyhedral stellar stratified space* consists of

- a CW cylindrically normal stellar stratified space  $X$ ,
- a family of Euclidean polyhedral complexes  $\tilde{F}_\lambda$  indexed by  $\lambda \in P(X)$  and
- a family of homeomorphisms  $\alpha_\lambda : \tilde{F}_\lambda \rightarrow \overline{D_\lambda}$  indexed by  $\lambda \in P(X)$ , where  $\overline{D_\lambda}$  is the closure of the domain stellar cell  $D_\lambda$  for  $e_\lambda$  in a disk containing  $D_\lambda$ ,

satisfying the following conditions:

1. For each cell  $e_\lambda$ ,  $\alpha_\lambda : \tilde{F}_\lambda \rightarrow \overline{D_\lambda}$  is a subdivision of stratified space, where the stratification on  $\overline{D_\lambda}$  is defined by the cylindrical structure<sup>13</sup>.
2. For each pair  $e_\mu < e_\lambda$ , the parameter space  $P_{\mu,\lambda}$  is a locally cone-like space<sup>14</sup> and the composition

$$P_{\mu,\lambda} \times F_\mu \xrightarrow{1 \times \alpha_\mu} P_{\mu,\lambda} \times D_\mu \xrightarrow{b_{\mu,\lambda}} D_\lambda \xrightarrow{\alpha_\lambda^{-1}} F_\lambda$$

is a PL map<sup>15</sup>, where  $F_\lambda = \alpha_\lambda^{-1}(D_\lambda)$ .

<sup>12</sup>The interior in its affine hull.

<sup>13</sup>Definition 4.1

<sup>14</sup>Definition B.22

<sup>15</sup>Definition B.30

Each  $\alpha_\lambda$  is called a *polyhedral replacement* of the cell structure map of  $e_\lambda$ . The collection  $A = \{\alpha_\lambda\}_{\lambda \in P(X)}$  is called a *polyhedral structure* on  $X$ .

**Remark 4.8.** When  $X$  is cellular,  $\overline{D_\lambda} = D^{\dim e_\lambda}$ .

**Remark 4.9.** An analogous structure for cell complexes is introduced by A. Kirillov, Jr. in [KJ12] as PLCW complexes for a different purpose.  $M_\kappa$ -polyhedral complexes in the book [BH99] by Bridson and Haefliger are also closely related.

**Definition 4.10.** A *morphism of polyhedral stellar stratified spaces* from  $(X, \pi, \Phi, A)$  to  $(X', \pi', \Phi', A')$  consists of a morphism of stellar stratified spaces

$$(\mathbf{f}, \{f_\lambda\}) : (X, \pi, \Phi) \longrightarrow (X', \pi', \Phi')$$

and a family of PL maps  $\tilde{f}_\lambda : \tilde{F}_\lambda \rightarrow \tilde{F}'_{f(\lambda)}$  for  $\lambda \in P(X)$  that are compatible with polyhedral structures.

Any polyhedral complex is polyhedral. More generally, we have the following criterion.

**Lemma 4.11.** *Let  $X$  be a subspace of  $\mathbb{R}^N$  equipped with a structure of cylindrically normal CW stellar stratified space whose parameter spaces  $P_{\mu,\lambda}$  are locally cone-like spaces. Suppose, for each  $e_\lambda \in P(X)$ , there exists a polyhedral complex  $\tilde{F}_\lambda$  and a homeomorphism  $\alpha_\lambda : \tilde{F}_\lambda \rightarrow D^{\dim e_\lambda}$  such that the composition*

$$P_{\mu,\lambda} \times F_\mu \xrightarrow{1 \times \alpha_\mu} P_{\mu,\lambda} \times D_\mu \xrightarrow{b_{\mu,\lambda}} D_\lambda \xrightarrow{\varphi_\lambda} X \hookrightarrow \mathbb{R}^N$$

is a PL map, where  $F_\lambda = \alpha_\lambda^{-1}(D_\lambda)$ . Suppose further that  $\alpha_\lambda : \tilde{F}_\lambda \rightarrow D^{\dim e_\lambda}$  is a cellular subdivision, where the cell decomposition on  $D^{\dim e_\lambda}$  is the one in the definition of cylindrical structure. Then the collection  $\{\alpha_\lambda\}_{\lambda \in P(X)}$  defines a polyhedral structure on  $X$ .

*Proof.* For each pair  $e_\mu < e_\lambda$ , define  $\tilde{b}_{\mu,\lambda} : P_{\mu,\lambda} \times F_\mu \rightarrow F_\lambda$  to be the composition

$$\tilde{b}_{\mu,\lambda} : P_{\mu,\lambda} \times F_\mu \xrightarrow{1 \times \alpha_\mu} P_{\mu,\lambda} \times D_\mu \xrightarrow{b_{\mu,\lambda}} D_\lambda \xrightarrow{\alpha_\lambda^{-1}} F_\lambda.$$

Note that, when  $\mu = \lambda$ ,  $b_{\mu,\lambda}$  is the identity map. Thus the top horizontal composition in the following diagram is a PL map:

$$\begin{array}{ccccccc} F_\lambda & \xrightarrow{\alpha_\lambda} & D_\lambda & \xrightarrow{\varphi_\lambda} & X & \hookrightarrow & \mathbb{R}^N \\ \tilde{b}_{\mu,\lambda} \uparrow & & & & & & \parallel \\ P_{\mu,\lambda} \times F_\mu & \xrightarrow{1 \times \alpha_\mu} & P_{\mu,\lambda} \times D_\mu & \xrightarrow{b_{\mu,\lambda}} & D_\lambda & \xrightarrow{\varphi_\lambda} & X & \hookrightarrow & \mathbb{R}^N. \end{array}$$

The bottom horizontal composition is also a PL map by assumption and  $\tilde{b}_{\mu,\lambda}$  is an embedding when restricted to  $P_{\mu,\lambda} \times \alpha_\mu^{-1}(\text{Int} D^{\dim e_\mu})$ . Thus  $\tilde{b}_{\mu,\lambda}$  is also PL by Lemma B.32.  $\square$

**Example 4.12.** Consider the minimal cell decomposition  $S^2 = e^0 \cup e^2$ . We have an embedding  $f : S^2 \rightarrow \mathbb{R}^3$  whose image is the boundary  $\partial\Delta^3$  of the standard 3-simplex and  $f(e^0)$  is a vertex of  $\partial\Delta^3$ . This is a cylindrically normal cellular stratified space by Example 4.24. Let  $P$  be a 2-dimensional polyhedral complex in  $\mathbb{R}^2$  described in Figure 18. By collapsing the outer triangle,

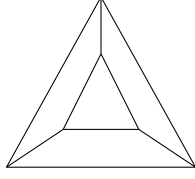


Figure 18: A polyhedral structure on  $S^2$

we obtain a map

$$\psi : P \longrightarrow \partial\Delta^3$$

whose restriction to the interior is a homeomorphism. Let  $\alpha : P \rightarrow D^2$  be a homeomorphism given by a radial expansion. Then maps  $f$  and  $\varphi$  can be chosen in such a way they make the following diagram commutative

$$\begin{array}{ccc} D^2 & \xrightarrow{\varphi} & S^2 \\ \alpha \uparrow & & \downarrow f \\ P & \xrightarrow{\psi} & \partial\Delta^3 \hookrightarrow \mathbb{R}^3. \end{array}$$

By Lemma 4.11, we obtain a structure of polyhedral cellular stratified space on  $S^2$  or  $\partial\Delta^3$ .  $\square$

**Definition 4.13.** A cellular stratified space satisfying the assumption of Lemma 4.11 is called a *Euclidean polyhedral cellular stratified space*.

Totally normal cellular stratified spaces form an important class of polyhedral cellular stratified spaces.

**Lemma 4.14.** *Any CW totally normal cellular stratified space has a polyhedral structure.*

*Proof.* Let  $X$  be a CW totally normal cellular stratified space. By definition, for each cell  $\varphi_\lambda : D_\lambda \rightarrow \bar{e}_\lambda$  in  $X$ , there exists a regular cell decomposition of  $D^{\dim e_\lambda}$  containing  $D_\lambda$  as a cellular stratified subspace. Since the barycentric subdivision of a regular cell complex has a structure of simplicial complex,  $D^{\dim e_\lambda}$  can be embedded in a Euclidean space as a finite simplicial complex  $\tilde{F}_\lambda$ . By induction on dimensions of cells, we may choose homeomorphisms  $\{\alpha_\lambda : \tilde{F}_\lambda \rightarrow D^{\dim e_\lambda}\}_{\lambda \in P(X)}$  in such a way the composition

$$F_\mu \xrightarrow{\alpha_\mu} D_\mu \xrightarrow{b} D_\lambda \xrightarrow{\alpha_\lambda^{-1}} F_\lambda$$

is a PL map for each lift  $b : D_\mu \rightarrow D_\lambda$  of the cell structure map of  $e_\mu$  for each pair  $e_\mu < e_\lambda$ . Thus we obtain a polyhedral structure.  $\square$

Conversely, the polyhedrality condition provides us with a useful criterion for a stratification to be totally normal. The following fact first appeared in [BGRT].

**Proposition 4.15.** *Let  $X$  be a normal CW complex embedded in  $\mathbb{R}^N$  for some  $N$ . Suppose for each cell  $e \subset X$  with cell structure map  $\varphi$ , there exists a Euclidean polyhedral complex  $F$  and a homeomorphism  $\alpha : F \rightarrow D^{\dim e}$  such that the composition*

$$F \xrightarrow{\alpha} D^{\dim e} \xrightarrow{\varphi} X \subset \mathbb{R}^N$$

*is a PL map. Then any regular cellular stratified subspace of  $X$  is polyhedral.*

*Proof.* Let  $A$  be a regular cellular stratified subspace of  $X$ . It suffices to find polyhedral replacements of cell structures for  $A$ . For an  $n$ -cell  $e$  in  $A$ , let  $\varphi : D^n \rightarrow X$  be the cell structure of  $e$  in  $X$ . The cell structure for  $e$  in  $A$  is, by definition, given by

$$\varphi_A = \varphi|_{D_A} : D_A = \varphi^{-1}(\bar{e} \cap A) \longrightarrow A.$$

By assumption, there exists a polyhedral complex  $P$  and a homeomorphism  $\alpha : P \rightarrow D^n$  with  $\varphi \circ \alpha : P \rightarrow X \hookrightarrow \mathbb{R}^N$  a PL map.  $\square$

**Example 4.16.** The product cell decomposition of  $(S^n)^k$  induced by the minimal cell decomposition on  $S^n$  is polyhedral. In [BGR], a subdivision of the product cell decomposition containing  $\text{Conf}_k(S^n)$  as a stratified subspace was defined by using the stratification on  $(\mathbb{R}^n)^\ell$  associated with the braid arrangement  $\mathcal{A}_{\ell-1}$  for  $1 \leq \ell \leq k$ .

The author expected that the induced stratification on  $\text{Conf}_k(S^n)$  to be totally normal. It turned out that the stratification is much more complicated than the author imagined.  $\square$

We propose the following strategy to prove a given cellular stratification on  $X$  is cylindrically normal or totally normal:

1. Embed  $X$  into a cylindrically normal or totally normal cell complex  $\tilde{X}$ .
2. Find an appropriate subdivision of  $\tilde{X}$  which includes  $X$  as a stratified subspace.

In fact, it is easy to prove that, if  $X$  is polyhedral,  $X$  can be embedded in a cell complex as a stratified subspace.

**Lemma 4.17.** *Let  $X$  be a polyhedral cellular stratified space. Given a pair of cells  $e_\mu \subset \partial e_\lambda$ , the structure map*

$$b_{\mu,\lambda} : P_{\mu,\lambda} \times D_\mu \longrightarrow D_\lambda$$

*has a unique extension to the whole disks*

$$\overline{b_{\mu,\lambda}} : P_{\mu,\lambda} \times D^{\dim e_\mu} \longrightarrow D^{\dim e_\lambda}.$$

*Proof.* By Lemma B.32.  $\square$

These maps allow us to construct a canonical closure of any polyhedral cellular stratified space.

**Definition 4.18.** Let  $X$  be a polyhedral cellular stratified space with cells  $\{e_\lambda\}_{\lambda \in P(X)}$ . Define a cell complex  $U(X)$  by

$$U(X) = \left( \coprod_{\lambda \in P(X)} D^{\dim e_\lambda} \right) / \sim$$

where the relation  $\sim$  is the equivalence relation generated by the following relation: For  $x \in D^{\dim e_\lambda}$  and  $y \in D^{\dim e_\mu}$ ,  $x \sim y$  if  $e_\mu \subset \partial e_\lambda$  and there exists  $z \in P_{\mu,\lambda}$  such that  $\overline{b_{\mu,\lambda}}(z, y) = x$ .

There is a canonical inclusion

$$i : X \hookrightarrow U(X).$$

This space  $U(X)$  is called the *cellular closure* of  $X$ .

By definition, we have the following.

**Lemma 4.19.** *When  $X$  is an polyhedral cellular stratified space,  $U(X)$  is a cylindrically normal CW complex containing  $X$  as a cellular stratified subspace.*

**Example 4.20.** In the case of the punctured torus in Example 3.15,  $U(X)$  is obtained by gluing parallel edges in  $I^2$  and is homeomorphic to  $T^2$ . In other words,  $U(X)$  is obtained by closing the hole in  $X$ .  $\square$

The existence of cellular closure implies that cell structures of a polyhedral cellular stratified space are bi-quotient<sup>16</sup>.

**Corollary 4.21.** *Let  $X$  be a polyhedral cellular stratified space. Then any cell structure  $\varphi_\lambda : D_\lambda \rightarrow \overline{e_\lambda}$  is bi-quotient.*

*Proof.* For an  $n$ -cell  $\varphi_\lambda : D_\lambda \rightarrow \overline{e_\lambda}$  in  $X$ , let  $\widetilde{\varphi}_\lambda : D^n \rightarrow U(X)$  the extension. Since  $\widetilde{\varphi}_\lambda$  is proper, it is bi-quotient and hence is hereditarily quotient. By Lemma A.5,  $\varphi_\lambda$  is also hereditarily quotient.

For  $y \in \overline{e_\lambda} \subset X$ , the fiber  $\varphi_\lambda^{-1}(y)$  can be identified with one of parameter spaces by the proof of Lemma 4.4, which is a cellular stratified subspace of a regular cell decomposition of  $\partial D^n$ . Thus the boundary  $\partial\varphi_\lambda^{-1}(y)$  is compact. The result follows from 3 in Lemma A.8.  $\square$

**Corollary 4.22.** *Any polyhedral cellular stratified space is paracompact.*

### 4.3 Examples of Cylindrically Normal Cellular Stratified Spaces

Here is a collection of examples and nonexamples of cylindrically normal cellular stratified spaces.

**Example 4.23.** A stellar stratified space  $X$  is totally normal if and only if it is strictly cylindrically normal and each parameter space  $P_{\mu,\lambda}$  is a finite set (with discrete topology).

Consider the cell decomposition of  $D^3$  in Example 3.17. It is easily seen to be cylindrically normal with finite parameter spaces. However, it is not strictly cylindrical, as we have seen in Example 3.17. In other words, a cylindrically normal cell complex with finite parameter spaces is a PLCW complex, if it satisfies the PL requirement in the definition of PLCW complexes.  $\square$

**Example 4.24.** Consider the minimal cell decomposition of  $S^2 = \mathbb{C}P^1$  in Example 3.19. The trivial stratification on  $\partial D^2$  and the canonical inclusion

$$b_{0,2} : S^1 \times D^0 \longrightarrow \partial D^2 \subset D^2$$

define a cylindrical structure on  $S^2$  with  $P_{0,2} = S^1$ , for we have a commutative diagram

$$\begin{array}{ccc} D^2 & \xrightarrow{\varphi_2} & \mathbb{C}P^1 \\ \uparrow b_{0,2} & & \uparrow \varphi_0 \\ S^1 \times D^0 & \xrightarrow{\text{pr}_2} & D^0. \end{array}$$

We may also consider  $S^2$  as a stellar stratified space, by choosing a regular cell decomposition of  $\partial D^2$ . For example, we may use the minimal regular cell decomposition of  $\partial D^2$ .

Then we have embeddings

$$\begin{aligned} b_{0,+} & : D^0 \times D^0 \longrightarrow \partial D^2 \\ b_{0,-} & : D^0 \times D^0 \longrightarrow \partial D^2 \\ b_{1,+} & : D^0 \times D^1 \longrightarrow \partial D^2 \\ b_{1,-} & : D^0 \times D^1 \longrightarrow \partial D^2 \end{aligned}$$

---

<sup>16</sup>Definition A.1.

corresponding to cells  $e_+^0 \cup e_-^0 \cup e_+^1 \cup e_-^1$  in  $\partial D^2$ . And we obtain a cylindrical structure as a stellar stratified space.  $\square$

**Example 4.25.** Let us extend the cylindrically normal cell decomposition on  $S^2 = \mathbb{C}P^1$  to  $\mathbb{C}P^2$ . The minimal cell decomposition of  $\mathbb{C}P^2$  is given by

$$\mathbb{C}P^2 = S^2 \cup e^4 = e^0 \cup e^2 \cup e^4.$$

Consider the cell structure map of the 4-cell

$$\varphi_4 = \tilde{\eta} : D^4 \longrightarrow \mathbb{C}P^2$$

whose restriction to the boundary is the Hopf map

$$\eta : S^3 \longrightarrow S^2.$$

This is a fiber bundle with fiber  $S^1$  and thus the cell decomposition  $S^2 = e^0 \cup e^2$  induces a decomposition

$$S^3 \cong e^0 \times S^1 \cup e^2 \times S^1,$$

as we have seen in Example 2.16.

Let

$$\begin{aligned} \varphi_0 : D^0 &\rightarrow \overline{e^0} \subset S^2 \\ \varphi_2 : D^2 &\rightarrow \overline{e^2} \subset S^2 \end{aligned}$$

be the cell structure maps of  $e^0$  and  $e^2$ , respectively. We have a trivialization

$$t : \varphi_2^*(S^3) \xrightarrow{\cong} D^2 \times S^1.$$

Let  $b_{2,4} : S^1 \times D^2 \rightarrow S^3 = \partial D^4$  be the composition

$$S^1 \times D^2 \xrightarrow{t^{-1}} \varphi_2^*(S^3) \xrightarrow{\tilde{\varphi}_2} S^3,$$

then we have the following commutative diagram

$$\begin{array}{ccccc} S^3 & \xlongequal{\quad} & S^3 & \xrightarrow{\eta} & \mathbb{C}P^1 \\ & & \uparrow \tilde{\varphi}_2 & & \uparrow \varphi_2 \\ & & \varphi_2^*(S^3) & \longrightarrow & D^2 \\ & \nearrow t^{-1} & & & \parallel \\ S^1 \times D^2 & \xrightarrow{\quad} & & \xrightarrow{\text{pr}_2} & D^2 \\ & \uparrow b_{2,4} & & & \end{array}$$

Let  $b_{0,4} : S^1 \times D^0 \rightarrow S^3$  be the inclusion of the fiber over  $e^0$ . Then we have

$$\partial D^4 = b_{0,4}(S^1 \times D^0) \cup b_{2,4}(S^1 \times D^2).$$

Let  $P_{0,2} = P_{2,4} = P_{0,4} = S^1$  and define

$$c_{0,2,4} : P_{2,4} \times P_{0,2} \rightarrow P_{0,4}$$

by the multiplication of  $S^1$ .

Let us check that the above data define a cylindrical structure on  $\mathbb{CP}^2 = e^0 \cup e^2 \cup e^4$ . It remains to verify the commutativity of the diagram

$$\begin{array}{ccc} P_{2,4} \times P_{0,2} \times D^0 & \xrightarrow{1 \times b_{0,2}} & P_{2,4} \times D^2 \\ c \times 1 \downarrow & & \downarrow b_{2,4} \\ P_{0,4} \times D^0 & \xrightarrow{b_{0,4}} & D^4. \end{array}$$

In other words, we need to show the restriction of  $b_{2,4}$  to

$$b_{2,4}|_{S^1 \times S^1} : S^1 \times S^1 \longrightarrow S^3 = \partial D^4$$

is given by the multiplication of  $S^1$  followed by the inclusion of the fiber  $\eta^{-1}(e^0)$ . Recall that the Hopf map  $\eta$  is given by

$$\eta(z_1, z_2) = (2|z_1|^2 - 1, 2z_1\bar{z}_2),$$

where we regard

$$\begin{aligned} S^2 &= \{(x, z) \in \mathbb{R} \times \mathbb{C} \mid x^2 + |z|^2 = 1\} \\ S^3 &= \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}, \end{aligned}$$

and, on  $U_+ = S^2 - \{(1, 0)\}$ , the local trivialization

$$\varphi_+ : \eta^{-1}(U_+) \longrightarrow U_+ \times S^1$$

is given by

$$\varphi_+(z_1, z_2) = \left( 2|z_1|^2 - 1, 2z_1\bar{z}_2, \frac{z_2}{|z_2|} \right).$$

The inverse of  $\varphi_+$  is given by

$$\varphi_+^{-1}(x, z, w) = \left( \frac{zw}{2\sqrt{\frac{1-x}{2}}}, w\sqrt{\frac{1-x}{2}} \right).$$

Define

$$\text{wrap} : D^2 \longrightarrow S^2$$

by

$$\text{wrap}(z) = \left( 2|z| - 1, 2\sqrt{|z|(1-|z|)} \frac{z}{|z|} \right);$$

then the restriction of  $\text{wrap}$  to  $\text{Int}(D^2)$  is a homeomorphism onto  $S^2 \setminus \{(1, 0)\}$ . The map  $b_{2,4}$  is defined by the composition

$$S^1 \times \text{Int}(D^2) \xrightarrow{1 \times \text{wrap}} S^1 \times U_+ \cong U_+ \times S^1 \xrightarrow{\varphi_+^{-1}} \eta^{-1}(U_+) \hookrightarrow S^3,$$



which is given by

$$\begin{aligned}
b_{2,4}(w, z) &= \varphi_+^{-1} \left( 2|z| - 1, 2\sqrt{|z|(1-|z|)} \frac{z}{|z|}, w \right) \\
&= \left( \frac{2\sqrt{|z|(1-|z|)} \frac{z}{|z|} w}{2\sqrt{\frac{1-(2|z|-1)}{2}}}, w\sqrt{\frac{1-(2|z|-1)}{2}} \right) \\
&= \left( \frac{zw}{\sqrt{|z|}}, w\sqrt{1-|z|} \right).
\end{aligned}$$

From this calculation, we see  $b_{2,4}(w, z) \rightarrow (zw, 0)$  as  $|z| \rightarrow 1$ .

Thus this is a cylindrically normal cellular stratification.  $\square$

**Example 4.26.** There is an alternative way of describing the cylindrical structure in the above Example. Recall that complex projective spaces are typical examples of quasitoric manifolds. Define an action of  $T^n = (S^1)^n$  on  $\mathbb{C}P^n$  by

$$(t_1, \dots, t_n) \cdot [z_0, \dots, z_n] = [z_0, t_1 z_1, \dots, t_n z_n].$$

As we have seen in Example 2.11, this action induces a stratifications on  $\mathbb{C}P^n$  which descends to  $\mathbb{C}P^n/T^n$

$$\begin{aligned}
\pi_{\mathbb{C}P^n} &: \mathbb{C}P^n \longrightarrow I(T^n) \\
\pi_{\mathbb{C}P^n/T^n} &: \mathbb{C}P^n/T^n \longrightarrow I(T^n).
\end{aligned}$$

The quotient space  $\mathbb{C}P^n/T^n$  is known to be homeomorphic to  $\Delta^n$  and the stratification  $\pi_{\mathbb{C}P^n/T^n}$  can be identified with the stratification  $\pi_n$  on  $\Delta^n$  in Example 2.10. This stratification, however, does not induce the minimal cell decomposition of  $\mathbb{C}P^n$ . The other stratification  $\pi_n^{\max}$  on  $\Delta^n$  defined in Example 2.10 induces the minimal cell decomposition on  $\mathbb{C}P^n$  by the composition

$$\mathbb{C}P^n \xrightarrow{p} \mathbb{C}P^n/T^n \cong \Delta^n \xrightarrow{\pi_n^{\max}} [n].$$

Let us show that this cell decomposition is cylindrically normal. To this end, we first rewrite  $\mathbb{C}P^n$  by using the construction introduced in [DJ91] by Davis and Januszkiewicz. Given a simple polytope<sup>17</sup>  $P$  of dimension  $n$  and a function  $\lambda : \{\text{codimension-1 faces in } P\} \rightarrow \mathbb{C}^n$  satisfying certain conditions, they constructed a space  $M(\lambda)$  with  $T^n$ -action. Suppose  $P = \Delta^n$  and define

$$\lambda_n(C_i) = \begin{cases} (1, \dots, 1), & i = 0 \\ (\underbrace{0, \dots, 0}_{i-1}, 1, \underbrace{0, \dots, 0}_{n-i}), & i = 1, \dots, n, \end{cases}$$

where  $C_i$  is the codimension-1 face with vertices in  $[n] - \{i\}$ . In this case,  $M(\lambda_n)$  can be described as

$$M(\lambda_n) = (T^n \times \Delta^n) / \sim,$$

where the equivalence relation  $\sim$  is generated by the following relations: Let  $p = (p_0, \dots, p_n) \in \Delta^n$ .

<sup>17</sup>A  $d$ -dimensional convex polytope is said to be *simple* if each vertex is adjacent to exactly  $d$  edges.

- When  $p_i = 0$  for  $1 \leq i \leq n$ ,

$$(t_1, \dots, t_i, \dots, t_n; p_0, \dots, p_n) \sim (t_1, \dots, t'_i, \dots, t_n; p_0, \dots, p_n)$$

for any  $t_i, t'_i \in S^1$ .

- When  $p_0 = 0$ ,

$$(t_1, \dots, t_n; 0, p_1, \dots, p_n) \sim (\omega t_1, \dots, \omega t_n; 0, p_1, \dots, p_n)$$

for any  $\omega \in S^1$ .

An explicit homeomorphism  $p_n : \mathbb{C}P^n \rightarrow M(\lambda_n)$  and its inverse  $q_n$  are given by

$$\begin{aligned} p_n([z_0 : \dots : z_n]) &= \begin{cases} \left[ \frac{z_1/z_0}{|z_1/z_0|}, \dots, \frac{z_n/z_0}{|z_n/z_0|}, \frac{|z_0|^2}{\sum_{i=0}^n |z_i|^2}, \dots, \frac{|z_n|^2}{\sum_{i=0}^n |z_i|^2} \right] & z_0 \neq 0 \\ \left[ \frac{z_1}{|z_1|}, \dots, \frac{z_n}{|z_n|}, 0, \frac{|z_1|^2}{\sum_{i=0}^n |z_i|^2}, \dots, \frac{|z_n|^2}{\sum_{i=0}^n |z_i|^2} \right] & z_0 = 0 \end{cases} \\ q_n([z_1, \dots, z_n; x_0, \dots, x_n]) &= [\sqrt{x_0}, \sqrt{x_1}z_1, \dots, \sqrt{x_n}z_n]. \end{aligned}$$

Under this identification, the minimal cell decomposition on  $\mathbb{C}P^n$  can be described as

$$\mathbb{C}P^n \cong M(\lambda_n) = \bigcup_{i=1}^n (T^n/T^{n-i} \times (\Delta^i \setminus \Delta^{i-1})) / \sim.$$

Regard  $D^{2n} = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid \sum_{i=1}^n |z_i|^2 \leq 1\}$  and define a map

$$\varphi_{2n} : D^{2n} \longrightarrow M(\lambda_n)$$

by

$$\varphi_{2n}(z_1, \dots, z_n) = \left[ \frac{z_1}{|z_1|}, \dots, \frac{z_n}{|z_n|}; 1 - \sum_{i=1}^n |z_i|^2, |z_1|^2, \dots, |z_n|^2 \right].$$

This is a cell structure map for the  $2n$ -cell. For  $m < n$ , define

$$b_{2m, 2n} : S^1 \times D^{2m} \longrightarrow D^{2n}$$

by

$$b_{2m, 2n}(\omega, z_1, \dots, z_m) = \left( 0, \dots, 0, \omega \sqrt{1 - \sum_{i=1}^m |z_i|^2}, \omega z_1, \dots, \omega z_m \right).$$

Then each  $b_{2m, 2n}|_{S^1 \times \text{Int}(D^{2m})}$  is a homeomorphism onto its image and we have a stratification

$$\partial D^{2n} = S^{2n-1} = \bigcup_{m=0}^{n-1} b_{2m, 2n}(S^1 \times \text{Int}(D^{2m})).$$

Furthermore the diagram

$$\begin{array}{ccc} & & D^{2n} \xrightarrow{\varphi_{2n}} M(\lambda_n) \\ & & \uparrow \\ S^1 \times D^{2m} & \xrightarrow{b_{2m, 2n}} & S^{2n-1} \\ \text{pr}_2 \downarrow & & \uparrow \\ D^{2m} & \xrightarrow{\varphi_{2m}} & M(\lambda_m), \end{array}$$

is commutative, where the inclusion  $M(\lambda_m) \hookrightarrow M(\lambda_n)$  is given by

$$[t_1, \dots, t_m; p_0, \dots, p_m] \mapsto [1, \dots, 1, t_1, \dots, t_m; 0, \dots, 0, p_0, \dots, p_m].$$

Now define  $P_{2m,2n} = S^1$  for  $m < n$ . The group structure of  $S^1$  defines a map

$$c_{2\ell,2m,2n} : P_{2m,2n} \times P_{2\ell,2n} \longrightarrow P_{2\ell,2n}$$

making the diagram

$$\begin{array}{ccc} S^1 \times S^1 \times D^{2\ell} & \xrightarrow{1 \times b_{2\ell,2m}} & S^1 \times D^{2m} \\ c_{2\ell,2m,2n} \times 1 \downarrow & & \downarrow b_{2m,2n} \\ S^1 \times D^{2\ell} & \xrightarrow{b_{2\ell,2n}} & D^{2n} \end{array}$$

commutative and we have a cylindrical structure.

More generally, Davis and Januszkiewicz [DJ91] proved that any quasitoric manifold  $M$  of dimension  $2n$  can be expressed as  $M \cong M(\lambda)$  for a simple convex polytope  $P$  of dimension  $n$  and a function  $\lambda$ . The right hand side is a space constructed as a quotient of  $T^n \times P$  under an equivalence relation analogous to the case of  $\mathbb{C}P^n$ . Davis and Januszkiewicz proved in §3 of their paper that there is a “perfect Morse function” on  $M(\lambda)$  which induces a cell decomposition of  $M(\lambda)$  or  $M$  with exactly  $h_i(P)$  cells of dimension  $2i$ , where  $(h_0(P), \dots, h_n(P))$  is the  $h$ -vector of  $P$ . It seems very likely that the above construction of a cylindrical structure on  $\mathbb{C}P^n$  can be extended to quasitoric manifolds.  $\square$

**Example 4.27.** In the same paper, Davis and Januszkiewicz introduced the notion of small covers as a real analogue of quasitoric manifolds by replacing  $S^1$  by  $\mathbb{Z}_2$ . Small covers have many properties in common with quasitoric (or torus) manifolds.

For example, we have  $\mathbb{R}P^n / (\mathbb{Z}_2)^n \cong \Delta^n$  and the stratification  $\pi_n^{\max}$  on  $\Delta^n$  induces the minimal cell decomposition of  $\mathbb{R}P^n$ . An argument analogous to the case of  $\mathbb{C}P^n$  can be used to prove that this stratification is totally normal.  $\square$

**Example 4.28.** Let  $X = D^3$  and consider the cell decomposition

$$X = e_1^0 \cup e_2^0 \cup e_1^1 \cup e_2^1 \cup e_1^2 \cup e_2^2 \cup e^3$$

given as follows. The interior of  $D^3$  is the unique 3-cell and the boundary  $S^2$  is cut into two

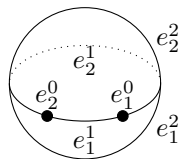


Figure 19: A regular cell decomposition of  $S^2$

2-cells by the equator, which is cut into two 1-cells by two 0-cells on it.

Consider the map between  $S^2$  given by collapsing the shaded region in the figure below (the wedge of two 2-disks embedded in  $S^2$ ) “vertically” and expanding the remaining part of  $S^2$  continuously. Extend it to a continuous map  $\varphi_3 : D^3 \rightarrow D^3$ . It defines a 3-cell structure on  $e^3$ .

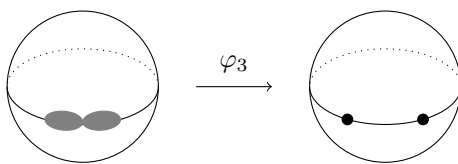


Figure 20: A 3-cell structure which is not cylindrically normal

Let  $P_{(1,1),3} = D^1$ . We have a continuous map

$$b : P_{(1,1),3} \times D^1 \longrightarrow \partial D^3$$

making the diagram

$$\begin{array}{ccc} D^3 & \xrightarrow{\varphi_3} & X \\ \uparrow b & & \uparrow \varphi_{1,1} \\ P_{(1,1),3} \times D^1 & \longrightarrow & D^1 \end{array}$$

commutative. However,  $b$  is not a homeomorphism when restricted to  $P_{(1,1),3} \times \text{Int}(D^1)$ . And this is not a cylindrical structure.  $\square$

In Examples 4.24 and Example 4.25, the restrictions of cell structure maps to the boundary spheres are fiber bundles onto their images. These facts seem to be closely related to the existence of cylindrical structures in these examples. Of course, there are many cylindrically normal cellular stratified spaces that do not have such bundle structures. We may be able to characterize cylindrically normal cellular stratified spaces by using an appropriate notion of stratified fiber bundles. See [Dav78] for example.

**Example 4.29.** Let  $X$  be the subspace of the unit 3-disk  $D^3$  obtained by removing the interior of the 3-disk of radius  $\frac{1}{2}$  centered at the origin. It has a cell decomposition with two 0-cells, a 1-cell, two 2-cells, and a 3-cell depicted as follows. Cells of dimension at most 1 are regular. The

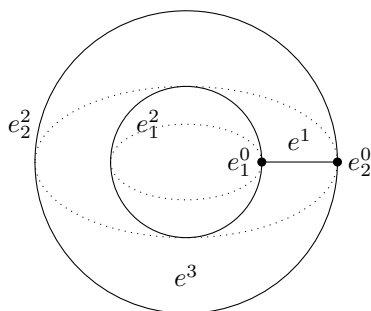


Figure 21: A cell decomposition of thick sphere

cell structure maps for 2-cells are given as in Example 3.19. The restriction of the cell structure map  $\varphi_3$  for the 3-cell to  $S^2$  is given by collapsing a middle part of a sphere to a segment.

This cell decomposition is cylindrically normal, because of the “local triviality” of the middle band. However the restriction  $\varphi_3|_{S^2}$  is not a fiber bundle onto its image.  $\square$

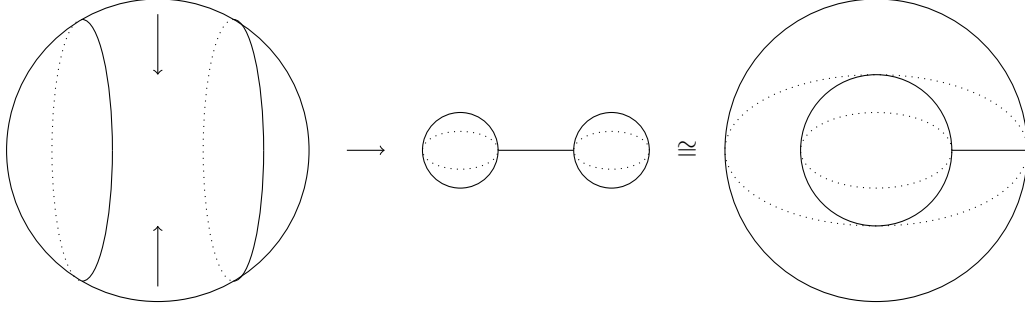


Figure 22: A 3-cell structure on thick sphere

**Example 4.30.** We have seen in Example 3.16 that the standard cell decomposition of the geometric realization of a  $\Delta$ -set is totally normal. Let us consider the geometric realization of a simplicial set  $X$ . Define

$$P(X) = \prod_{n=0}^{\infty} \left( X_n \setminus \bigcup_{i=0}^n s_i(X_{n-1}) \right)$$

to be the set of nondegenerate simplices. For  $\sigma, \tau \in P(X)$ , define  $\tau \leq \sigma$  if there exists an injective morphism  $u : [m] \rightarrow [n]$  with  $X(u)(\sigma) = \tau$ . Define

$$\pi_X : |X| \longrightarrow P(X)$$

analogously to the case of  $\Delta$ -sets. Then  $\pi_X$  is a cellular stratification. In other words, cells in  $|X|$  are in one-to-one correspondence to nondegenerate simplices.

Suppose  $\tau \leq \sigma$  in  $P(X)$ . Then the set

$$\mathcal{P}(\tau, \sigma) = \{u : [m] \rightarrow [n] \mid u \text{ injective and } X(u)(\sigma) = \tau\}$$

is nonempty. For  $u, v \in \mathcal{P}(\tau, \sigma)$ , define  $u \leq v$  if and only if  $u(i) \leq v(i)$  for all  $i \in [m]$ . Let us denote the order complex of this poset by  $P_{\tau, \sigma}$ . This is a simplicial complex whose simplices are indexed by chains in  $\mathcal{P}(\tau, \sigma)$  and can be written as

$$P_{\tau, \sigma} = \bigcup_{k=0}^{\infty} \bigcup_{\mathbf{u} \in N_k(\mathcal{P}(\tau, \sigma))} \Delta^k \times \{\mathbf{u}\}.$$

For a  $k$ -chain  $\mathbf{u} = \{u_0 < \dots < u_k\} \in N_k(\mathcal{P}(\tau, \sigma))$ , define a map

$$\beta^{\mathbf{u}} : [k] \times [m] \longrightarrow [n]$$

by

$$\beta^{\mathbf{u}}(i, j) = u_i(j).$$

This map induces an affine map

$$b^{\mathbf{u}} : (\Delta^k \times \{\mathbf{u}\}) \times \Delta^m \longrightarrow \Delta^n.$$

These maps can be glued together to give us a map

$$b_{\tau, \sigma} : P_{\tau, \sigma} \times \Delta^m \longrightarrow \Delta^n.$$

For  $\sigma_0 < \sigma_1 < \sigma_2$  in  $P(X)$ , the composition

$$\mathcal{P}(\sigma_1, \sigma_2) \times \mathcal{P}(\sigma_0, \sigma_1) \longrightarrow \mathcal{P}(\sigma_0, \sigma_2)$$

is a morphism of posets and induces a map

$$c_{\sigma_0, \sigma_1, \sigma_2} : P_{\sigma_1, \sigma_2} \times P_{\sigma_0, \sigma_1} \longrightarrow P_{\sigma_0, \sigma_2}.$$

It is straightforward to check that these maps satisfy the requirements of a cylindrical structure. Thus the standard cell decomposition of the geometric realization  $|X|$  is cylindrically normal.  $\square$

## 5 Topological Face Categories and Their Classifying Spaces

Recall that the collection of all cells in a regular cell complex  $X$  forms a poset whose order complex is homeomorphic to  $X$ . We also have a face poset for any cellular stratified space. When  $X$  is a non-regular cell complex or a cellular stratified space, however, we cannot expect to recover the homotopy type of  $X$  from its face poset, as we will see in Example 5.3.

The aim of this section is to show that there is a canonical way to construct an acyclic topological category<sup>18</sup>  $C(X)$  from a cylindrically normal cellular stratified space  $X$  and that its classifying space  $BC(X)$  has the same homotopy type as  $X$  under appropriate conditions.

### 5.1 Face Categories

There are several ways to construct a category from a cellular or stellar stratified space. A naive idea is the following.

**Definition 5.1.** For a cellular or a stellar stratified space  $(X, \pi, \Phi)$  and cells  $\varphi_\mu : D_\mu \rightarrow \overline{e_\mu}$  and  $\varphi_\lambda : D_\lambda \rightarrow \overline{e_\lambda}$  with  $e_\mu \subset \overline{e_\lambda}$ , define  $F(X)(e_\mu, e_\lambda)$  to be the set of all maps  $b : D_\mu \rightarrow D_\lambda$  making the following diagram commutative

$$\begin{array}{ccc} D_\lambda & \xrightarrow{\varphi_\lambda} & \overline{e_\lambda} \\ \uparrow b & & \uparrow \downarrow \\ D_\mu & \xrightarrow{\varphi_\mu} & \overline{e_\mu} \end{array}$$

The set  $F(X)(e_\mu, e_\lambda)$  is topologized by the compact-open topology as a subspace of  $\text{Map}(D_\mu, D_\lambda)$ .  $F(X)(e_\mu, e_\lambda)$  is defined to be empty if  $e_\mu \not\subset \overline{e_\lambda}$ .

By defining the set of objects to be cells in  $X$  and morphisms from  $e_\mu$  to  $e_\lambda$  to be  $F(X)(e_\mu, e_\lambda)$ , we obtain a topological category  $F(X)$ . The composition is given by the composition of maps. This topological category  $F(X)$  is called the *naive face category* of  $X$ . It is also denoted by  $F(X, \pi)$  or  $F(X, \pi, \Phi)$ .

**Lemma 5.2.** *The naive face category  $F(X)$  is an acyclic category. When  $X$  is regular,  $F(X)$  is a poset and coincides with the face poset  $P(X)$ . In particular, when  $X$  is a regular cell complex, our construction coincides with the classical face poset construction.*

<sup>18</sup>See Appendix C for basics of topological categories.

*Proof.* If both  $F(X)(e, e')$  and  $F(X)(e', e)$  are nonempty, we have  $\dim e = \dim e'$ , since the existence of a morphism  $e \rightarrow e'$  in  $F(X)$  implies  $\dim e \leq \dim e'$ . The compatibility of lifts with cell structure maps implies that the only case we have a morphism is  $e = e'$  and the morphism should be the identity. Thus  $F(X)$  is acyclic.

When  $X$  is regular, the regularity implies that there is at most one morphism between two objects. Hence it is a poset.  $\square$

Even when  $X$  is not regular, we have the underlying poset<sup>19</sup>, since  $F(X)$  is an acyclic category. Obviously it is isomorphic to the face poset  $P(X, \pi) = \text{Im } \pi$ .

**Example 5.3.** Consider the minimal cell decomposition

$$\pi_n : S^n = e^0 \cup e^n \longrightarrow \{0 < n\}.$$

This is a typical non-regular cell complex. The face poset  $P(S^n, \pi_n)$  is a totally ordered set of two elements and its order complex is homeomorphic to an interval  $BP(S^n, \pi_n) \cong [0, 1]$ . The homotopy type of  $S^n$  cannot be recovered from its face poset.

On the other hand, the face category  $F(S^n, \pi_n)$  has more information. It has two objects  $e^0$  and  $e^n$ . When  $n = 1$ , as we have seen in Example 3.13, the set of morphisms from  $e^0$  to  $e^1$  is given by

$$F(S^1, \pi_1)(e^0, e^1) = \{b_1, b_{-1}\}.$$

We also have the identity morphism on each object. The resulting category is depicted in Figure 23. When  $n > 1$ , there are infinitely many morphisms from  $e^0$  to  $e^n$  parametrized by  $\partial D^n$ . And

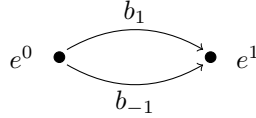


Figure 23: The face category of  $S^1 = e^0 \cup e^1$

we have a homeomorphism

$$F(S^n, \pi_n)(e^0, e^n) \cong S^{n-1}.$$

$\square$

In the above example of  $S^n$ , the compact-open topology on the morphism space  $F(S^n, \pi_n)(e^0, e^n)$  can be replaced with a more understandable topology of  $S^{n-1}$ . In general, we cannot expect such simplicity. Under the assumption of cylindrical normality, however, we may define a smaller face category.

**Definition 5.4.** Let  $X$  be a cylindrically normal stellar stratified space. Define a category  $C(X)$  as follows. Objects are cells in  $X$ . For each pair  $e_\mu \subset \bar{e}_\lambda$ , define

$$C(X)(e_\mu, e_\lambda) = P_{\mu, \lambda}.$$

The composition of morphisms is given by

$$c_{\lambda_0, \lambda_1, \lambda_2} : P_{\lambda_1, \lambda_2} \times P_{\lambda_0, \lambda_1} \longrightarrow P_{\lambda_0, \lambda_2}.$$

The category  $C(X)$  is called the *cylindrical face category* of  $X$ .

<sup>19</sup>Definition C.6

**Lemma 5.5.** *For any cylindrically normal stellar stratified space  $X$ , its face category  $C(X)$  is an acyclic topological category. the maps  $b_{\mu,\lambda}$  induces a continuous functor*

$$b : C(X) \longrightarrow F(X),$$

*which is natural with respect to morphisms of cylindrically normal stellar stratified spaces.*

*Furthermore the underlying poset<sup>20</sup> of  $C(X)$  is also  $P(X)$  and the diagram*

$$\begin{array}{ccc} C(X) & \xrightarrow{b} & F(X) \\ & \searrow & \swarrow \\ & P(X) & \end{array}$$

*is commutative.*

*Proof.* The continuity of

$$b_{\mu,\lambda} : P_{\mu,\lambda} \times D_\mu \longrightarrow D_\lambda$$

implies the continuity of its adjoint

$$\text{ad}(b_{\mu,\lambda}) : P_{\mu,\lambda} \longrightarrow \text{Map}(D_\mu, D_\lambda),$$

which factors through  $F(X)(e_\mu, e_\lambda)$ . It is immediate to verify that these maps form a continuous functor

$$b : C(X) \longrightarrow F(X).$$

Morphisms of cylindrically normal stellar stratified spaces are required to be compatible with maps  $b_{\mu,\lambda}$  and thus the functor  $b$  is natural with respect to morphisms of cylindrically normal stellar stratified spaces. The commutativity of the triangle is obvious from the definition.  $\square$

**Example 5.6.** Consider the minimal cell decomposition on  $\mathbb{C}P^2$

$$\mathbb{C}P^2 = e^0 \cup e^2 \cup e^4.$$

It is shown in Example 4.25 that it has a cylindrical structure. The cylindrical face category  $C(\mathbb{C}P^2)$  has three objects,  $e^0$ ,  $e^2$ , and  $e^4$ . We have seen that

$$F(\mathbb{C}P^2)(e^0, e^2) = F(S^2, \pi_2)(e^0, e^2) \cong S^1 = C(\mathbb{C}P^2)(e^0, e^2).$$

Since the attaching map of  $e^4$  is the Hopf map

$$\eta : S^3 \longrightarrow \mathbb{C}P^1,$$

we have

$$F(\mathbb{C}P^2)(e^0, e^4) \cong \eta^{-1}(e^0) \cong S^1 = C(\mathbb{C}P^2)(e^0, e^4).$$

By using the local trivialization

$$\eta^{-1}(e^2) \cong e^2 \times S^1,$$

we see that  $F(\mathbb{C}P^2)(e^2, e^4)$  is the set of sections of the trivial bundle

$$D^2 \times S^1 \longrightarrow D^2$$

---

<sup>20</sup>Definition C.6



and thus  $F(\mathbb{CP}^2)(e^2, e^4) = \text{Map}(D^2, S^1)$ . On the other hand, we have

$$C(\mathbb{CP}^2)(e^2, e^4) = S^1$$

by definition. The composition

$$C(\mathbb{CP}^2)(e^2, e^4) \times C(\mathbb{CP}^2)(e^0, e^2) \longrightarrow C(\mathbb{CP}^2)(e^0, e^4)$$

is given by the multiplication of  $S^1$ , as is shown in Example 4.25.

In general, Example 4.26 says that the face category  $C(\mathbb{CP}^n)$  of the minimal cell decomposition of  $\mathbb{CP}^n$  can be described as a “poset enriched by  $S^1$ ” in the sense that, for any pair of

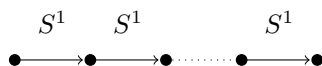


Figure 24: The face category of  $\mathbb{CP}^n$

objects  $e^{2k}, e^{2m}$  ( $k < m$ ), the space of morphisms  $C(\mathbb{CP}^n)(e^{2k}, e^{2m})$  is  $S^1$  and the composition of morphisms is given by the group structure of  $S^1$ .  $\square$

Recall that the order complex of the face poset of a regular cell complex  $X$  is the barycentric subdivision of  $X$ . With this fact in mind, we introduce the following notation.

**Definition 5.7.** Let  $X$  be a cylindrically normal stellar stratified space. Define its *barycentric subdivision*  $\text{Sd}(X)$  to be the classifying space of the cylindrical face category

$$\text{Sd}(X) = BC(X).$$

**Remark 5.8.** There is a notion of barycentric subdivision  $\text{Sd}(C)$  of a small category  $C$ . A good reference is a paper [dH08] by del Hoyo. See also Noguchi’s papers [Nog11, Nog13]. We will show that, for a totally normal stellar stratified space  $X$ , there is an isomorphism of categories  $\text{Sd}(C(X)) \cong C(\text{Sd}(X))$  in §7.

When  $X$  is not a regular cell complex, we usually do not have a homeomorphism between  $\text{Sd}(X)$  and  $X$ .

**Example 5.9.** Consider  $X = \mathbb{R}^n$ . This is a regular totally normal cellular stratification consisting of a single  $n$ -cell. The barycentric subdivision is a single point.  $\square$

**Example 5.10.** Consider the minimal cell decomposition  $\pi_n$  of  $S^n$ . When  $n = 1$ , it is easy to see that  $\text{Sd}(S^1, \pi_1)$  is the cell complex in Figure 25 and is homeomorphic to  $S^1$ . Note that



Figure 25:  $\text{Sd}(S^1, \pi_1)$

this complex is obtained by subdividing the 1-cell in  $\pi_1$  and can be regarded as the barycentric subdivision of  $\pi_1$ .

When  $n > 1$ ,  $C(S^n, \pi_n)$  is a topological category with nontrivial topology on  $C(S^n, \pi_n)(e^0, e^n)$ . Since we have a homeomorphism

$$C(S^n, \pi_n)(e^0, e^n) \cong S^{n-1},$$

it is easy to determine  $\text{Sd}(S^n, \pi_n)$  and we have

$$\text{Sd}(S^n, \pi_n) = BC(S^n, \pi_n) \cong \Sigma(S^{n-1}) \cong S^n.$$

Again we recovered  $S^n$ . □

**Example 5.11.** Consider  $X = \text{Int}D^n \cup \{(1, 0)\}$  with the obvious stratification.

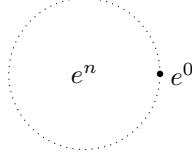


Figure 26:  $\text{Int}D^n \cup \{(1, 0)\}$

This is a regular cellular stratification and  $\text{Sd}(X)$  is a 1-simplex  $[0, 1]$ . We have an embedding

$$i : [0, 1] \longrightarrow X$$

by

$$i(t) = (1 - t)(1, 0) + t(0, 0).$$

See Figure 27. Obviously  $i([0, 1])$  is a strong deformation retract of  $X$ . □

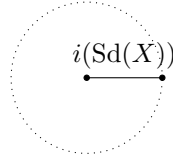


Figure 27:  $i(\text{Sd}(\text{Int}D^2 \cup \{(1, 0)\}))$

**Example 5.12.** Consider the punctured torus in Example 3.15. There is a totally normal cellular stratification on  $X = S^1 \times S^1 \setminus e^0 \times e^0$  induced from the product cell decomposition  $\pi_1^2$

$$S^1 \times S^1 = e^0 \times e^0 \cup e^0 \times e^1 \cup e^1 \times e^0 \cup e^1 \times e^1.$$

Let

$$\varphi_{1,1} : D_{1,1} = [-1, 1]^2 \setminus \{(-1, -1), (-1, 1), (1, -1), (1, 1)\} \longrightarrow X$$

be the cell structure map of the 2-cell in  $X$  and

$$\begin{aligned} \varphi_{0,1} & : D_{0,1} = (-1, 1) \longrightarrow X \\ \varphi_{1,0} & : D_{1,0} = (-1, 1) \longrightarrow X \end{aligned}$$

be the cell structure maps for 1-cells.

As we have seen in Example 3.15, there are two ways to lift each cell structure map of a 1-cell and these four lifts cover  $\partial D_{1,1}$ .

$$\partial D_{1,1} = b'_{0,1}(D_{0,1}) \cup b''_{0,1}(D_{0,1}) \cup b'_{1,0}(D_{1,0}) \cup b''_{1,0}(D_{1,0}).$$

The cylindrical face category  $C(X)$  consists of three objects  $e^0 \times e^1$ ,  $e^1 \times e^0$ ,  $e^1 \times e^1$ . Nontrivial morphisms are

$$\begin{aligned} C(X)(e^0 \times e^1, e^1 \times e^1) &= \{b'_{0,1}, b''_{0,1}\}, \\ C(X)(e^1 \times e^0, e^1 \times e^1) &= \{b'_{1,0}, b''_{1,0}\}. \end{aligned}$$

By allowing multiple edges, we can draw a ‘‘Hasse diagram’’ of this acyclic category as in Figure 28.

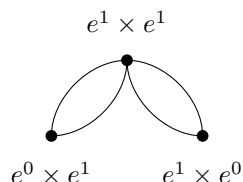


Figure 28: The barycentric subdivision of the punctured torus

Obviously, the classifying space of this category is the wedge of two circles,

$$\text{Sd}(X) = BC(X) = S^1 \vee S^1.$$

It is easy to find an embedding of  $\text{Sd}(X)$  into  $X$  by using lifts of cell structure maps. The images of 0 under the four maps  $b'_{0,1}$ ,  $b''_{0,1}$ ,  $b'_{1,0}$ ,  $b''_{1,0}$  constitute four points in  $\partial D_{1,1}$  that are mapped to a single point under  $\varphi_{1,1}$ . By connecting each of these four points and  $(0, 0)$  in  $D_{1,1}$  by a segment, respectively, we obtain a 1-dimensional stratified space  $\tilde{K}$  in  $D_{1,1}$ . See Figure 29.

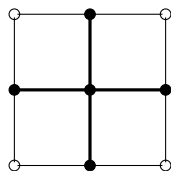


Figure 29: A thin subcomplex  $\tilde{K}$  in  $D_{1,1}$

The complex  $\tilde{K}$  in Figure 29 corresponds to cells in  $\text{Sd}(X)$  and we have an embedding

$$\text{Sd}(X) \hookrightarrow X.$$

Figure 29 can be also used to construct a deformation retraction of  $X$  onto  $\text{Sd}(X)$ . □

The above examples show that, when a cellular stratification  $\pi$  is totally normal, or more generally, cylindrically normal, the barycentric subdivision  $\text{Sd}(X, \pi)$  is closely related to  $X$ .

The work of Cohen, Jones, and Segal [CJS] suggests that one should analyze the nerve of the face category of a cylindrically normal stellar stratified space by using the underlying poset functor

$$\pi : C(X, \pi) \longrightarrow P(X, \pi).$$

The following easily verifiable fact will be used in the next section.

**Lemma 5.13.** *For a cylindrically normal stellar stratified space  $(X, \pi)$ , consider the induced morphism of simplicial sets<sup>21</sup>*

$$N(\pi) : N(C(X, \pi)) \longrightarrow N(P(X, \pi)).$$

For each  $n$ -chain in the face poset  $\mathbf{e} = (e_{\lambda_0}, \dots, e_{\lambda_n}) \in N_n(P(X, \pi))$ , we have

$$N(\pi)_n^{-1}(\mathbf{e}) = P_{\lambda_{n-1}, \lambda_n} \times \cdots \times P_{\lambda_0, \lambda_1}.$$

Consequently the space of  $n$ -chains has the following decomposition

$$N_n(C(X, \pi)) = \coprod_{\mathbf{e} \in N_n(P(X, \pi))} \{\mathbf{e}\} \times P_{\lambda_{n-1}, \lambda_n} \times \cdots \times P_{\lambda_0, \lambda_1}.$$

The space of nondegenerate  $n$ -chains is, therefore, given by

$$\bar{N}_n(C(X, \pi)) = \coprod_{\mathbf{e} \in \bar{N}_n(P(X, \pi))} \{\mathbf{e}\} \times P_{\lambda_{n-1}, \lambda_n} \times \cdots \times P_{\lambda_0, \lambda_1}.$$

By using the face category, we may regard a cylindrically normal stellar stratified space as a functor.

**Definition 5.14.** For a cylindrically normal stellar stratified space  $X$ , define a functor

$$D^X : C(X) \longrightarrow \mathbf{Spaces}$$

by assigning the domain  $D_\lambda$  to each cell  $\varphi_\lambda : D_\lambda \rightarrow \bar{e}_\lambda$ . For a morphism  $p \in C(X)(e_\mu, e_\lambda) = P_{\mu, \lambda}$ , define  $D^X(p) = b_{\mu, \lambda}(p) \in \text{Map}(D_\mu, D_\lambda)$ .

The following is an extension of Proposition 2.47 of [FMT15].

**Proposition 5.15.** *For a CW cylindrically normal stellar stratified space  $X$ , the functor  $D^X$  is a continuous functor<sup>22</sup> and we have a natural homeomorphism*

$$\text{colim}_{C(X)} D^X \xrightarrow{\cong} X.$$

*Proof.* The continuity of the functor  $D^X$  is obvious from the definition.

By definition,  $\text{colim}_{C(X)} D^X$  is a quotient space of  $\text{tot}(D^X) = D(X)$ . Let  $\sim_c$  be the defining equivalence relation so that  $\text{colim}_{C(X)} D^X = D(X)/\sim_c$ . On the other hand, by Lemma 2.55, we have a description of  $X$  as a quotient of  $D(X)$

$$X \cong D(X)/\sim_\Phi,$$

where the relation  $\sim_\Phi$  is defined by  $x \sim_\Phi y$  if and only if  $\varphi_\mu(x) = \varphi_\lambda(y)$  for  $x \in D_\mu$  and  $y \in D_\lambda$ .

It remains to verify that  $\sim_c$  is identical to  $\sim_\Phi$ . The proof is essentially the same as that of Proposition 2.47 in [FMT15]. Details are omitted.  $\square$

<sup>21</sup>Here we forget the topology on  $C(X, \pi)$  temporarily.

<sup>22</sup>See Definition C.10 for the definitions of the continuity of a functor to  $\mathbf{Spaces}$  and its colimit.

## 5.2 Barycentric Subdivisions of Cellular Stratified Spaces

As we can see from the examples in §5.1, we often have an embedding of the barycentric subdivision  $\text{Sd}(X)$  in  $X$ , even if  $X$  is neither a cell complex nor regular. In this section, we first show that such an embedding always exists for any cylindrically normal cellular stratified space. And then we show that it often admits a strong deformation retraction.

In order to describe our embedding, we use the language of  $\Delta$ -spaces<sup>23</sup>.

**Theorem 5.16.** *Let  $(X, \pi)$  be a cylindrically normal CW stellar stratified space. There exists an embedding*

$$i_X : \text{Sd}(X, \pi) \hookrightarrow X,$$

*which is natural with respect to strict morphisms of cylindrically normal stellar stratified spaces. Furthermore, when all cells in  $X$  are closed,  $i_X$  is a homeomorphism onto  $X$ .*

The construction of  $i_X$  and the proof of the fact that  $i_X$  is an embedding are essentially the same as those of Proposition 2.51 in [FMT15]. We record them for the convenience of the reader.

**Definition 5.17.** We construct  $i_X$  as follows. Since the face category  $C(X, \pi) = C(X)$  is acyclic,  $\text{Sd}(X) = BC(X) = \|\overline{N}(C(X))\|$ . Thus it suffices to construct a series of maps

$$i_n : \overline{N}_n(C(X)) \times \Delta^n \longrightarrow X$$

making the following diagram commutative for all  $i$

$$\begin{array}{ccc} \overline{N}_n(C(X, \pi)) \times \Delta^{n-1} & \xrightarrow{1 \times d^i} & \overline{N}_n(C(X, \pi)) \times \Delta^n \\ d_i \times 1 \downarrow & & \downarrow i_n \\ \overline{N}_{n-1}(C(X, \pi)) \times \Delta^{n-1} & \xrightarrow{i_{n-1}} & X. \end{array}$$

Let us construct  $i_n$  by induction on  $n$ . The set  $\overline{N}_0(C(X))$  is the set of cells for  $\pi$ . For each cell  $e_\lambda$  with cell structure map  $\varphi_\lambda : D_\lambda \rightarrow \overline{e}_\lambda \subset X$ , define

$$i_0(e_\lambda) = \varphi_\lambda(0)$$

and we obtain a map

$$i_0 : \overline{N}_0(C(X)) \cong \overline{N}_0(C(X)) \times \Delta^0 \hookrightarrow X.$$

Note that this map makes the diagram

$$\begin{array}{ccc} \overline{N}_0(C(X, \pi)) \times \Delta^0 & \xrightarrow{i_0} & X \\ \downarrow z_0 & \nearrow \Phi & \\ D(X) & & \end{array}$$

commutative, where  $z_0$  is induced by the inclusion of  $\Delta^0$  to the center of each disk and  $D(X)$  and  $\Phi$  are defined in Lemma 2.55.

<sup>23</sup>A  $\Delta$ -space is a simplicial space without degeneracies. See Definition B.14.

Suppose that we have constructed a map

$$i_{k-1} : \overline{N}_{k-1}(C(X)) \times \Delta^{k-1} \longrightarrow X$$

satisfying the above compatibility conditions and that the restriction of  $i_{k-1}$  to  $\overline{N}_{k-1}(C(X)) \times \text{Int}(\Delta^{k-1})$  is an embedding. Suppose, further, that there exists a map

$$z_{k-1} : \overline{N}_{k-1}(C(X)) \times \Delta^{k-1} \longrightarrow D(X)$$

making the diagram

$$\begin{array}{ccc} \overline{N}_{k-1}(C(X)) \times \Delta^{k-1} & \xrightarrow{i_{k-1}} & X \\ \downarrow z_{k-1} & \nearrow \Phi & \\ D(X) & & \end{array}$$

commutative. We construct an embedding

$$z_k : \overline{N}_k(C(X)) \times \Delta^k \longrightarrow D(X)$$

satisfying the compatibility conditions corresponding to those of  $i_k$  and define  $i_k$  to be  $\Phi \circ z_k$ .

Under the decomposition in Lemma 5.13

$$\overline{N}_k(C(X)) = \coprod_{e \in \overline{N}_k(P(X))} \{e\} \times \overline{N}(\pi)_k^{-1}(e),$$

it suffices to construct a map

$$z_e : \overline{N}(\pi)_k^{-1}(e) \times \Delta^k \longrightarrow D_{\lambda_k}$$

for each nodegenerate  $k$ -chain  $e : e_{\lambda_0} < \dots < e_{\lambda_k}$  in  $P(X)$ . The maps  $\{z_e\}$  should satisfy the following conditions:

1. For each  $0 \leq j < k$ , the following diagram is commutative:

$$\begin{array}{ccc} \overline{N}(\pi)_k^{-1}(e) \times \Delta^k & \xrightarrow{z_e} & D_{\lambda_k} \\ \uparrow 1 \times d^j & & \uparrow z_{d_j(e)} \\ \overline{N}(\pi)_k^{-1}(e) \times \Delta^{k-1} & \xrightarrow{d_j \times 1} & \overline{N}(\pi)_{k-1}^{-1}(d_j(e)) \times \Delta^{k-1}. \end{array} \quad (2)$$

2. When  $j = k$ , the following diagram is commutative:

$$\begin{array}{ccc} \overline{N}(\pi)_k^{-1}(e) \times \Delta^k & \xrightarrow{z_e} & D_{\lambda_k} \\ \uparrow 1 \times d^k & & \uparrow b_{\lambda_{k-1}, \lambda_k} \\ \overline{N}(\pi)_k^{-1}(e) \times \Delta^{k-1} & & \\ \parallel & & \\ P_{\lambda_{k-1}, \lambda_k} \times \overline{N}(\pi)_{k-1}^{-1}(d_k(e)) \times \Delta^{k-1} & \xrightarrow{1 \times z_{d_k(e)}} & P_{\lambda_{k-1}, \lambda_k} \times D_{\lambda_{k-1}}. \end{array} \quad (3)$$

By the inductive assumption, we have an embedding

$$z_{d_k(\mathbf{e})} : \overline{N}(\pi)_{k-1}^{-1}(d_k(\mathbf{e})) \times \Delta^{k-1} \longrightarrow D_{\lambda_{k-1}}$$

corresponding to the  $(k-1)$ -chain  $d_k(\mathbf{e}) : e_{\lambda_0} < \cdots < e_{\lambda_{k-1}}$ . Note that

$$\overline{N}(\pi)_k^{-1}(\mathbf{e}) = P_{\lambda_{k-1}, \lambda_k} \times \overline{N}(\pi)_{k-1}^{-1}(d_k(\mathbf{e})).$$

Compose with  $b_{\lambda_{k-1}, \lambda_k}$  and we obtain a map

$$\begin{aligned} \overline{N}(\pi)_k^{-1}(\mathbf{e}) \times \Delta^{k-1} &= P_{\lambda_{k-1}, \lambda_k} \times \overline{N}(\pi)_{k-1}^{-1}(d_k(\mathbf{e})) \times \Delta^{k-1} \\ &\xrightarrow{1 \times z_{d_k(\mathbf{e})}} P_{\lambda_{k-1}, \lambda_k} \times D_{\lambda_{k-1}} \\ &\xrightarrow{b_{\lambda_{k-1}, \lambda_k}} \partial D_{\lambda_k}. \end{aligned}$$

Since  $D_{\lambda_k}$  is an aster<sup>24</sup>, it can be extended to an embedding

$$z_{\mathbf{e}} : \overline{N}(\pi)_k^{-1}(\mathbf{e}) \times \Delta^k = \overline{N}(\pi)_k^{-1}(\mathbf{e}) \times \Delta^{k-1} * \mathbf{v}_k \longrightarrow \partial D_{\lambda_k} * 0 \subset D_{\lambda_k}$$

by

$$z_{\mathbf{e}}(\mathbf{p}, (1-t)\mathbf{s} + t\mathbf{v}_k) = (1-t)z_{d_k(\mathbf{e})}(\mathbf{p}, \mathbf{s}) + t \cdot 0 = (1-t)z_{d_k(\mathbf{e})}(\mathbf{p}, \mathbf{s}),$$

where  $\mathbf{v}_k = (0, \dots, 0, 1)$  is the last vertex in  $\Delta^k$ . Recall that the join operation  $*$  used in this paper is defined by connecting points by line segments<sup>25</sup> and is not the same as the join operation used in algebraic topology, for example in Milnor's paper [Mil56].

By definition,  $z_{\mathbf{e}}$  makes the diagram (3) commutative. Let us verify that  $z_{\mathbf{e}}$  also makes the diagram (2) commutative for  $0 \leq j < k$ . By the inductive assumption, the following diagram is commutative:

$$\begin{array}{ccc} \overline{N}(\pi)_{k-1}^{-1}(d_k(\mathbf{e})) \times \Delta^{k-2} & \xrightarrow{d_j \times 1} & \overline{N}(\pi)_{k-2}^{-1}(d_j d_k(\mathbf{e})) \times \Delta^{k-2} & \xlongequal{\quad} & \overline{N}(\pi)_{k-2}^{-1}(d_{k-1} d_j(\mathbf{e})) \times \Delta^{k-2} \\ \downarrow 1 \times d^j & & \downarrow z_{d_j d_k(\mathbf{e})} & \swarrow z_{d_{k-1} d_j(\mathbf{e})} & \\ \overline{N}(\pi)_{k-1}^{-1}(d_k(\mathbf{e})) \times \Delta^{k-1} & \xrightarrow{z_{d_k(\mathbf{e})}} & D_{\lambda_{k-1}} & & \end{array}$$

Under the identification  $\Delta^k = \Delta^{k-1} * \mathbf{v}_k$ ,  $d^j : \Delta^{k-1} \rightarrow \Delta^k$  can be identified with the composition

$$\Delta^{k-2} * \mathbf{v}_{k-1} \cong \Delta^{k-2} * \mathbf{v}_k \xrightarrow{d^j * 1_{\mathbf{v}_k}} \Delta^{k-1} * \mathbf{v}_k.$$

On the other hand, since  $j < k$ , we have

$$\begin{aligned} \overline{N}(\pi)_k(\mathbf{e}) &= P_{\lambda_{k-1}, \lambda_k} \times \overline{N}(\pi)_{k-1}^{-1}(d_k(\mathbf{e})) \\ \overline{N}(\pi)_{k-1}(d_j(\mathbf{e})) &= P_{\lambda_{k-1}, \lambda_k} \times \overline{N}(\pi)_{k-2}^{-1}(d_{k-1} d_j(\mathbf{e})), \end{aligned}$$

and the face operator  $d_j : \overline{N}(\pi)_k(\mathbf{e}) \rightarrow \overline{N}(\pi)_{k-1}(d_j(\mathbf{e}))$  coincides with the map

$$P_{\lambda_{k-1}, \lambda_k} \times \overline{N}(\pi)_{k-1}^{-1}(d_k(\mathbf{e})) \xrightarrow{1 \times d_j} P_{\lambda_{k-1}, \lambda_k} \times \overline{N}(\pi)_{k-2}^{-1}(d_{k-1} d_j(\mathbf{e})) = P_{\lambda_{k-1}, \lambda_k} \times \overline{N}(\pi)_{k-1}^{-1}(d_{k-1} d_j(\mathbf{e})).$$

<sup>24</sup>Definition 2.47

<sup>25</sup>Definition B.20.

Thus we obtain the commutative diagram

$$\begin{array}{ccc}
\overline{N}(\pi)_k^{-1}(\mathbf{e}) \times \Delta^{k-1} & \xrightarrow{d_j \times 1} & \overline{N}(\pi)_{k-1}^{-1}(d_j(\mathbf{e})) \times \Delta^{k-1} \\
\parallel & & \parallel \\
P_{\lambda_{k-1}, \lambda_k} \times \overline{N}(\pi)_{k-1}^{-1}(d_k(\mathbf{e})) \times \Delta^{k-1} & \xrightarrow{1 \times d_j \times 1} & P_{\lambda_{k-1}, \lambda_k} \times \overline{N}(\pi)_{k-2}^{-1}(d_{k-1}d_j(\mathbf{e})) \times \Delta^{k-1} \\
\downarrow 1 \times 1 \times d^j & & \downarrow 1 \times \tilde{z}_{d_{k-1}d_j(\mathbf{e})} \\
P_{\lambda_{k-1}, \lambda_k} \times \overline{N}(\pi)_{k-1}^{-1}(d_k(\mathbf{e})) \times \Delta^k & \xrightarrow{1 \times \tilde{z}_{d_k(\mathbf{e})}} & P_{\lambda_{k-1}, \lambda_k} \times ((D_{\lambda_{k-1}} \times \{0\}) * (0, 1))
\end{array}$$

where we embed  $D_{\lambda_{k-1}}$  into  $D_{\lambda_{k-1}} \times \mathbb{R}$  as  $D_{\lambda_{k-1}} \times \{0\}$  and  $\tilde{z}_{d_k(\mathbf{e})}$  is defined by

$$\tilde{z}_{d_k(\mathbf{e})}(\mathbf{p}, (1-t)\mathbf{s} + t\mathbf{v}_k) = (1-t)(z_{d_k(\mathbf{e})}(\mathbf{p}, \mathbf{s}), 0) + t(0, 1).$$

The map  $\tilde{z}_{d_{k-1}d_j(\mathbf{e})}$  is defined analogously. We also have an extension of  $b_{\lambda_{k-1}, \lambda_k}$

$$\tilde{b}_{\lambda_{k-1}, \lambda_k} : P_{\lambda_{k-1}, \lambda_k} \times ((D_{\lambda_{k-1}} \times \{0\}) * (0, 1)) \longrightarrow D_{\lambda_k}.$$

By definition, we have the commutative diagram

$$\begin{array}{ccc}
P_{\lambda_{k-1}, \lambda_k} \times \overline{N}(\pi)_{k-2}^{-1}(d_{k-1}d_j(\mathbf{e})) \times \Delta^{k-1} & \xlongequal{\quad} & \overline{N}(\pi)_{k-1}^{-1}(d_j(\mathbf{e})) \times \Delta^{k-1} \\
\downarrow 1 \times \tilde{z}_{d_{k-1}d_j(\mathbf{e})} & & \downarrow z_{d_j(\mathbf{e})} \\
P_{\lambda_{k-1}, \lambda_k} \times ((D_{\lambda_{k-1}} \times \{0\}) * (0, 1)) & \xrightarrow{\tilde{b}_{\lambda_{k-1}, \lambda_k}} & D_{\lambda_k} \\
\uparrow 1 \times \tilde{z}_{d_k(\mathbf{e})} & & \uparrow z_{\mathbf{e}} \\
P_{\lambda_{k-1}, \lambda_k} \times \overline{N}(\pi)_{k-1}^{-1}(d_k(\mathbf{e})) \times \Delta^k & \xlongequal{\quad} & \overline{N}(\pi)_k^{-1}(\mathbf{e}) \times \Delta^k.
\end{array}$$

This completes the proof of the commutativity of the diagram (2). And we obtain a map

$$z_k : \overline{N}_k(C(X)) \times \Delta^k \longrightarrow D(X)$$

for all  $k$ . By composing with  $\Phi : D(X) \rightarrow X$ , we obtain a family of continuous maps

$$\{i_k : \overline{N}_k(C(X)) \times \Delta^k \longrightarrow X\}_{k \geq 0}$$

that are compatible with  $d_i$ , which induces a continuous map

$$i_X : \text{Sd}(X) = \|\overline{N}(C(X))\| \longrightarrow X.$$

The next step is to prove that  $i_X$  is an embedding.

*Proof of Theorem 5.16.* Let us show that  $i_X : BC(X) \rightarrow \text{Im}(i_X)$  is a bijective closed map, hence is a homeomorphism. The bijectivity is obvious from the construction. In order to show that  $i_X$



is a closed map, consider the diagram

$$\begin{array}{ccc}
\coprod_{n, \mathbf{e} \in \overline{N}_n(P(X))} \{\mathbf{e}\} \times \overline{N}(\pi)_n^{-1}(\mathbf{e}) \times \Delta^n & \xrightarrow{\coprod_{\mathbf{e}} z_{\mathbf{e}}} & D(X) \\
\downarrow p & & \downarrow \Phi \\
\|\overline{N}(C(X))\| & \xrightarrow{i_X} & X
\end{array}$$

whose vertical arrows are quotient maps. For simplicity, let us denote  $z = \coprod_{\mathbf{e}} z_{\mathbf{e}}$ . For  $A \subset \text{Sd}(X)$ , we show that

$$\Phi^{-1}(i_X(A)) = z(p^{-1}(A)), \quad (4)$$

which immediately implies that  $i_X$  is a closed map onto its image, since  $\Phi$  is a quotient map and  $z$  is an embedding.

The commutativity of the above diagram implies

$$\Phi^{-1}(i_X(A)) \supset z(p^{-1}(A)).$$

On the other hand, suppose  $x \in \Phi^{-1}(i_X(A))$ . Then there exist  $a \in A$  and  $\lambda \in \Lambda$  with  $\Phi(x) = i_X(a)$  and  $x \in D_\lambda$ . Write  $a = p(\mathbf{b}, \mathbf{t})$  for  $\mathbf{b} \in \overline{N}(\pi)_k^{-1}(\mathbf{e})$  and  $\mathbf{t} \in \Delta^k$ .

If  $x \in \text{Int}(D_\lambda)$ , we have

$$\varphi_\lambda(x) = i_X(a) = i_X(p(\mathbf{b}, \mathbf{t})) = \Phi(z_{\mathbf{e}}(\mathbf{b}, \mathbf{t})).$$

Since  $x \in \text{Int}(D_\lambda)$ ,  $\mathbf{t}$  is of the form  $\mathbf{t} = (1-t)\mathbf{s} + t\mathbf{v}_k$  for some  $0 < t < 1$  and  $\mathbf{s} \in \Delta^{k-1}$ . This implies that  $z_{\mathbf{e}}(\mathbf{b}, \mathbf{t}) \in \text{Int}(D_\lambda)$  and  $\varphi_\lambda(x) = \varphi_\lambda(z_{\mathbf{e}}(\mathbf{b}, \mathbf{t}))$ . Thus  $x = z_{\mathbf{e}}(\mathbf{b}, \mathbf{t})$  with  $p(\mathbf{b}, \mathbf{t}) = a \in A$ . In other words,  $x \in z(p^{-1}(A))$ .

Suppose  $x \in \partial D_\lambda$  and let  $e_\mu$  be a cell containing  $\varphi_\lambda(x)$ . By the cylindrical normality,  $\partial D_\lambda$  is a stratified space and

$$b_{\mu, \lambda} : P_{\mu, \lambda} \times D_\mu \longrightarrow \partial D_\lambda$$

is a strict morphism of stratified space. Choose  $(b, y) \in P_{\mu, \lambda} \times D_\mu$  with  $b_{\mu, \lambda}(b, y) = x$ . Then  $\varphi_\mu(y) = \varphi_\lambda(x) = \Phi(x) \in i_X(A)$ . Since  $y \in \text{Int}(D_\mu)$ , the above argument implies that there exists  $(\mathbf{b}', \mathbf{t}') \in p^{-1}(A)$  with  $y = z_{\pi(\mathbf{b}')}(\mathbf{b}', \mathbf{t}')$ . Define  $\mathbf{b} = (b, \mathbf{b}')$  and  $\mathbf{t} = d^k(\mathbf{t}')$ . Then the defining relation of the geometric realization of a  $\Delta$ -space implies that

$$p(\mathbf{b}, \mathbf{t}) = [b, d^k(\mathbf{t}')] = [d_k(b, \mathbf{t}')] = [b', \mathbf{t}'] = p(\mathbf{b}', \mathbf{t}') \in A.$$

On the other hand, the commutativity of (3) implies that

$$x = b_{\mu, \lambda}(b, y) = b_{\mu, \lambda}(b, z_{p(\mathbf{b}')}(\mathbf{b}', \mathbf{t}')) = z_{p(b, \mathbf{b}')} (p(b, \mathbf{b}'), d^k(\mathbf{t}')) = z_{p(\mathbf{b})}(\mathbf{b}, \mathbf{t}).$$

And we have  $x \in z(p^{-1}(A))$ . This completes the proof of (4).

Finally when all cells are closed, each  $z_k$  is surjective and thus  $i_X$  is a homeomorphism.  $\square$

The next task is to show that the image of  $i_X$  is a strong deformation retract of  $X$  under a suitable condition. The idea of proof is essentially the same as Theorem 5.16. We construct deformation retractions on each cell and glue them together. In order to define cell-wise deformation retraction, we assume the polyhedrality.

**Theorem 5.18.** *For a polyhedral stellar stratified space  $X$ , the image of embedding  $i_X : \text{Sd}(X, \pi) \rightarrow X$  is a strong deformation retract of  $X$ . The deformation retraction can be taken to be natural with respect to strict morphisms of polyhedral stellar stratified spaces.*

**Corollary 5.19.** *For a totally normal cellular stratified space  $(X, \pi)$ , the map  $i_X$  embeds  $\text{Sd}(X, \pi)$  into  $X$  as a strong deformation retract.*

The following fact is essential.

**Lemma 5.20.** *Let  $\pi$  be a regular cell decomposition of  $S^{n-1}$  and  $L \subset S^{n-1}$  be a stratified subspace. Let  $\tilde{\pi}$  be the cellular stratification on  $K = \text{Int}D^n \cup L$  obtained by adding  $\text{Int}D^n$  as an  $n$ -cell. Then there is a deformation retraction  $H$  of  $K$  to  $i_K(\text{Sd}(K, \tilde{\pi}))$ . Furthermore if a deformation retraction  $h$  of  $L$  onto  $i_L(\text{Sd}(L))$  is given,  $H$  can be taken to be an extension of  $h$ .*

This Lemma can be proved by using good old simplicial topology. It was first proved in a joint work with Basabe, González, and Rudyak [BGR] but removed from the published version [BGR14]. The proof is now contained in [FMT15] as Appendix A.

Now we are ready to prove Theorem 5.18.

*Proof of Theorem 5.18.* Let us show the embedding  $i_X$  constructed in the proof of Theorem 5.16 has a homotopy inverse.

Since  $X$  is polyhedral, each cell  $\varphi_\lambda : D_\lambda \rightarrow \bar{e}_\lambda$  has a polyhedral replacement  $\alpha_\lambda : \tilde{F}_\lambda \rightarrow D^{\dim e_\lambda}$ . For simplicity, we identify  $D_\lambda$  with  $\alpha_\lambda^{-1}(D_\lambda)$ . Thus  $D_\lambda$  is a stratified subspace of a polyhedral complex and  $b_{\mu,\lambda} : P_{\mu,\lambda} \times D_\mu \rightarrow D_\lambda$  is a PL map.

We construct, by induction on  $k$ , a PL homotopy

$$H_\lambda : D_\lambda \times [0, 1] \longrightarrow D_\lambda$$

for each  $k$ -cell  $e_\lambda$  satisfying the following conditions:

1. It is a strong deformation retraction of  $D_\lambda$  onto  $i_{D_\lambda}(\text{Sd}(D_\lambda))$ , where the stratification on  $D_\lambda$  is given by adding  $\text{Int}(D_\lambda)$  as a unique  $k$  cell to the stratification on  $\partial D_\lambda$ .
2. The diagram

$$\begin{array}{ccc} D_\lambda \times [0, 1] & \xrightarrow{H_\lambda} & D_\lambda \\ \uparrow b_{\mu,\lambda} \times 1 & & \uparrow b_{\mu,\lambda} \\ P_{\mu,\lambda} \times D_\mu \times [0, 1] & \xrightarrow{1 \times H_\mu} & P_{\mu,\lambda} \times D_\mu \end{array} \quad (5)$$

is commutative for any pair  $e_\mu \subset \bar{e}_\lambda$ .

When  $k = 0$ , the homotopy is the canonical projection. Suppose we have constructed  $H_\mu$  for all  $i$ -cells  $e_\mu$  with  $i \leq k - 1$ . We would like to extend them to  $k$ -cells. For a  $k$ -cell  $e_\lambda$  with stellar structure map  $\varphi_\lambda : D_\lambda \rightarrow X$  and a cell  $e_\mu$  with  $e_\mu \subset \bar{e}_\lambda$ , consider the diagram

$$\begin{array}{ccc} P_{\mu,\lambda} \times D_\mu \times [0, 1] & \xrightarrow{1 \times H_\mu} & P_{\mu,\lambda} \times D_\mu \\ \downarrow b_{\mu,\lambda} \times 1 & & \downarrow b_{\mu,\lambda} \\ b_{\mu,\lambda}(P_{\mu,\lambda} \times D_\mu) \times [0, 1] & \xrightarrow{\quad \quad \quad} & b_{\mu,\lambda}(P_{\mu,\lambda} \times D_\mu). \end{array}$$

Since  $b_{\mu,\lambda}$  is a homeomorphism when restricted to  $P_{\mu,\lambda} \times \text{Int}(D_\mu)$  and  $b_{\mu,\lambda}$  and  $H_\mu$  are assumed to be PL, we have the dotted arrow making the diagram commutative by Lemma B.32.

The decomposition

$$\partial D_\lambda = \bigcup_{e_\mu \subset \overline{e_\lambda}} b_{\mu,\lambda}(P_{\mu,\lambda} \times D_\mu)$$

allows us to glue these homotopies together. Thus we obtain a homotopy

$$H_\lambda : \partial D_\lambda \times [0, 1] \longrightarrow \partial D_\lambda.$$

By Lemma 5.20, this homotopy can be extended to a strong deformation retraction

$$H_\lambda : D_\lambda \times [0, 1] \longrightarrow D_\lambda$$

of  $D_\lambda$  onto  $i_{D_\lambda}(\text{Sd}(D_\lambda))$ . This completes the inductive step.

The second condition above (5) and the CW condition imply that these deformation retractions can be assembled together to give a strong deformation retraction

$$H : X \times [0, 1] \longrightarrow X$$

of  $X$  onto  $i_X(\text{Sd}(X))$ . □

The construction of the higher order Salvetti complex in [BZ92, DCS00] can be regarded as a corollary to Theorem 5.18.

**Example 5.21.** Consider the stratification on  $\mathbb{R}^n \otimes \mathbb{R}^\ell$

$$\pi_{\mathcal{A} \otimes \mathbb{R}^\ell} : \mathbb{R}^n \otimes \mathbb{R}^\ell \longrightarrow \text{Map}(L, S_\ell)$$

in Example 2.9. As we have seen in Example 2.41, this is a normal cellular stratification. It is also regular and polyhedral.

It contains

$$\text{Lk}(\mathcal{A} \otimes \mathbb{R}^\ell) = \bigcup_{i=1}^k H_i \otimes \mathbb{R}^\ell$$

as a cellular stratified subspace. The complement

$$M(\mathcal{A} \otimes \mathbb{R}^\ell) = \mathbb{R}^n \otimes \mathbb{R}^\ell \setminus \text{Lk}(\mathcal{A} \otimes \mathbb{R}^\ell)$$

is also a cellular stratified subspace. Since  $\pi_{\mathcal{A} \otimes \mathbb{R}^\ell}$  is regular, we have

$$C(M(\mathcal{A} \otimes \mathbb{R}^\ell)) = P(M(\mathcal{A} \otimes \mathbb{R}^\ell)) = P(\mathcal{A} \otimes \mathbb{R}^\ell) \setminus P(\text{Lk}(\mathcal{A} \otimes \mathbb{R}^\ell)).$$

By Theorem 5.18,  $BC(M(\mathcal{A} \otimes \mathbb{R}^\ell))$  can be embedded in the complement  $M(\mathcal{A} \otimes \mathbb{R}^\ell)$  as a strong deformation retract. This simplicial complex  $BC(M(\mathcal{A} \otimes \mathbb{R}^\ell))$  is nothing but the higher order Salvetti complex in [BZ92, DCS00]. □

The next example is totally normal but not regular.

**Example 5.22.** By Example 3.14, a graph  $X$  can be regarded as a totally normal cellular stratified space. As is described in [FMT15], we may define a totally normal cellular stratification on the  $k$ -fold product  $X^k$  by using the braid arrangements, which can be restricted to a totally normal cellular stratification on  $\text{Conf}_k(X)$ . By Corollary 5.19,  $BC(\text{Conf}_k(X))$  can be embedded in  $\text{Conf}_k(X)$  as a  $\Sigma_k$ -equivariant strong deformation retract.

$BC(\text{Conf}_k(X))$  is a regular CW complex model for  $\text{Conf}_k(X)$ . This is better than Abrams' model in the sense that it works for all graphs. □

The cellular stratification on  $\mathbb{C}P^n$  in Example 4.26 is cylindrically normal and Theorem 5.16 applies.

**Example 5.23.** Consider the cylindrically normal cellular stratification on  $\mathbb{C}P^n$  in Example 4.25 and Example 4.26. By Theorem 5.16,  $BC(\mathbb{C}P^n)$  is homeomorphic to  $\mathbb{C}P^n$ . Let us compare the description of  $BC(\mathbb{C}P^n)$  with the one in Example 4.26 as the Davis-Januszkiewicz construction  $M(\lambda_n)$ .

As we have seen in Example 5.6, the cylindrical face category  $C(\mathbb{C}P^n)$  can be obtained from the underlying poset  $[n]$  of  $C(\mathbb{C}P^n)$  by replacing the set of morphisms by  $S^1$ . Compositions of morphisms are given by the group structure of  $S^1$ .

By Lemma 5.13, we have

$$\overline{N}_k(C(\mathbb{C}P^n)) = \coprod_{\mathbf{e} \in \overline{N}_k([n])} \{\mathbf{e}\} \times (S^1)^k$$

and the classifying space of  $C(\mathbb{C}P^n)$  can be described in the form

$$\begin{aligned} BC(\mathbb{C}P^n) &= \|\overline{N}(C(\mathbb{C}P^n))\| \\ &= \left( \coprod_k \coprod_{\mathbf{e} \in \overline{N}_k([n])} \{\mathbf{e}\} \times (S^1)^k \times \Delta^k \right) / \sim. \end{aligned}$$

Note that a nondegenerate  $k$ -chain  $\mathbf{e}$  in the poset  $[n]$  can be described by a strictly increasing sequence of nonnegative integers  $\mathbf{e} = (i_0, \dots, i_k)$  with  $i_k \leq n$ . For an element  $(\mathbf{e}; t_1, \dots, t_k; p_0, \dots, p_k)$  in  $\overline{N}_k(C(\mathbb{C}P^n))$ , let  $\tilde{\mathbf{t}}$  be an element of  $(S^1)^n$  obtained from  $(t_1, \dots, t_k)$  by inserting 1 in such a way that each  $t_j$  is placed between  $i_{j-1}$ -th and  $i_j$ -th positions. Then there exists a face operator  $d_I : N_n(C(\mathbb{C}P^n)) \rightarrow N_k(C(\mathbb{C}P^n))$  such that

$$(\mathbf{e}; t_1, \dots, t_k) = (d_I(0, 1, \dots, n), d_I(\tilde{\mathbf{t}})).$$

And we have

$$(\mathbf{e}; t_1, \dots, t_k; p_0, \dots, p_k) \sim (0, \dots, n; \tilde{\mathbf{t}}; d^I(p_0, \dots, p_k)).$$

Note that  $d^I(p_0, \dots, p_k)$  is obtained from  $(p_0, \dots, p_k)$  by inserting 0 in appropriate coordinates. Thus any point in  $BC(\mathbb{C}P^n)$  can be represented by a point in  $(S^1)^n \times \Delta^n$  and  $BC(\mathbb{C}P^n)$  can be written as

$$BC(\mathbb{C}P^n) = (S^1)^n \times \Delta^n / \sim. \quad (6)$$

The relation here is not exactly the same as the defining relation of  $M(\lambda_n)$ .

Define a map

$$s_n : M(\lambda_n) \longrightarrow BC(\mathbb{C}P^n)$$

by

$$s_n([t_1, \dots, t_n; p_0, \dots, p_n]) = [t_1, t_1^{-1}t_2, \dots, t_{n-1}^{-1}t_n; p_0, \dots, p_n].$$

It is left to the reader to verify that  $s_n$  is a well-defined homeomorphism making the diagram

$$\begin{array}{ccc} M(\lambda_n) & \xrightarrow{s_n} & BC(\mathbb{C}P^n) \\ \downarrow & & \downarrow B\pi_{C(\mathbb{C}P^n)} \\ \Delta^n & \xlongequal{\quad} & B([n]) \xlongequal{\quad} BP(\mathbb{C}P^n) \end{array}$$

commutative, where  $\pi_{C(\mathbb{C}P^n)} : C(\mathbb{C}P^n) \rightarrow P(\mathbb{C}P^n)$  is the canonical projection onto the underlying poset<sup>26</sup>. Thus we see that  $BC(\mathbb{C}P^n)$  coincides with  $M(\lambda_n)$  up to a homeomorphism.  $\square$

<sup>26</sup>Definition C.6

## 6 Basic Constructions on Cellular Stratified Spaces

We study the following operations on cellular and stellar stratified spaces in this section:

- taking stratified subspaces, cellular stratified subspaces, and stellar stratified subspaces,
- taking products of cellular stratified spaces, and
- taking subdivisions of cellular and stellar stratified spaces.

### 6.1 Stratified Subspaces

In this section, we consider the problem of restricting cellular stratifications to subspaces. Obviously the category of stratified spaces is closed under taking complements of stratified subspaces.

**Lemma 6.1.** *Let  $X$  be a stratified space and  $A$  be a stratified subspace. Then the complement  $X \setminus A$  is also a stratified subspace of  $X$ .*

This is one of the most useful facts when we work with stratifications on configuration spaces and complements of arrangements. On the other hand, when  $X$  is a CW complex and  $A$  is a subcomplex,  $A$  is always closed. This is no longer true for cellular stratified spaces. A typical example is the case of the complement  $M(\mathcal{A} \otimes \mathbb{R}^\ell) = \mathbb{R}^n \otimes \mathbb{R}^\ell \setminus \text{Lk}(\mathcal{A} \otimes \mathbb{R}^\ell)$  of an arrangement in Example 5.21.

Let us consider cell structures on subspaces. Suppose  $X$  is a cellular stratified space and  $A$  is a stratified subspace. In order to incorporate cell structures, we need to specify a cell structure for each cell contained in  $A$ .

**Definition 6.2.** Let  $X$  be a topological space and  $A$  be a subspace. For a hereditarily quotient<sup>27</sup>  $n$ -cell structure  $\varphi : D \rightarrow \bar{e} \subset X$  with  $e \subset A$ , define an  $n$ -cell structure on  $e$  in  $A$  to be  $(D_A, \varphi|_{D_A})$  where  $D_A = \varphi^{-1}(\bar{e} \cap A)$ .

The hereditarily-quotient assumption guarantees the restriction is a quotient map by Lemma A.5.

**Lemma 6.3.** *Let  $X$  be a topological space and  $A$  be a subset of  $X$ . If  $(D, \varphi)$  is a hereditarily quotient  $n$ -cell structure on  $e \subset X$  and  $e \subset A$ , then  $(D_A, \varphi|_{D_A})$  defined above is an  $n$ -cell structure on  $e$  in  $A$ .*

**Definition 6.4.** Let  $(X, \pi)$  be a cellular stratified space. A stratified subspace  $(A, \pi|_A)$  of  $X$  is said to be a *cellular stratified subspace*, provided cell structures on cells in  $A$  are given as indicated in Definition 6.2. When the inclusion  $A \hookrightarrow X$  is a strict morphism,  $A$  is said to be a *strict cellular stratified subspace*.

We need to take cell decompositions of domains of cells into account for stellar stratified subspaces.

**Lemma 6.5.** *Let  $X$  be a topological space,  $A$  a subspace, and  $\varphi : D \rightarrow \bar{e}$  a stellar  $n$ -cell of  $X$  with  $e \subset A$ . When  $D_A = \varphi^{-1}(\bar{e} \cap A)$  is a strict stratified subspace of  $D$  and  $\varphi$  is hereditarily quotient, the restriction  $\varphi|_{D_A} : D_A \rightarrow A$  defines a stellar  $n$ -cell structure on  $e$  in  $A$ .*

**Definition 6.6.** Let  $(X, \pi)$  be a stellar stratified space and  $A$  a subspace of  $X$ . If the assumption of Lemma 6.5 is satisfied for each cell in  $A$ ,  $(A, \pi|_A)$  is said to be a *stellar stratified subspace* of  $X$ .

---

<sup>27</sup>Definition A.2.

**Remark 6.7.** Let  $X$  be a cellular or stellar stratified space and  $A$  be a cellular or stellar stratified subspace. Then  $A$  is a strict stratified subspace of  $X$ .

The following fact can be regarded as a generalization of the fact that any subcomplex of a regular cell complex is regular.

**Proposition 6.8.** *Any stellar stratified subspace  $A$  of a cylindrically normal stellar stratified space  $X$  is cylindrically normal. Furthermore the parameter space for a pair of cells  $e_\mu < e_\lambda$  in  $A$  can be identified with the parameter space for the same pair when regarded as cells in  $X$ .*

*Proof.* Let  $e_\mu < e_\lambda$  be a pair of cells in  $A$ . Let  $\varphi_\mu : D_\mu \rightarrow X$  and  $\varphi_\lambda : D_\lambda \rightarrow X$  be the stellar structures for these cells in  $X$ . When regarded as cells in  $A$ , their stellar structures are denoted by

$$\begin{aligned}\varphi_{A,\mu} &: D_{A,\mu} \longrightarrow A \\ \varphi_{A,\lambda} &: D_{A,\lambda} \longrightarrow A,\end{aligned}$$

respectively.

We need to show that the structure map

$$b_{\mu,\lambda} : P_{\mu,\lambda} \times D_\mu \longrightarrow D_\lambda$$

of cylindrical structure for the pair in  $X$  can be restricted to

$$b_{\mu,\lambda}|_{P_{\mu,\lambda} \times D_{A,\mu}} : P_{\mu,\lambda} \times D_{A,\mu} \longrightarrow D_{A,\lambda}.$$

This can be verified by the commutativity of the diagram

$$\begin{array}{ccccc} P_{\mu,\lambda} \times D_{A,\mu} & \xrightarrow{\quad\quad\quad} & D_{A,\lambda} & \xrightarrow{\varphi_{A,\lambda}} & \overline{e_\lambda \cap A} \\ & \searrow & \downarrow & & \swarrow \\ & P_{\mu,\lambda} \times D_\mu & \xrightarrow{b_{\mu,\lambda}} & D_\lambda & \xrightarrow{\varphi_\lambda} & \bar{e}_\lambda \\ & \downarrow \text{pr}_2 & & & \uparrow & \\ & D_\mu & \xrightarrow{\varphi_\mu} & \bar{e}_\mu & & \\ & \swarrow & & & \searrow & \\ D_{A,\mu} & \xrightarrow{\varphi_{A,\mu}} & & & \bar{e}_\mu \cap A & \end{array}$$

and the definition of stellar structures on  $A$ . □

**Example 6.9.** Consider the stratification in Example 5.21. The link  $\text{Lk}(\mathcal{A} \otimes \mathbb{R}^\ell)$  and the complement  $M(\mathcal{A} \otimes \mathbb{R}^\ell)$  are both cellular stratified subspaces of  $(\mathbb{R}^n \otimes \mathbb{R}^\ell, \pi_{\mathcal{A} \otimes \mathbb{R}^\ell})$ , which is regular, hence cylindrically normal. □

Thanks to Corollary 4.21 and Lemma A.7, cell structures in a polyhedral stellar stratified space  $X$  are hereditarily quotient. Thus any stellar stratified subspace  $A$  inherits a cylindrically normal structure with structure maps satisfying the PL conditions in the definition of polyhedral cellular stratification. The problem is the CW condition. It is easy to see that the closure finiteness condition can be restricted freely. The question is when a stratified subspace of a CW stratified subspace inherits the weak topology.

**Lemma 6.10.** *A closed or an open stratified subspace  $A$  of a CW stratified space  $X$  is CW.*

*Proof.* This follows from the corresponding property of weak topology. □

By Lemma 4.19, any polyhedral cellular stratified space  $X$  can be embedded in a CW complex  $U(X)$ . In general, however, a cellular stratified subspace  $A$  of  $X$  is neither closed nor open in  $U(X)$  and it is not easy to verify the weak topology condition. One of the practical conditions is the locally finiteness. The CW condition is guaranteed by Proposition 2.21.

**Proposition 6.11.** *Let  $X$  be a polyhedral cellular stratified space. Any locally finite cellular stratified subspace  $A$  is polyhedral.*

We end this section by an example which shows another difference between cellular stratified subspaces and subcomplexes. In the case of CW complexes, the colimit of an increasing sequence of finite subcomplexes

$$X_0 \subset X_1 \subset \cdots \subset \operatorname{colim}_n X$$

is automatically a CW complex. This is not true for cellular stratified spaces.

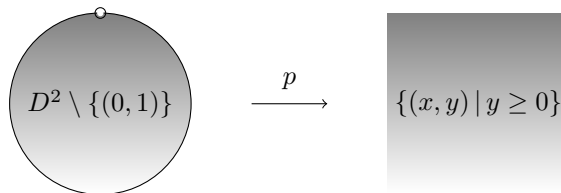
**Example 6.12.** Consider the space

$$X = \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \cup \mathbb{Z} \times \{0\}.$$

The homeomorphism

$$p : D^2 \setminus \{(0, 1)\} \longrightarrow \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$$

given by extending the stereographic projection  $S^1 \setminus \{(0, 1)\} \rightarrow \mathbb{R}$ ,



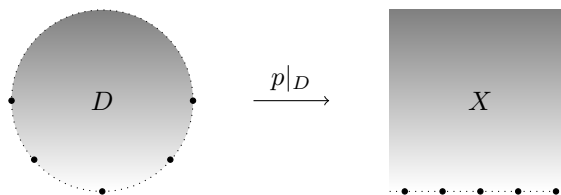
defines a 2-cell structure on

$$e^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \subset X$$

by restricting  $p$  to  $D = p^{-1}(X)$ .

Each  $\{(n, 0)\} \subset \mathbb{Z} \times \{0\}$  can be regarded as a 0-cell  $e_n^0$ . And we have a cellular stratification on  $X$

$$X = \left( \bigcup_{n \in \mathbb{Z}} e_n^0 \right) \cup e^2.$$



$X$  is a colimit of

$$X_n = \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \cup \{i \in \mathbb{Z} \mid |i| \leq n\} \times \{0\}.$$

Each  $X_n$  is a finite stratified subspace of  $X$ , hence is CW. But  $X$  is not CW. □

## 6.2 Products

In this section, we study products of stratifications, cell structures, and cellular stratified spaces and deduce a couple of conditions under which we may take products.

It is not difficult to define a stratification on the product of two stratified spaces, as we have seen in Lemma 2.13. We have to be careful, however, when we take products of cellular stratified spaces. Even in the category of CW complexes, there is a well-known difficulty in taking products. Given CW complexes  $X$  and  $Y$ , we need to impose the local-finiteness on  $X$  or  $Y$  or to redefine a topology on  $X \times Y$  in order to make  $X \times Y$  into a CW complex.

In the case of cellular stratified spaces, we have another difficulty because of our requirement on cell structures. The product of two quotient maps may not be a quotient map.

**Definition 6.13.** Let  $(X, \pi_X, \Phi_X)$  and  $(Y, \pi_Y, \Phi_Y)$  be cellular stratified spaces and consider the product stratification

$$\pi_X \times \pi_Y : X \times Y \longrightarrow P(X) \times P(Y)$$

on  $X \times Y$  in Lemma 2.13. For a pair of cells  $e_\lambda \subset X$  and  $e_\mu \subset Y$ , consider the composition

$$\varphi_{\lambda,\mu} : D \cong D_\lambda \times D_\mu \xrightarrow{\varphi_\lambda \times \varphi_\mu} \overline{e_\lambda} \times \overline{e_\mu} = \overline{e_\lambda \times e_\mu} \subset X \times Y,$$

where  $D$  is the subspace of  $D^{\dim e_\lambda + \dim e_\mu}$  defined by pulling back  $D_\lambda \times D_\mu$  via the standard homeomorphism

$$D^{\dim e_\lambda + \dim e_\mu} \cong D^{\dim e_\lambda} \times D^{\dim e_\mu}.$$

If  $\varphi_{\lambda,\mu}$  is a quotient map for each pair of cells, the resulting cellular stratification is called the *product cellular stratification* on  $X \times Y$ .

The above definition is incomplete. Unless we have a general criterion for  $\varphi_{\lambda,\mu}$  to be a quotient map, this definition is useless.

By Lemma A.7 and Proposition A.13, we can take products of bi-quotient cell structures. The question is when a cell structure is a bi-quotient map. By Corollary A.15 and Corollary 4.21, we obtain the following practical conditions under the assumption of cylindrical normality.

**Proposition 6.14.** *Let  $X$  and  $Y$  be cylindrically normal cellular stratified spaces. If they satisfy one of the following conditions, any product  $e_\lambda \times e_\mu$  has the product cell structure for  $\lambda \in P(X)$  and  $\mu \in P(Y)$ :*

1. *All parameter spaces are compact.*
2. *Polyhedral.*

*Proof.* If  $X$  or  $Y$  is polyhedral, all cell structures are bi-quotient by Corollary 4.21. By Lemma 4.4, each fiber of a cell structure in a cylindrically normal cellular stratified space can be identified with a parameter space. Thus, when all parameter spaces are compact, the cell structures are bi-quotient by Corollary A.15.  $\square$

If  $X$  and  $Y$  satisfy one of the above conditions, the product  $X \times Y$  has a structure of cellular stratified space. It is reasonable to expect that  $X \times Y$  is again cylindrically normal.

**Theorem 6.15.** *Let  $X$  and  $Y$  be cylindrically normal cellular stratified spaces satisfying one of the conditions in Proposition 6.14. Then the product  $X \times Y$  is a cylindrically normal cellular stratified space.*



*Proof.* Let  $\{\varphi_\lambda^X : D_\lambda \rightarrow \overline{e_\lambda}\}$  and  $\{\varphi_\mu^Y : D_\mu \rightarrow \overline{e_\mu}\}$  be cell structures on  $X$  and  $Y$ , respectively. Cylindrical structures on  $X$  and  $Y$  are denoted by  $\{b_{\lambda,\lambda'}^X : P_{\lambda,\lambda'}^X \times D_\lambda \rightarrow D_{\lambda'}\}$  and  $\{b_{\mu,\mu'}^Y : P_{\mu,\mu'}^Y \times D_\mu \rightarrow D_{\mu'}\}$ , respectively.

For a pair of cell structures  $\varphi_\lambda^X : D_\lambda \rightarrow X$  and  $\varphi_\mu^Y : D_\mu \rightarrow Y$ , consider the product cell structure

$$\varphi_{\lambda,\mu} : D_{\lambda,\mu} \cong D_\lambda \times D_\mu \xrightarrow{\varphi_\lambda^X \times \varphi_\mu^Y} \overline{e_\lambda} \times \overline{e_\mu}.$$

For  $(\lambda, \mu) \leq (\lambda', \mu')$ , define

$$P_{(\lambda,\mu),(\lambda',\mu')} = P_{\lambda,\lambda'}^X \times P_{\mu,\mu'}^Y.$$

When  $(\lambda, \mu) < (\lambda', \mu')$ , we have either  $\lambda < \lambda'$  or  $\mu < \mu'$  and thus the image of the composition

$$\begin{aligned} P_{(\lambda,\mu),(\lambda',\mu')} \times D_{\lambda,\mu} &\cong P_{\lambda,\lambda'}^X \times P_{\mu,\mu'}^Y \times D_\lambda \times D_\mu \\ &\xrightarrow{\cong} P_{\lambda,\lambda'}^X \times D_\lambda \times P_{\mu,\mu'}^Y \times D_\mu \\ &\xrightarrow{b_{\lambda,\lambda'}^X \times b_{\mu,\mu'}^Y} D_{\lambda'} \times D_{\mu'} \\ &\cong D_{\lambda',\mu'} \end{aligned}$$

lies in  $\partial D_{\lambda,\mu} \cong (D_\lambda \times \partial D_\mu) \cup (\partial D_\lambda \times D_\mu)$ . And we obtain a map

$$b_{(\lambda,\mu),(\lambda',\mu')} : P_{(\lambda,\mu),(\lambda',\mu')} \times D_{\lambda,\mu} \longrightarrow \partial D_{\lambda',\mu'}.$$

The composition operations

$$P_{(\lambda_1,\mu_1),(\lambda_2,\mu_2)} \times P_{(\lambda_0,\mu_0),(\lambda_1,\mu_1)} \longrightarrow P_{(\lambda_0,\mu_0),(\lambda_2,\mu_2)}$$

are defined in an obvious way.

It is straightforward to verify that these maps define a cylindrical structure on  $X \times Y$  under the product stratification and the product cell structures.  $\square$

Let us consider the CW conditions on products next. The closure finiteness condition is automatic.

**Lemma 6.16.** *If  $X$  and  $Y$  are stratified spaces satisfying the closure finiteness condition, then so is  $X \times Y$ .*

As is the case of CW complexes [Dow52], the product of two CW stratifications may not satisfy the weak topology condition. In the case of CW complexes, the local finiteness of  $X$  implies that  $X \times Y$  is CW for any CW complex  $Y$ . The author does not know if an analogous fact holds for CW stratified spaces in general. The following obvious fact is still useful in many cases.

**Lemma 6.17.** *Let  $X$  and  $Y$  be CW stratified spaces. If both  $X$  and  $Y$  are locally finite, then  $X \times Y$  is a CW stratified space.*

*Proof.* The product of locally finite stratified spaces is again locally finite. The result follows from Proposition 2.21.  $\square$

Thus, by Theorem 6.15, the product  $X \times Y$  of locally finite cylindrically normal cellular stratified spaces  $X$  and  $Y$  is a CW cylindrically normal cellular stratified space. When  $X$  and  $Y$  are polyhedral, it is easy to verify that  $X \times Y$  inherits a polyhedral structure.

**Corollary 6.18.** *Let  $X$  and  $Y$  be polyhedral cellular stratified spaces. Suppose both  $X$  and  $Y$  are locally finite. Then the product  $X \times Y$  is a polyhedral cellular stratified space.*

Thus we may freely take finite products of Euclidean polyhedral cellular stratified spaces.

**Corollary 6.19.** *For any Euclidean polyhedral cellular stratified spaces  $X$  and  $Y$ , the product  $X \times Y$  is polyhedral.*

*Proof.* By Lemma 2.23,  $X$  and  $Y$  are locally finite. □

In Proposition 6.15 and Definition 6.13, we made an implicit choice of a homeomorphism

$$D^{m+n} \cong D^m \times D^n.$$

This procedure can be avoided by using cubes as domains of cell structures, in which case it is a reasonable idea to require the structure maps to be compatible with cubical structures. Thus we introduce the following variant of polyhedral cellular stratified spaces.

**Definition 6.20.** Let  $X$  be a CW cellular stratified space. We consider  $I^n = (\Delta^1)^n$  as a stratified space (regular cell complex) under the product stratification of the simplicial stratification on  $\Delta^1$ . A *cubical structure* on  $X$  consists of

- a cylindrical structure  $(\{b_{\mu,\lambda} : P_{\mu,\lambda} \times D_\mu \rightarrow \partial D_\lambda\}, \{c_{\lambda_0,\lambda_1,\lambda_2} : P_{\lambda_1,\lambda_2} \times P_{\lambda_0,\lambda_1} \rightarrow P_{\lambda_0,\lambda_2}\})$ ,
- a stratified subspace  $Q_\lambda$  of  $I^{\dim e_\lambda}$  for each  $\lambda \in P(X)$  under a suitable regular cellular subdivision of  $I^{\dim e_\lambda}$ , and
- a homeomorphism  $\alpha_\lambda : Q_\lambda \rightarrow D_\lambda$  for each  $\lambda \in P(X)$ ,

satisfying the following conditions:

1. Each  $P_{\mu,\lambda}$  is a stratified subspace of  $I^{\dim e_\lambda - \dim e_\mu}$ .
2. The composition

$$\tilde{b}_{\mu,\lambda} : P_{\mu,\lambda} \times Q_\mu \xrightarrow{1_{P_{\mu,\lambda}} \times \alpha_\mu} P_{\mu,\lambda} \times D_\mu \xrightarrow{b_{\mu,\lambda}} \partial D_\lambda \xrightarrow{\alpha_\lambda^{-1}} Q_\lambda$$

is a strict morphism of stratified spaces.

3. The map  $\tilde{b}_{\mu,\lambda}$  is an affine embedding onto its image when restricted to each face.

The family of maps  $\{\alpha_\lambda : Q_\lambda \rightarrow D_\lambda\}_{\lambda \in P(X)}$  is also called a *cubical structure*.

A cylindrically normal CW cellular stratified space equipped with a cubical structure is called a *cubically normal cellular stratified space*.

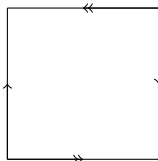
**Example 6.21.** The minimal cell decomposition of  $S^n$  is cubically normal. The radial expansion

$$\alpha_n : I^n \longrightarrow D^n$$

defines a cubical structure. The parameter space between the 0-cell and the  $n$ -cell is  $\partial I^n$  and is a stratified subspace of  $I^n$ . □

**Example 6.22.** Recall that any 1-dimensional cellular stratified space is totally normal, hence is cylindrically normal. They are cubically normal for an obvious reason, if they are CW. □

**Example 6.23.** Recall that  $\mathbb{R}P^2$  can be obtained by gluing the edges of  $I^2$  as follows.



This can be regarded as a description of a cell decomposition of  $\mathbb{R}P^2$ , consisting of two 0-cells  $e_1^0, e_2^0$ , two 1-cells  $e_1^1, e_2^1$  and a 2-cell  $e^2$ . This is obviously cubically normal.  $\square$

**Remark 6.24.** When the domains of cells are not globular<sup>28</sup>, the situation is more complicated and we do not discuss the product structure here.

### 6.3 Subdivisions of Cells

We defined the notion of cellular stratified subspace in Definition 6.4. It often happens that we need to subdivide cells before we take a stratified subspace. For example, the complement of an arrangement  $M(\mathcal{A} \otimes \mathbb{R}^\ell)$  in Example 5.21 was defined first by taking a subdivision of the trivial stratification on  $\mathbb{R}^n \otimes \mathbb{R}^\ell$  and then by taking the complement of  $\text{Lk}(\mathcal{A} \otimes \mathbb{R}^\ell)$ .

We have already defined subdivisions of stratified spaces in Definition 2.14. We impose the following “regularity condition” on the definition of subdivisions of cell structures.

**Definition 6.25.** Let  $(\pi, \Phi)$  be a cellular stratification on  $X$ . A *cellular subdivision* of  $(\pi, \Phi)$  consists of

- a subdivision of stratified spaces

$$\mathbf{s} = (1_X, \underline{\mathbf{s}}) : (X, \pi') \longrightarrow (X, \pi)$$

and

- a regular cellular stratification  $(\pi_\lambda, \Phi_\lambda)$  on the domain  $D_\lambda$  of each cell  $e_\lambda$  in  $(\pi, \Phi)$  containing  $\text{Int}(D_\lambda)$  as a strict stratified subspace,

satisfying the following conditions:

1. For each  $\lambda \in P(X, \pi)$ , the cell structure

$$\varphi_\lambda : (D_\lambda, \pi_\lambda) \longrightarrow (X, \pi')$$

of  $e_\lambda$  is a strict morphism of stratified spaces.

2. The maps

$$P(\varphi_\lambda) : P(\text{Int}(D_\lambda)) \longrightarrow P(X, \pi)$$

induced by the cell structures  $\{\varphi_\lambda\}$  give rise to a bijection

$$P(\Phi) = \coprod_{\lambda \in P(X, \pi)} P(\varphi) : \coprod_{\lambda \in P(X, \pi)} P(\text{Int}(D_\lambda), \pi_\lambda) \longrightarrow P(X, \pi').$$

**Remark 6.26.** The morphism  $\mathbf{s}$  induces a surjective map

$$P(\mathbf{s}) : P(X, \pi') \longrightarrow P(X, \pi),$$

---

<sup>28</sup>Definition 2.24

which gives rise to a decomposition

$$P(X, \pi') = \coprod_{\lambda \in P(X, \pi)} P(\mathbf{s})^{-1}(\lambda)$$

of sets. Since  $\varphi_\lambda : (D_\lambda, \pi_\lambda) \rightarrow (X, \pi')$  is a strict morphism, it induces an isomorphism of posets

$$P(\varphi_\lambda) : P(\text{Int}(D_\lambda), \pi_\lambda) \xrightarrow{\cong} P(\mathbf{s})^{-1}(\lambda).$$

In other words, the restriction of the bijection  $P(\Phi)$  in the second condition to each  $P(\text{Int}(D_\lambda), \pi_\lambda)$  is an embedding of posets, although  $P(\Phi)$  itself is rarely an isomorphism of posets.

**Remark 6.27.** Cellular subdivisions of stellar stratified spaces can be defined in a similar way. We may also define stellar subdivisions of cellular or stellar stratified spaces. We do not pursue such generalizations in this paper.

It is easy to verify that the category of cellular stratified spaces is closed under cellular subdivisions, when all cell structures are hereditarily quotient.

**Lemma 6.28.** *Let  $(X, \pi, \Phi)$  be a cellular stratified space whose cell structures are hereditarily quotient. Then any cellular subdivision of  $(X, \pi, \Phi)$  defines a structure of cellular stratified space on  $X$ , under which, for  $\lambda \in P(X, \pi)$  and  $\lambda' \in P(\text{Int}(D_\lambda), \pi_\lambda)$ , the composition*

$$D_{\lambda'} \xrightarrow{s_{\lambda'}} D_\lambda \xrightarrow{\varphi_\lambda} X$$

*is the cell structure of the cell  $e_{\lambda'}$  in  $P(X, \pi')$ , where  $s_{\lambda'} : D_{\lambda'} \rightarrow D_\lambda$  is the cell structure of the cell in  $P(D_\lambda, \pi_\lambda)$  indexed by  $\lambda'$ .*

*Proof.* By assumption, each cell structure  $\varphi_\lambda$  is hereditarily quotient and hence the composition  $\varphi_\lambda \circ s_{\lambda'} : D_{\lambda'} \rightarrow \overline{e_{\lambda'}}$  is a quotient map. Each new stratum is connected, since  $e_{\lambda'}$  is connected. It is also locally closed, since  $\varphi_\lambda|_{\text{Int}(D_\lambda)}$  is a homeomorphism onto its image. Other conditions can be verified immediately.  $\square$

**Remark 6.29.** The reader might want to define a subdivision of a cellular stratified space  $(X, \pi, \Phi)$  as a morphism

$$\mathbf{s} = (1_X, \underline{s}, \{s_{\lambda'}\}_{\lambda' \in P(X, \pi')}) : (X, \pi', \Phi') \longrightarrow (X, \pi, \Phi)$$

of cellular stratified spaces satisfying the condition that, for each cell  $e_\lambda$  in  $(\pi, \Phi)$ , the stratification of the interior of the domain  $D_\lambda$  is indexed by  $\underline{s}^{-1}(\lambda)$

$$\text{Int}(D_\lambda) = \bigcup_{\lambda' \in \underline{s}^{-1}(\lambda)} s_{\lambda'}(\text{Int}(D_{\lambda'})).$$

However, we also need to specify the behavior of each cell structure on the boundary  $\partial D_\lambda$ .

**Example 6.30.** Let  $\mathcal{A} = \{H_1, \dots, H_k\}$  be a hyperplane arrangement in  $\mathbb{R}^n$  defined by affine 1-forms  $L = \{\ell_1, \dots, \ell_k\}$ . The stratification  $\pi_{\mathcal{A} \otimes \mathbb{R}^\ell}$  on  $\mathbb{R}^n \otimes \mathbb{R}^\ell$  in Example 5.21 is one of the coarsest cellular stratifications containing  $\text{Lk}(\mathcal{A} \otimes \mathbb{R}^\ell) = \bigcup_{i=1}^k H_k \otimes \mathbb{R}^\ell$  as a stratified subspace.

This efficiency is achieved by sacrificing symmetry. The stratification  $\pi_{\mathcal{A} \otimes \mathbb{R}^\ell}$  is not compatible with the action of the symmetric group  $\Sigma_\ell$  on  $\mathbb{R}^\ell$ .

As is stated by Björner and Ziegler in [BZ92], we may subdivide  $\pi_{\mathcal{A} \otimes \mathbb{R}^\ell}$  by using the product sign vector  $S_1^\ell$ . Define

$$\pi_{\mathcal{A} \otimes \mathbb{R}^\ell}^s : \mathbb{R}^n \otimes \mathbb{R}^\ell \longrightarrow \text{Map}(L, S_1^\ell)$$

by

$$\pi_{\mathcal{A} \otimes \mathbb{R}^\ell}^s(x_1, \dots, x_\ell)(\ell_i) = (\text{sign}(\ell_i(x_1)), \dots, \text{sign}(\ell_i(x_\ell))).$$

Define a map  $c : S_1^\ell \rightarrow S_\ell$  of posets by

$$c(\varepsilon_1, \dots, \varepsilon_\ell) = \varepsilon_i e_i,$$

where  $i = \max \{j \mid \varepsilon_j \neq 0\}$ . This is surjective and the diagram

$$\begin{array}{ccc} \mathbb{R}^n \otimes \mathbb{R}^\ell & \xlongequal{\quad} & \mathbb{R}^n \otimes \mathbb{R}^\ell \\ \pi_{\mathcal{A} \otimes \mathbb{R}^\ell}^s \downarrow & & \downarrow \pi_{\mathcal{A} \otimes \mathbb{R}^\ell} \\ \text{Map}(L, S_1^\ell) & \xrightarrow{\text{Map}(L, c)} & \text{Map}(L, S_\ell) \end{array}$$

is commutative. Thus  $\pi_{\mathcal{A} \otimes \mathbb{R}^\ell}^s$  is a subdivision of the stratification  $\pi_{\mathcal{A} \otimes \mathbb{R}^\ell}$ . The regularity of the cellular stratification  $\pi_{\mathcal{A} \otimes \mathbb{R}^\ell}$  implies that this is a cellular subdivision. Note that  $(\mathbb{R}^n \otimes \mathbb{R}^\ell, \pi_{\mathcal{A} \otimes \mathbb{R}^\ell}^s)$  is a  $\Sigma_\ell$ -cellular stratified space.  $\square$

**Example 6.31.** Let  $\pi_{\min}$  be the minimal cell decomposition of  $S^2$ . By dividing the 2-cell by the equator, we obtain a subdivision  $\pi$  of  $\pi_{\min}$  as a stratified space. By dividing the domain of the cell structure of the 2-cell in  $\pi_{\min}$ , this subdivision can be made into a subdivision of cellular stratified spaces, as is depicted in Figure 30.

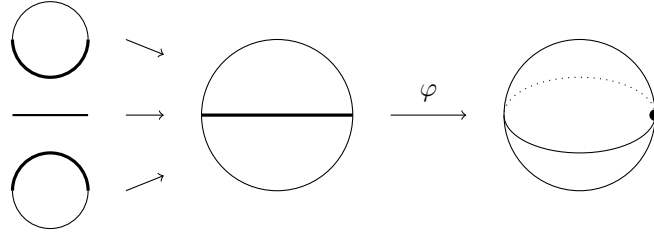


Figure 30: A subdivision of the minimal cell decomposition of  $S^2$ .

There is another way to make the stratification  $\pi$  into a cellular stratified space. Choose a small disk  $D$  in  $D^2$  touching  $\partial D^2$  at a point  $p$ . The small disk  $D$  is mapped homeomorphically

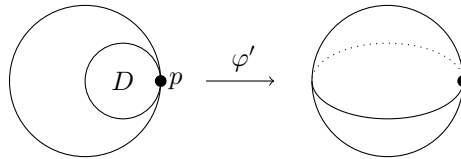


Figure 31: Not a subdivision of  $\pi_{\min}$ .

onto the lower hemisphere via  $\varphi'$ . There is a map

$$\psi : D^2 \longrightarrow D^2 \setminus \text{Int}(D)$$

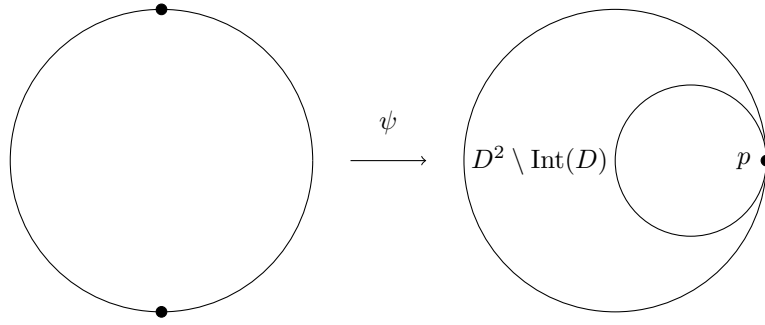


Figure 32: A definition of  $\psi$ .

such that the composition  $\varphi \circ \psi$  is a cell structure for the upper hemisphere. For example, such a map can be defined by identifying the two points in Figure 32 to  $p$ . The morphism of stratified spaces  $\pi \rightarrow \pi_{\min}$  also becomes a morphism of cellular stratified spaces with this cellular stratification on  $\pi$ , but it is not a subdivision of cellular stratified spaces, since the above stratification on the domain  $D^2$  is not a regular cellular stratification.  $\square$

We often define a cellular subdivision of a cellular stratified space  $(X, \pi, \Phi)$  by defining subdivisions of domains of cell structures. The problem is of course the compatibility.

**Lemma 6.32.** *Let  $(X, \pi, \Phi)$  be a cellular stratified space whose cell structures are hereditarily quotient. Suppose that a regular cellular stratification  $\pi_\lambda$  on each domain  $D_\lambda$  is given. Define a stratification  $\pi'$  on  $X$  by*

$$X = \bigcup_{\lambda \in P(X, \pi)} \bigcup_{\lambda' \in P(\text{Int}(D_\lambda))} \varphi_\lambda(e_{\lambda'}).$$

Suppose further that the following conditions are satisfied:

1. For each  $\lambda \in P(X)$ ,  $\text{Int}(D_\lambda)$  is a strict stratified subspace of  $D_\lambda$ .
2. For  $\lambda' \in P(D_\lambda)$  and  $\mu' \in P(D_\mu)$ ,

$$\varphi_\lambda \circ s_{\lambda'}(\text{Int}(D_{\lambda'})) \cap \varphi_\mu \circ s_{\mu'}(\text{Int}(D_{\mu'})) \neq \emptyset$$

implies

$$\varphi_\lambda \circ s_{\lambda'}(\text{Int}(D_{\lambda'})) = \varphi_\mu \circ s_{\mu'}(\text{Int}(D_{\mu'})).$$

For  $\lambda' \in P(\text{Int}(D_\lambda))$ , define a map  $\psi_{\lambda'} : D_{\lambda'} \rightarrow X$  by the composition

$$D_{\lambda'} \xrightarrow{s_{\lambda'}} D_\lambda \xrightarrow{\varphi_\lambda} X,$$

where  $s_{\lambda'}$  is the cell structure of  $e_{\lambda'}$ . Then these structures define a cellular stratification on  $X$  with  $\{\psi_{\lambda'}\}$  cell structure maps.

*Proof.* It suffices to verify that  $\varphi_\lambda$  is a strict morphism of cellular stratified spaces from  $(D_\lambda, \pi_\lambda)$  to  $(X, \pi')$ .

By assumption,

$$\varphi_\lambda|_{\text{Int}(D_\lambda)} : \text{Int}(D_\lambda) \longrightarrow e_\lambda$$

is an isomorphism of stratified spaces. We need to show that  $\varphi_\lambda$  is a strict morphism of stratified spaces on  $\partial D_\lambda$ . For a cell  $e_{\lambda'} \subset \partial D_\lambda$ ,  $\varphi(e_\lambda)$  is contained in  $X \setminus e_\lambda$  and there exist  $\mu \in P(X)$  and  $\mu' \in P(\text{Int}(D_\mu))$  such that

$$\varphi_\lambda \circ s_{\lambda'}(\text{Int}(D_{\lambda'})) \cap \varphi_\mu \circ s_{\mu'}(\text{Int}(D_{\mu'})) \neq \emptyset.$$

By assumption,

$$\varphi_\lambda \circ s_{\lambda'}(\text{Int}(D_{\lambda'})) = \varphi_\mu \circ s_{\mu'}(\text{Int}(D_{\mu'}))$$

and thus  $\varphi_\lambda : (D_\lambda, \pi_\lambda) \rightarrow (X, \pi')$  is a strict morphism of stratified spaces.  $\square$

**Definition 6.33.** We say that the above cellular stratification is *induced* by the family  $\{\pi_\lambda\}_{\lambda \in P(X)}$  of cellular stratifications on the domains of cells.

**Example 6.34.** Consider the cellular stratification  $A = e^1 \cup e^2$  of an open annulus  $A = S^1 \times (0, 1)$  in Figure 6.34.

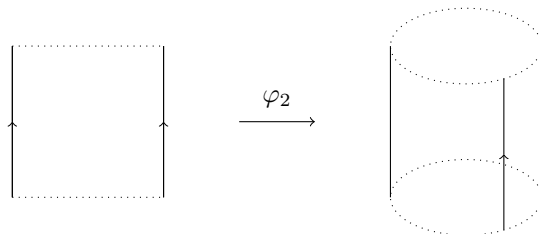


Figure 33: A cellular stratification on an open annulus.

We use  $(0, 1)$  and  $(0, 1) \times [0, 1]$  as the domains of the cell structures for the 1-cell and the 2-cell, respectively. The subdivisions of  $(0, 1)$  in the middle and of  $(0, 1) \times [0, 1]$  by horizontal cut in the middle induce a subdivision of this stratification.

The following “tilted subdivision” of  $(0, 1) \times [0, 1]$ , however, does not induce a subdivision of the annulus, since it does not satisfy the second condition of cellular subdivision.

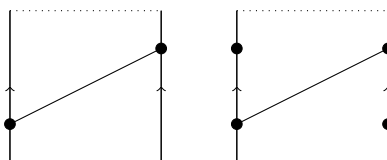


Figure 34: A tilted subdivision of an open annulus.

By subdividing the boundary further, we obtain an induced subdivision, as is shown in the right figure.  $\square$

For totally normal cellular stratified spaces, we require the following conditions.

**Definition 6.35.** Let  $(X, \pi, \Phi)$  be a totally normal cellular stratified space. We say a cellular subdivision  $(X, \pi', \Phi')$  is a *subdivision* of  $(X, \pi, \Phi)$  as a totally normal cellular stratified space if the following conditions are satisfied:

1. For each  $\lambda' \in P(X, \pi')$ , there exists a structure of regular cell complex on  $\partial D^{\dim e_{\lambda'}}$  containing  $\partial D_{\lambda'}$  as a strict stratified subspace.
2. For each  $b \in C(X, \pi)(e_\mu, e_\lambda)$ , the associated map on domains of cells is a strict morphism  $b : (D_\mu, \pi_\mu) \rightarrow (D_\lambda, \pi_\lambda)$  of cellular stratified spaces.

**Proposition 6.36.** *A cellular subdivision of a totally normal cellular stratified space satisfying the conditions in Definition 6.35 is totally normal.*

**Remark 6.37.** An analogous statement was stated as Proposition 2.45 in [FMT15] without the first condition of Definition 6.35. It was pointed out by Priyavrat Deshpande that this condition was missing. He also pointed out typos and errors in the proof, that are corrected in the following proof.

*Proof.* Let  $(X, \pi, \Phi)$  be a totally normal cellular stratified space and  $s : P(X, \pi') \rightarrow P(X, \pi)$  be a cellular subdivision of  $(X, \pi)$  satisfying the conditions in Definition 6.35. Its cell structure is denoted by  $\Phi'$ .

We need to verify that, for each cell  $e_{\lambda'}$  in  $(X, \pi', \Phi')$  and a cell  $e'$  in  $\partial D_{\lambda'}$ , there exists a cell  $e_{\mu'}$  in  $(X, \pi', \Phi')$  and a map  $b' : D_{\mu'} \rightarrow D_{\lambda'}$  making the diagram

$$\begin{array}{ccc}
 D_{\lambda'} & \xrightarrow{\varphi'_{\lambda'}} & \overline{e_{\lambda'}} \hookrightarrow X \\
 \uparrow b' & & \nearrow \\
 D_{\mu'} & \xrightarrow{\varphi'_{\mu'}} & \overline{e_{\mu'}}
 \end{array}$$

commutative and satisfying  $b'(\text{Int}(D_{\mu'})) = e'$ .

Let  $\lambda = s(\lambda')$ . By the definition of cellular subdivision,  $D_\lambda$  has a structure of regular cellular stratified space  $(D_\lambda, \pi_\lambda, \Phi_\lambda)$  under which  $e'$  is a cell. Under the identification

$$P(\text{Int}(D_\lambda), \pi_\lambda) \cong P(\mathbf{s})^{-1}(\lambda),$$

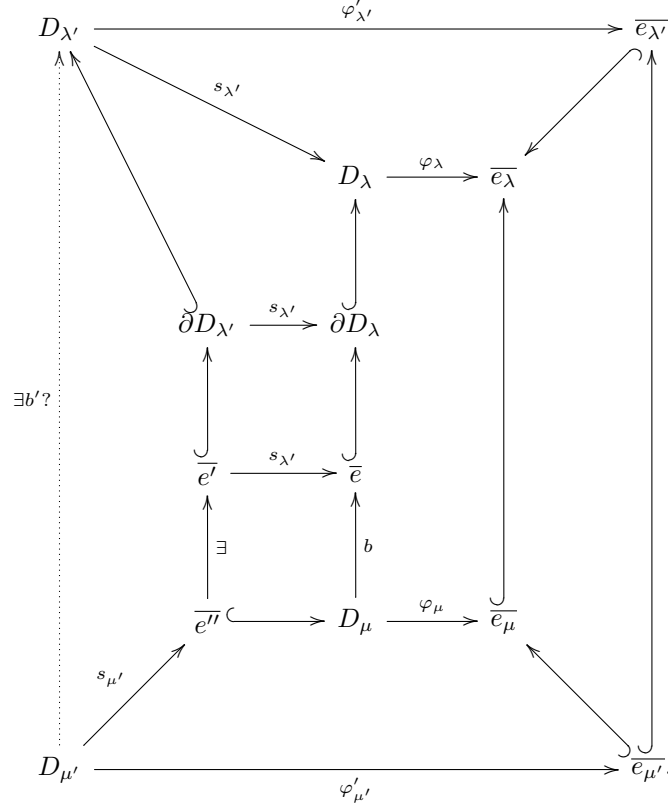
there exists a cell in  $(\text{Int}(D_\lambda), \pi_\lambda)$  corresponding to  $e_{\lambda'}$ . Let  $s_{\lambda'} : D_{\lambda'} \rightarrow D_\lambda$  be its characteristic map. Then the characteristic map for  $e_{\lambda'}$  in  $(X, \pi', \Phi')$  is given by the composition  $\varphi_{\lambda'} \circ s_{\lambda'}$ .

On the other hand, there exists a cell  $e$  in  $\partial D_\lambda$ , before subdivision, containing  $s_{\lambda'}(e')$  in its interior. By the total normality of  $(X, \pi, \Phi)$ , there exists a cell  $e_\mu$  in  $(X, \pi, \Phi)$  and a morphism  $b : e_\mu \rightarrow e_\lambda$  in  $C(X, \pi)$  with  $b(\text{Int}(D_\mu)) = e$  and  $\varphi_\lambda \circ b = \varphi_\mu$ . Since  $b$  is a strict morphism of stratified spaces and  $s_{\lambda'}(e')$  is contained in the interior of  $e$ , there exists a unique cell  $e''$  in  $(D_\mu, \pi_\mu, \Phi_\mu)$  such that  $e' = b(e'')$ .  $\varphi_\mu$  is also a strict morphism of stratified spaces and thus  $\varphi_\mu(e'')$  is a cell in  $(X, \pi', \Phi')$ . Let us denote this cell by  $e_{\mu'}$ . Then the characteristic map for  $e_{\mu'}$  is given by the composition  $\varphi_\mu \circ s_{\mu'}$ , where  $s_{\mu'} : D_{\mu'} \rightarrow D_\mu$  is the characteristic map for  $e''$ .

When  $\mu = \lambda$ , both  $e$  and  $e'$  are cells in  $P(D_\lambda, \pi_\lambda)$  and  $b$  is the identity map. Hence the regularity of  $(D_\lambda, \pi_\lambda, \Phi_\lambda)$  implies the existence of a unique map  $b' : D_{\mu'} \rightarrow D_{\lambda'}$  satisfying the required conditions.



Suppose  $\mu < \lambda$ . The relation among these cells is depicted in the following diagram.



The image of the composition  $b \circ s_{\mu'}$  is contained in the image of  $s_{\lambda'}$ , i.e.  $\bar{e}$ . The regularity of  $(D_{\lambda}, \pi_{\lambda}, \Phi_{\lambda})$  and the fact that  $b$  is an embedding imply that the above diagram can be completed by a map  $b'$ .  $\square$

**Remark 6.38.** A subdivision  $(s, \{(\pi_{\lambda}, \Phi_{\lambda})\})$  of a totally normal cellular stratified space  $(X, \pi, \Phi)$  gives rise to a functor

$$P(s) : C(X, \pi) \longrightarrow \mathbf{Posets}$$

in a canonical way. On objects it is defined by  $P(s)(\lambda) = P(D_{\lambda}, \pi_{\lambda})$ . The second condition in Definition 6.35 guarantees that this extends to a functor.

In general, for a functor  $F : C \rightarrow \mathbf{Cats}$  from a small category to the category  $\mathbf{Cats}$  of small categories, there is a construction of a single category  $\text{Gr}(F)$ , called the Grothendieck construction of  $F$ . A definition can be found in Thomason's paper [Tho79], for example.

It is easy to verify that, in the case of a subdivision of a totally normal cellular stratified space, we have an isomorphism categories

$$\text{Gr}(P(s)) \cong C(X, \pi').$$

The next step would be to study subdivisions of cylindrical and polyhedral structures but it is not easy to describe appropriate conditions on parameter spaces.

## 7 Duality

In [Sal87], Salvetti first constructed a simplicial complex  $\text{Sd}(M(\mathcal{A} \otimes \mathbb{C})) = BC(M(\mathcal{A} \otimes \mathbb{C}))$  modelling the homotopy type of the complement of the complexification of a real hyperplane arrangement  $\mathcal{A}$  and then defined a structure of regular cell complex by gluing simplices. This process reduces the number of cells and allows us to relate the combinatorics of the arrangement  $\mathcal{A}$  and the topology of Salvetti's model  $\text{Sd}(M(\mathcal{A} \otimes \mathbb{C}))$ . The process is closely related to the classical concept of duality in PL topology.

In this section, we introduce an analogous process for polyhedral stellar stratified spaces. With stellar stratifications, we are able to extend both Salvetti's construction and the duality in PL topology.

### 7.1 A Canonical Cellular Stratification on the Barycentric Subdivision

Let us first study stratifications on the barycentric subdivision  $\text{Sd}(X)$  of a cylindrically normal stellar stratified space  $X$ .

In the case of totally normal stellar stratified spaces, Lemma 5.5 and Lemma C.16 imply the following.

**Proposition 7.1.** *For a totally normal stellar stratified space  $X$ ,  $\text{Sd}(X)$  has a structure of regular cell complex.*

Let us extend this structure to a cellular stratification on  $\text{Sd}(X)$  for a cylindrically normal stellar stratified space  $X$ .

Given a cylindrically normal stellar stratified space  $X$ , we have an acyclic topological category  $C(X)$ . By forgetting the topology, Lemma C.16 and Lemma 3.16 give us a map

$$\pi_{\text{Sd}(X)} : \text{Sd}(X) = BC(X) = \|\overline{N}(C(X))\| \xrightarrow{\pi_{\overline{N}(C(X))}} \coprod_k \overline{N}_k(C(X)).$$

As we will see in Lemma 7.33, the set  $\coprod_k \overline{N}_k(C(X))$  can be identified with the set of objects of the barycentric subdivision of the acyclic category  $C(X)$  and has a structure of poset. The partial order is defined as follows.

**Lemma 7.2.** *For  $\mathbf{b} \in \overline{N}_k(C(X))$  and  $\mathbf{b}' \in \overline{N}_\ell(C(X))$ , regarded as functors  $\mathbf{b} : [k] \rightarrow C(X)$  and  $\mathbf{b}' : [\ell] \rightarrow C(X)$ , define  $\mathbf{b} \leq \mathbf{b}'$  if and only if there exists a morphism  $\varphi : [k] \rightarrow [\ell]$  in  $\Delta_{\text{inj}}$  such that  $\mathbf{b}' \circ \varphi = \mathbf{b}$ . Then the relation  $\leq$  is a partial order.*

*Proof.* See Lemma 7.33. □

Let us verify that the map  $\pi_{\text{Sd}(X)}$  defines a cellular stratification on  $\text{Sd}(X)$ .

**Proposition 7.3.** *For a cylindrically normal stellar stratified space  $X$ ,  $\pi_{\text{Sd}(X)}$  defines a cellular stratification.*

*Proof.* Let  $X$  be a cylindrically normal stellar stratified space. For each nondegenerate  $k$ -chain  $\mathbf{b} \in \overline{N}_k(C(X))$ ,  $\pi_{\text{Sd}(X)}^{-1}(\mathbf{b})$  is homeomorphic to  $\text{Int}(\Delta^k)$ . The closure of  $\pi_{\text{Sd}(X)}^{-1}(\mathbf{b})$  in  $\text{Sd}(X)$  is an identification space of  $\Delta^n$ . Since the identification is defined only on the boundary,  $\pi_{\text{Sd}(X)}^{-1}(\mathbf{b})$  is open in its closure and is locally closed. For nondegenerate chains  $\mathbf{b}$  and  $\mathbf{b}'$ ,  $\pi_{\text{Sd}(X)}^{-1}(\mathbf{b}) \subset \overline{\pi_{\text{Sd}(X)}^{-1}(\mathbf{b}'})$  if and only if  $\pi_{\text{Sd}(X)}^{-1}(\mathbf{b})$  is included in  $\pi_{\text{Sd}(X)}^{-1}(\mathbf{b}')$  as a face. In other words, this holds if and only if there exists a sequence of face operators mapping  $\mathbf{b}'$  to  $\mathbf{b}$ , which is equivalent to saying  $\mathbf{b} \leq \mathbf{b}'$ . Thus  $\pi_{\text{Sd}(X)}$  is a stratification in the sense of Definition 2.1.

Under the standard homeomorphism  $\Delta^k \cong D^k$ , we obtain a continuous map

$$\varphi_{\mathbf{b}} : D^k \cong \{\mathbf{b}\} \times \Delta^k \hookrightarrow \prod_k \overline{N}_k(C(X)) \times \Delta^k \longrightarrow \|\overline{N}(C(X))\| \cong \text{Sd}(X).$$

By the compactness of  $D^k$ ,  $\varphi_{\mathbf{b}}$  is a quotient map onto its image, which is the closure of  $\pi_{\text{Sd}(X)}^{-1}(\mathbf{b})$ . The boundary of  $\pi_{\text{Sd}(X)}^{-1}(\mathbf{b})$  consists of points in  $\text{Sd}(X)$  of the form  $[\mathbf{b}, t]$  with  $t \in \partial\Delta^k$ . Under the defining relation, such a point is equivalent to a point in  $\overline{N}_\ell(X) \times \Delta^\ell$  for  $\ell < k$ . Thus we obtain a cellular stratification.  $\square$

Note that this cellular stratification is rarely a CW stratification, when parameter spaces have non-discrete topology.

**Example 7.4.** Consider the minimal cell decomposition  $\pi_n : S^n = e^0 \cup e^n$  of the  $n$ -sphere. We have

$$\begin{aligned} \overline{N}_0(C(S^n)) &= C(S^n)_0 = \{e^0, e^n\} \\ \overline{N}_1(C(S^n)) &= C(S^n)(e^0, e^n) = S^{n-1} \\ \prod_{k \geq 2} \overline{N}_k(C(S^n)) &= \emptyset. \end{aligned}$$

Thus  $\text{Sd}(S^n; \pi_n)$  has only 0-cells and 1-cells. There are two 0-cells  $v(e_0)$  and  $v(e^n)$  corresponding to  $e^0$  and  $e^n$ . There are infinitely many 1-cells parametrized by the equator  $S^{n-1}$ . For  $x \in S^{n-1}$ , the 1-cell  $e_x^1$  corresponding to  $x$  is given by the great half circle from the south pole to the north pole through  $x$  under the identification  $\text{Sd}(S^n; \pi_n) \cong S^n$ .

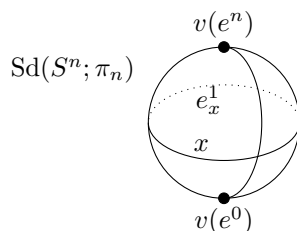


Figure 35: The canonical cellular stratification on  $\text{Sd}(S^n, \pi_n)$ .

$\square$

## 7.2 Stars

As Example 7.4 shows, the cellular stratification on  $\text{Sd}(X)$  defined in the previous section is not very useful. However, Example 7.4 also suggests that by gluing cells in  $\text{Sd}(X)$  together, we may construct a good cellular stratification. In the classical PL topology, such a construction is called *star*<sup>29</sup>.

**Definition 7.5.** Let  $X$  be a stellar stratified space. For  $x \in X$ , define

$$\text{St}(x; X) = \bigcup_{x \in e_\lambda} e_\lambda.$$

<sup>29</sup>See Definition B.18 for the classical definition.

For a subset  $A \subset X$ , define

$$\text{St}(A; X) = \bigcup_{x \in A} \text{St}(x; X).$$

The stratified subspace  $\text{St}(A; X)$  is called the *star* of  $A$  in  $X$ .

**Example 7.6.** In the cellular stratification on  $\text{Sd}(S^n; \pi_n)$  in Example 7.4, the star  $\text{St}(v(e^n); \text{Sd}(S^n; \pi_n))$  coincides with  $S^n \setminus \{v(e^0)\}$  and we recover the original cellular stratification on  $S^n$  as

$$\text{Sd}(S^n) = v(e^0) \cup \text{St}(v(e^n); \text{Sd}(S^n; \pi_n)).$$

Note that we also have another cellular stratification

$$\text{Sd}(S^n) = v(e^n) \cup \text{St}(v(e^0); \text{Sd}(S^n; \pi_n)),$$

which can be regarded as a dual to the original cellular stratification.  $\square$

We introduce analogous constructions for acyclic categories.

**Definition 7.7.** Let  $C$  be an acyclic topological category and  $x$  an object of  $C$ . The nondegenerate nerve  $\overline{N}(x \downarrow C)$  of the comma category<sup>30</sup>  $x \downarrow C$  is denoted by  $\text{St}_{\geq x}(C)$  and is called the *upper star* of  $x$  in  $C$ .

The full subcategory of  $x \downarrow C$  consisting of  $(x \downarrow C)_0 \setminus \{1_x\}$  is denoted by  $C_{>x}$ . The nondegenerate nerve  $\overline{N}(C_{>x})$  is denoted by  $\text{Lk}_{>x}(C)$  and is called the *upper link* of  $x$  in  $C$ .

The functor induced by the target map in  $C$  is denoted by

$$t_x : C_{>x} \subset x \downarrow C \longrightarrow C.$$

The induced map of  $\Delta$ -spaces is also denoted by

$$t_x : \text{Lk}_{>x}(C) \subset \text{St}_{\geq x}(C) \longrightarrow \overline{N}(C).$$

Dually the nondegenerate nerves of the comma category  $C \downarrow x$  and of its full subcategory  $C_{<x}$  consisting of  $(C \downarrow x)_0 \setminus \{1_x\}$  are denoted by  $\text{St}_{\leq x}(C)$  and  $\text{Lk}_{<x}(C)$  and called the *lower star* and *lower link* of  $x$  in  $C$ , respectively. We also have a functor and a map

$$\begin{aligned} s_x & : C_{<x} \longrightarrow C \\ s_x & : \text{Lk}_{<x}(C) \longrightarrow \overline{N}(C) \end{aligned}$$

induced by the source map.

**Remark 7.8.** The notation  $C_{>x}$  and its definition is borrowed from Kozlov's book [Koz08]. Note that  $\text{Lk}_{>x}(C)$  is different from the usual link of  $x$  in  $\overline{N}(C)$  in general.

We have the following description.

**Lemma 7.9.** *For an acyclic topological category  $C$  and an object  $x \in C_0$ , we have the following identification*

$$\text{Lk}_{>x}(C)_k \cong \begin{cases} \coprod_{x \neq y} C(x, y), & k = 0 \\ \{\mathbf{u} \in \overline{N}_{k+1}(C) \mid s(\mathbf{u}) = x\}, & k > 0, \end{cases}$$

under which the face operators

$$d_i^{\text{Lk}} : \text{Lk}_{>x}(C)_k \longrightarrow \text{Lk}_{>x}(C)_{k-1}$$

are identified as  $d_i^{\text{Lk}} = d_{i+1}$ , where  $d_{i+1}$  is the  $(i+1)$ -st face operator in  $\overline{N}(C)$ .

<sup>30</sup> $x \downarrow C$  is the category whose objects are morphisms  $u$  in  $C$  with  $s(u) = x$ . Morphisms are morphisms in  $C$  making the obvious triangle commutative.

**Example 7.10.** Consider the poset  $[2] = \{0 < 1 < 2\}$  regarded as a category  $0 \rightarrow 1 \rightarrow 2$ . The category  $[2]_{>1}$  has a unique object  $1 \rightarrow 2$  and no nontrivial morphism. Thus  $\text{Lk}_{>1}([2])$  is a single point. Under the map  $t_1 : \text{Lk}_{>1}([2]) \rightarrow \overline{N}([2]) \cong \Delta^2$ ,  $\text{Lk}_{>1}([2])$  can be identified with the vertex 2 in  $\Delta^2$ . On the other hand, the usual link of 1 in  $\Delta^2$  is the 1-simplex spanned by vertices 0 and 2.  $\square$

**Example 7.11.** Consider the face category  $C(S^1; \pi_1)$  of the minimal cell decomposition of  $S^1$ .  $C(S^1; \pi_1)_{>e^0}$  consists of two objects  $(C(S^1; \pi_1)_{>e^0})_0 = C(S^1; \pi_1)(e^0, e^1) = \{b_-, b_+\}$  and no nontrivial morphisms. Thus  $\text{St}_{\geq e^0}(C(S^1; \pi_1))$  is the cell complex  $[-1, 1] = \{-1\} \cup (-1, 0) \cup \{0\} \cup (0, 1) \cup \{1\}$  and  $\text{Lk}_{>e^0}(C(S^1; \pi_1))$  is  $S^0$ . The map  $t_{e^0}$  maps the boundary  $\partial[-1, 1] = S^0$  to  $v(e^2)$  in  $\text{Sd}(S^1)$  and defines a 1-cell structure.  $\square$

Note that, the comma category  $x \downarrow C$  has an initial object  $1_x$ .

**Lemma 7.12.** For an acyclic topological category  $C$  and an object  $x \in C_0$ , we have a homeomorphism

$$\|\text{St}_{\geq x}(C)\| \cong \{1_x\} * \|\text{Lk}_{>x}(C)\|.$$

*Proof.* Define a map  $h_x : \|\text{St}_{\geq x}(C)\| \rightarrow \{1_x\} * \|\text{Lk}_{>x}(C)\|$  as follows. For  $[(\mathbf{u}, \mathbf{t})] \in \|\text{St}_{\geq x}(C)\| = \|\overline{N}(x \downarrow C)\|$ , choose a representative  $(\mathbf{u}, \mathbf{t}) \in \overline{N}_k(x \downarrow C) \times \Delta^k$ . Here we regard  $\mathbf{u}$  as a sequence of composable  $k+1$  morphisms in  $C$  starting from  $x$ ;

$$\mathbf{u} : x \xrightarrow{u_0} x_0 \xrightarrow{u_1} x_1 \xrightarrow{u_2} \cdots \xrightarrow{u_k} x_k$$

with  $u_1, \dots, u_k$  non-identity morphisms. When  $u_0$  is not the identity morphism,  $\mathbf{u}$  belongs to  $\text{Lk}_{>x}(C)_k$  and  $[(\mathbf{u}, \mathbf{t})]$  can be regarded as an element of  $\|\text{Lk}_{>x}(C)\|$ . Define

$$h_x[(\mathbf{u}, \mathbf{t})] = 01_x + 1[(\mathbf{u}, \mathbf{t})].$$

When  $u_0 = 1_x$ , write  $\mathbf{t} = t_0 0 + (1 - t_0)\mathbf{t}'$  under the identification  $\Delta^k \cong \{0\} * \Delta^{k-1}$  and define

$$h_x[(\mathbf{u}, \mathbf{t})] = t_0 1_x + (1 - t_0)[(\mathbf{u}', \mathbf{t}')],$$

where

$$\mathbf{u}' : x \xrightarrow{u_1} x_1 \xrightarrow{u_2} \cdots \xrightarrow{u_k} x_k$$

is the  $(k-1)$ -chain obtained from  $\mathbf{u}$  by removing  $u_0$ . Since  $\mathbf{u}$  is a nondegenerate chain,  $u_1$  is not the identity morphism and  $\mathbf{u}' \in \text{Lk}_{>x}(C)_{k-1}$ .

Since the set of objects  $C_0$  has the discrete topology, the decomposition

$$\begin{aligned} & \overline{N}_k(x \downarrow C) \\ &= \left( \coprod_{x < x_1 < \cdots < x_k} C(x, x) \times C(x, x_1) \times \cdots \times C(x_{k-1}, x_k) \right) \\ & \quad \amalg \left( \coprod_{x < x_0 < \cdots < x_k} C(x, x_0) \times \cdots \times C(x_{k-1}, x_k) \right) \\ &= (\{1_x\} \times \text{Lk}_{>x}(x \downarrow C)) \amalg \text{Lk}_{>x}(x \downarrow C) \end{aligned}$$

is a decomposition of topological spaces. The map  $h_x$  is continuous on each component of  $\overline{N}_k(x \downarrow C) \times \Delta^k$  and defines a continuous map

$$h_x : \|\text{St}_{\geq x}(C)\| \longrightarrow \{1_x\} * \|\text{Lk}_{>x}(C)\|.$$

It is easy to define an inverse to  $h_x$ , and thus  $h_x$  is a homeomorphism.  $\square$

### 7.3 Canonical Stellar Stratifications on the Barycentric Subdivision

Let us now go back to the discussion of stellar stratifications on  $\text{Sd}(X)$ .

**Definition 7.13.** Let  $X$  be a cylindrically normal stellar stratified space. Define a map

$$\pi_{X^{\text{op}}} : \text{Sd}(X) \longrightarrow P(X)$$

by the composition

$$\text{Sd}(X) \xrightarrow{\pi_{\text{Sd}(X)}} \coprod_k \overline{N}_k(C(X)) \xrightarrow{s} C(X)_0 = P(X),$$

where  $s$  is the map induced by the source map in the category  $C(X)$  and is also called the *source map*.

By definition<sup>31</sup>,  $\pi_{\text{Sd}(X)}$  is a subdivision of  $\pi_{X^{\text{op}}}$  if  $\pi_{X^{\text{op}}}$  defines a stratification on  $\text{Sd}(X)$ . Each inverse image  $\pi_{X^{\text{op}}}^{-1}(\lambda)$  can be described by using the upper stars.

**Definition 7.14.** For a cell  $e_\lambda$  in a cylindrically normal stellar stratified space  $X$ , define

$$D_\lambda^{\text{op}} = \|\text{St}_{\geq e_\lambda}(C(X))\|$$

We also denote

$$D_\lambda^{\text{op}, \circ} = \|\text{St}_{\geq e_\lambda}(C(X))\| \setminus \|\text{Lk}_{> e_\lambda}(C(X))\|,$$

where  $\|\text{Lk}_{> e_\lambda}(C(X))\|$  is regarded as the bottom subspace of the cone under the identification

$$\|\text{St}_{\geq e_\lambda}(C(X))\| \cong \{1_{e_\lambda}\} * \|\text{Lk}_{> e_\lambda}(C(X))\|$$

in Lemma 7.12.

**Lemma 7.15.** Let  $X$  be a cylindrically normal stellar stratified space. In the stratification  $\pi_{X^{\text{op}}}$ , each stratum is given by the image of  $D_\lambda^{\text{op}, \circ}$  under the map  $t_\lambda = t_{e_\lambda} : \|\text{St}_{\geq e_\lambda}(C(X))\| \rightarrow \text{Sd}(X)$  defined in Definition 7.7. Namely

$$\pi_{X^{\text{op}}}^{-1}(\lambda) = t_\lambda(D_\lambda^{\text{op}, \circ}).$$

Hence the closure of each stratum is given by

$$\overline{\pi_{X^{\text{op}}}^{-1}(\lambda)} = t_\lambda(D_\lambda^{\text{op}}).$$

*Proof.* Elements in  $D_\lambda^{\text{op}, \circ}$  are those which are represented by  $(\mathbf{u}, t_0 0 + (1 - t_0) \mathbf{t}')$ , where  $t_0 < 1$  and  $\mathbf{u}$  begins with the identity morphism  $1_{e_\lambda}$ . Therefore  $s(t_\lambda(\mathbf{u})) = e_\lambda$  and  $t_\lambda(D_\lambda^{\text{op}, \circ}) \subset \pi_{X^{\text{op}}}^{-1}(\lambda)$ . Conversely, by choosing a representative  $(\mathbf{u}, \mathbf{t})$  of  $[(\mathbf{u}, \mathbf{t})] \in \pi_{X^{\text{op}}}^{-1}(\lambda)$  with  $\mathbf{t} \in \text{Int}(\Delta^k)$ , we see  $\pi_{X^{\text{op}}}^{-1}(\lambda) \subset t_\lambda(D_\lambda^{\text{op}, \circ})$ .

Let  $p : \coprod_k \overline{N}_k(C(X)) \times \Delta^k \rightarrow \text{Sd}(X)$  be the projection. Then the topology on  $\text{Sd}(X)$  is the weak topology defined by the covering

$$\{p(C(X)(e_{\lambda_{k-1}}, e_{\lambda_k}) \times \cdots \times C(X)(e_{\lambda_0}, e_{\lambda_1}) \times \Delta^k)\}$$

Thus the closure of

$$t_\lambda(D_\lambda^{\text{op}, \circ}) = p\left(\coprod_k \coprod_{\lambda < \lambda_0 < \cdots < \lambda_k} C(X)(e_{\lambda_{k-1}}, e_{\lambda_k}) \times \cdots \times C(X)(e_\lambda, e_{\lambda_0}) \times (\Delta^k \setminus d^0(\Delta^{k-1}))\right)$$

---

<sup>31</sup>Definition 2.14

is given by adding  $C(X)(e_\lambda, e_{\lambda_1}) \times \cdots \times C(X)(e_{\lambda_{k-1}}, e_{\lambda_k}) \times d^0(\Delta^{k-1})$ . And we have

$$\overline{\pi_{X^{\text{op}}}^{-1}(\lambda)} = \overline{t_\lambda(D_\lambda^{\text{op}, \circ})} = t_\lambda(D_\lambda^{\text{op}}).$$

□

**Corollary 7.16.** *For a cylindrically normal stellar stratified space  $X$ ,  $\pi_{X^{\text{op}}}$  is a stratification whose face poset is the opposite  $P(X)^{\text{op}}$  of  $P(X)$ .*

*Proof.* The fact that  $\pi_{X^{\text{op}}}^{-1}(\lambda)$  is locally closed for each  $\lambda \in P(X)$  follows from the description in Lemma 7.15. It also says that  $\pi_{X^{\text{op}}}^{-1}(\lambda)$  is connected, since it contains the barycenter  $v(e_\lambda)$  of  $e_\lambda$ .

The compatibility with the partial order in  $P(X)^{\text{op}}$  also follows from the description of the boundary in Lemma 7.15. □

**Definition 7.17.** The stratification  $\pi_{X^{\text{op}}}$  is called the *stellar dual* of  $\pi_X$ .

Thus when  $\|\text{Lk}_{>e_\lambda}(C(X))\|$  can be embedded in a sphere  $S^{N-1}$  in such a way that the closure  $\|\overline{\text{Lk}_{>e_\lambda}(C(X))}\|$  is a finite cell complex containing  $\|\text{Lk}_{>e_\lambda}(C(X))\|$  as a strict cellular stratified subspace,  $\|\text{St}_{\geq e_\lambda}(C(X))\|$  can be regarded as an aster in  $D^N$ .

**Proposition 7.18.** *Let  $X$  be a finite polyhedral relatively compact cellular stratified space. Then  $\text{Sd}(X)$  has a structure of stellar stratified space whose underlying stratification is  $\pi_{X^{\text{op}}}$  and the face poset is  $P(\text{Sd}(X), \pi_{X^{\text{op}}}) = P(X, \pi_X)^{\text{op}}$ .*

*Proof.* For  $\lambda \in P(X)$ , consider the upper star  $\text{St}_{\geq e_\lambda}(C(X))$  and the upper link  $\text{Lk}_{>e_\lambda}(C(X))$  of  $\lambda$  in  $C(X)$ . Since compact locally cone-like spaces can be expressed as a union of a finite number of simplices<sup>32</sup>, each parameter space has a structure of a finite polyhedral complex. And the comma category  $e_\lambda \downarrow C(X)$  is a cellular category whose morphism spaces are finite cell complexes. By Lemma C.17 and the finiteness assumption,  $\|\text{St}_{\geq e_\lambda}(C(X))\| = B(e_\lambda \downarrow C(X))$  is a finite cell complex and  $\|\text{Lk}_{>e_\lambda}(C(X))\|$  is a subcomplex. Choose an embedding  $\|\text{Lk}_{>e_\lambda}(C(X))\| \hookrightarrow S^{N-1}$ . Then  $D_\lambda^{\text{op}} = \{1_{e_\lambda}\} * \|\text{Lk}_{>e_\lambda}(C(X))\|$  is embedded in  $D^N$ .

By definition,

$$t_{e_\lambda} : D_\lambda^{\text{op}} \longrightarrow \overline{\pi_{X^{\text{op}}}^{-1}(\lambda)} \subset \text{Sd}(X)$$

is a quotient map. The fact that  $t_{e_\lambda}$  is a homeomorphism onto  $\pi_{X^{\text{op}}}^{-1}(\lambda)$  when restricted to  $\text{Int}(D_\lambda^{\text{op}}) = D_\lambda^{\text{op}, \circ}$  follows easily from the description of elements in  $D_\lambda^{\text{op}, \circ}$  in the proof of Lemma 7.15. □

**Remark 7.19.** The three assumptions, i.e. local-polyhedrality, finiteness, and relative-compactness, are imposed only for the purpose of the existence of an embedding of  $\|\text{Lk}_{>e_\lambda}(C(X))\|$  in a sphere. If we relax the definition of a stellar cell  $\varphi_\lambda : D_\lambda \rightarrow \overline{e_\lambda}$  by dropping the embeddability of the domain  $D_\lambda$  in a disk, we do not need to require these conditions.

The next problem is to define a cylindrical structure for the stellar dual  $(\text{Sd}(X), \pi_{X^{\text{op}}})$ .

**Theorem 7.20.** *Let  $X$  be a finite polyhedral stellar stratified space. Suppose all parameter spaces  $P_{\mu, \lambda}$  are compact. For  $\lambda \leq^{\text{op}} \mu$  in  $P(\text{Sd}(X), \pi_{X^{\text{op}}}) = P(X, \pi_X)^{\text{op}}$ , define  $P_{\lambda, \mu}^{\text{op}} = P_{\mu, \lambda}$ . Then the stellar structure in Proposition 7.18 and parameter spaces  $\{P_{\lambda, \mu}^{\text{op}}\}$  make  $(\text{Sd}(X), \pi_{X^{\text{op}}})$  into a polyhedral stellar stratified space.*

---

<sup>32</sup>Theorem 2.11 in [RS72]

*Proof.* It remain to construct PL structure maps

$$\begin{aligned} b_{\lambda,\mu}^{\text{op}} &: P_{\lambda,\mu}^{\text{op}} \times D_{\lambda}^{\text{op}} \longrightarrow D_{\mu}^{\text{op}} \\ \circ^{\text{op}} &: P_{\lambda_1,\lambda_0}^{\text{op}} \times P_{\lambda_2,\lambda_1}^{\text{op}} \longrightarrow P_{\lambda_2,\lambda_0}^{\text{op}}. \end{aligned}$$

for  $\lambda \leq^{\text{op}} \mu$  and  $\lambda_2 \leq^{\text{op}} \lambda_1 \leq^{\text{op}} \lambda_0^{\text{op}}$ .

The composition map  $\circ^{\text{op}}$  is obviously given by the composition in  $X$  under the identification

$$P_{\lambda_1,\lambda_0}^{\text{op}} \times P_{\lambda_2,\lambda_1}^{\text{op}} = P_{\lambda_0,\lambda_1} \times P_{\lambda_1,\lambda_2} \cong P_{\lambda_1,\lambda_2} \times P_{\lambda_0,\lambda_1}.$$

The action  $b_{\lambda,\mu}^{\text{op}}$  of the parameter space is given by the following composition

$$\begin{aligned} P_{\lambda,\mu}^{\text{op}} \times D_{\lambda}^{\text{op}} &= P_{\mu,\lambda} \times B(e_{\lambda} \downarrow C(X)) \\ &= C(X)(e_{\mu}, e_{\lambda}) \times B(e_{\lambda} \downarrow C(X)) \\ &\cong B(e_{\lambda} \downarrow C(X)) \times C(X)(e_{\mu}, e_{\lambda}) \\ &\longrightarrow B(e_{\mu} \downarrow C(X)) = D_{\mu}^{\text{op}}, \end{aligned}$$

where the last arrow is given by the composition in  $C(X)$ .

The compatibilities of  $b_{\mu,\lambda}$  and  $\circ$  in  $X$  implies that  $b_{\lambda,\mu}^{\text{op}}$  and  $\circ^{\text{op}}$  satisfy the conditions for cylindrical structure. Obviously this is polyhedral.  $\square$

**Definition 7.21.** The stratified space  $(\text{Sd}(X), \pi_{X^{\text{op}}})$  equipped with the stellar and polyhedral structures defined in Theorem 7.20 is called the *stellar dual of  $X$*  and is denoted by  $X^{\text{op}}$ .

The following fundamental example shows that cells in  $X^{\text{op}}$  are usually stellar and not cellular.

**Example 7.22.** Consider the standard regular cellular stratification on  $\Delta^2$  as a simplicial complex. The barycentric subdivision  $\text{Sd}(\Delta^2)$  is a simplicial complex depicted in Figure 36

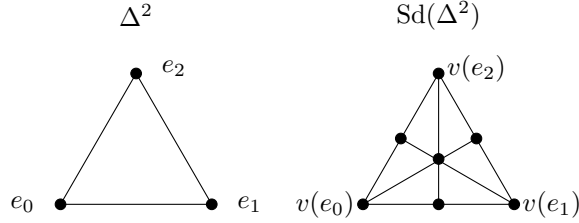


Figure 36:  $\Delta^2$  and its barycentric subdivision.

The stellar stratification on  $\text{Sd}(\Delta^2)$ , i.e.  $(\Delta^2)^{\text{op}}$  is given by Figure 37.  $(\Delta^2)^{\text{op}}$  consists of one stellar 0-cell, three stellar 1-cells, and three stellar 2-cells. The stellar structure for the 2-cell at the left bottom in  $D(\Delta^2)$  is indicated in Figure 37 as  $t_0$ .

Note that  $\|\text{Lk}_{>e_0}(C(\Delta^2))\|$  consists of two 1-simplices. It is embedded in  $S^1$ . The domain for this stellar structure map is the circular sector in the left. The middle point in the arc of the circular sector in  $D_0^{\text{op}}$  is mapped to the barycenter of  $\Delta^2$  and two radii in  $D_0^{\text{op}}$  are mapped to the half edges touching the lower left vertex of  $\Delta^2$ .

The barycentric subdivision of  $(\Delta^2)^{\text{op}}$  can be easily seen to be isomorphic to  $\text{Sd}(\Delta^2)$  as simplicial complexes. And we have  $((\Delta^2)^{\text{op}})^{\text{op}} \cong \Delta^2$  as simplicial complexes. For example, the 2-cell corresponding to the unique 0-cell in  $(\Delta^2)^{\text{op}}$  is the whole triangle.  $\square$



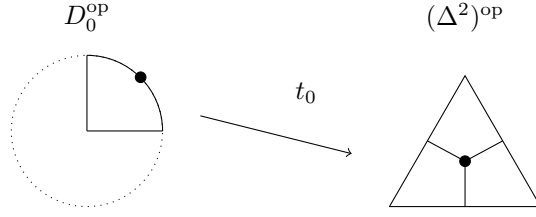


Figure 37: The stellar structures on  $(\Delta^2)^{\text{op}}$ .

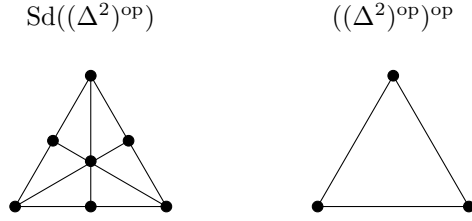


Figure 38: The dual of the dual of  $\Delta^2$ .

The next example shows that, when  $X$  contains non-closed cells, the process of taking the double dual  $((X)^{\text{op}})^{\text{op}}$  slims down  $X$  while retaining the stratification.

**Example 7.23.** Consider the 1-dimensional stellar stratified space  $Y$  in Figure 39. It consists of

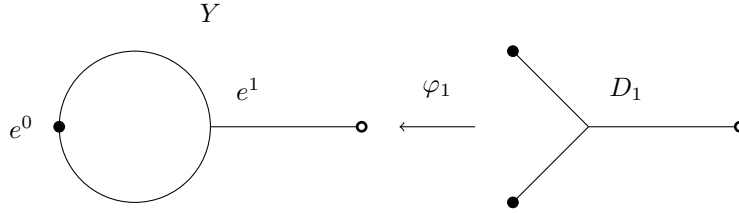


Figure 39: A 1-dimensional stellar stratified space  $Y$ .

a 0-cell  $e^0$  and a stellar 1-cell  $e^1$  whose domain  $D_1$  is a graph of the shape of  $Y$  with one vertex removed. The barycentric subdivision  $\text{Sd}(Y)$  is the minimal regular cell decomposition of  $S^1$ , as is shown in Figure 40. Both  $Y^{\text{op}}$  and  $(Y^{\text{op}})^{\text{op}}$  are the minimal cell decomposition of  $S^1$ . Note that the embedding  $i_Y$  in Theorem 5.16 embeds  $(Y^{\text{op}})^{\text{op}}$  in  $Y$  as a stellar stratified space.  $\square$

As the above example suggests, the embedding  $i_X$  in Theorem 5.16 is an embedding of stellar stratified spaces if the domain is regarded as  $(X^{\text{op}})^{\text{op}}$ . Furthermore, when all cells are closed, we can always recover  $X$  from this stellar structure.

Note that we may define this stellar stratification on  $\text{Sd}(X)$  directly without using  $(-)^{\text{op}}$ .

**Definition 7.24.** Let  $X$  be a cylindrically normal stellar stratified space. For  $\lambda \in P(X)$ , define

$$\begin{aligned} D_\lambda^{\text{Sal}(X)} &= \|\text{St}_{\leq e_\lambda}(C(X))\| \\ D_\lambda^{\text{Sal}(X), \circ} &= \|\text{St}_{\leq e_\lambda}(C(X))\| \setminus \|\text{Lk}_{< e_\lambda}(C(X))\|. \end{aligned}$$

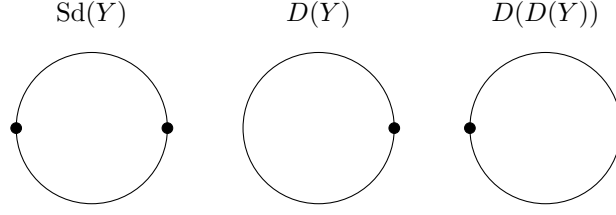


Figure 40: The stellar dual of  $Y$ .

The following fact is dual to Lemma 7.15. The proof is omitted.

**Lemma 7.25.** *Let  $X$  be a cylindrically normal stellar stratified space. Define a map*

$$\pi_{\text{Sal}(X)} : \text{Sd}(X) \longrightarrow P(X)$$

by the composition

$$\text{Sd}(X) \xrightarrow{\pi_{\text{Sd}(X)}} \prod_k \overline{N}_k(C(X)) \xrightarrow{t} C(X)_0 = P(X),$$

where  $t$  is the target map. Then  $\pi_{\text{Sal}(X)}$  is a stratification whose strata and their closures are given by

$$\begin{aligned} \pi_{\text{Sal}(X)}^{-1}(\lambda) &= s_\lambda \left( D_\lambda^{\text{Sal}(X), \circ} \right) \\ \overline{\pi_{\text{Sal}(X)}^{-1}(\lambda)} &= s_\lambda \left( D_\lambda^{\text{Sal}(X)} \right), \end{aligned}$$

where  $s_\lambda = s_{e_\lambda}$  is the map defined in Definition 7.7.

**Definition 7.26.** The stellar stratified space  $(\text{Sd}(X), \pi_{\text{Sal}})$  defined above is called the *Salvetti complex* of  $X$  and is denoted by  $\text{Sal}(X)$ .

**Remark 7.27.** When the three assumptions in Proposition 7.18 are satisfied, we have  $\text{Sal}(X) = (X^{\text{op}})^{\text{op}}$  as stellar stratified spaces.

**Theorem 7.28.** *Let  $(X, \pi_X)$  be a cylindrically normal stellar stratified space. Then  $\text{Sal}(X)$  has a structure of cylindrically normal stellar stratified space. When  $X$  is relatively compact, the embedding  $i_X : \text{Sal}(X) \hookrightarrow X$  is an embedding of cylindrically normal stellar stratified spaces. When all cells in  $X$  are closed,  $i_X$  is an isomorphism of cylindrically normal stellar stratified spaces. When  $X$  is a finite polyhedral stellar stratified space satisfying the assumptions of Proposition 7.18, we have  $\text{Sal}(X) = (X^{\text{op}})^{\text{op}}$  as stellar stratified spaces.*

**Definition 7.29.** When we regard  $\text{Sd}(X)$  as  $\text{Sal}(X)$ , the embedding  $i_X$  is denoted by  $i_{\text{Sal}(X)} : \text{Sal}(X) \hookrightarrow X$ .

In order to prove Theorem 7.28, we use the following reformulation of the construction of  $i_X$ .

**Lemma 7.30.** *Let  $X$  be a CW cylindrically normal stellar stratified space  $X$ . For each stellar cell  $e_\lambda$ , there exists an embedding*

$$z_\lambda : D_\lambda^{\text{Sal}(X)} \longrightarrow D_\lambda$$

making the diagram

$$\begin{array}{ccc}
P_{\mu,\lambda} \times D_{\mu}^{\text{Sal}(X)} & \xrightarrow{b_{\mu,\lambda}^{\text{Sal}(X)}} & D_{\lambda}^{\text{Sal}(X)} \\
\downarrow 1 \times z_{\mu} & & \downarrow z_{\lambda} \\
P_{\mu,\lambda} \times D_{\mu} & \xrightarrow{b_{\mu,\lambda}} & D_{\lambda}
\end{array} \tag{7}$$

commutative, where  $b_{\mu,\lambda}^{\text{Sal}(X)}$  is defined by the composition

$$\begin{aligned}
P_{\mu,\lambda} \times D_{\mu}^{\text{Sal}(X)} &= P_{\mu,\lambda} \times B(C(X) \downarrow e_{\mu}) \\
&= C(X)(e_{\mu}, e_{\lambda}) \times B(C(X) \downarrow e_{\mu}) \\
&\longrightarrow B(C(X) \downarrow e_{\lambda}).
\end{aligned}$$

Here the last map is induced by the composition in  $C(X)$ .

These maps  $\{z_{\lambda}\}$  can be glued together to define an embedding  $\text{Sal}(X) \hookrightarrow X$ , which can be identified with  $i_X$  in Theorem 5.16

*Proof.* Note that we have the following decomposition

$$\bar{N}_k(C(X)) = \coprod_{\lambda \in P(X)} \bar{N}_k(C(X) \downarrow e_{\lambda}).$$

The restrictions of the maps  $z_k : \bar{N}_k(C(X)) \times \Delta^k \rightarrow D(X)$  to  $\bar{N}_k(C(X) \downarrow e_{\lambda})$  define a map

$$z_{\lambda} : \|\bar{N}(C(X) \downarrow e_{\lambda})\| = D_{\lambda}^{\text{op}} \longrightarrow D_{\lambda}.$$

The fact that these maps make the diagram (7) commutative can be verified by investigating the construction of  $z_k$  and is omitted.  $\square$

*Proof of Theorem 7.28.* Lemma 7.30 says that  $i_X$  is an embedding of stellar stratified spaces.

When all cells in  $X$  are closed,  $i_X : \text{Sd}(X) \rightarrow X$  is a homeomorphism. The above argument implies that this map defines an isomorphism  $i_{\text{Sal}(X)} : \text{Sal}(X) \xrightarrow{\cong} X$  of cylindrically normal stellar stratified spaces.

Suppose that  $X$  satisfies the assumption of Proposition 7.18. By Theorem 7.20, we have an isomorphism of categories  $C(X^{\text{op}}) \cong C(X)^{\text{op}}$ . Thus

$$D_{\lambda}^{(X^{\text{op}})^{\text{op}}} = B(e_{\lambda} \downarrow C(X^{\text{op}})) \cong B(e_{\lambda} \downarrow C(X)^{\text{op}}) \cong B(C(X) \downarrow e_{\lambda})$$

and the stellar structure map  $t_{\lambda} : D_{\lambda}^{(X^{\text{op}})^{\text{op}}} \rightarrow \text{Sd}(X^{\text{op}})$  can be identified with  $s_{\lambda} : D_{\lambda}^{\text{Sal}(X)} \rightarrow \text{Sd}(X)$  under the identification  $\text{Sd}(D(X)) \cong B(C(X)^{\text{op}}) \cong B(C(X)) = \text{Sd}(X)$ . And we have an identification  $D(D(X)) \cong \text{Sal}(X)$ .  $\square$

The following example justifies the name for  $\text{Sal}(X)$ .

**Example 7.31.** For a real hyperplane arrangement  $\mathcal{A}$ , the structure of regular cell complex on the Salvetti complex  $\text{Sal}(\mathcal{A})$  for the complexification of  $\mathcal{A}$  defined in [Sal87] is nothing but  $D(D(M(\mathcal{A} \otimes \mathbb{C})))$ . For example, in the case of the arrangement  $\mathcal{A} = \{\{0\}\}$  in  $\mathbb{R}$ , the stratification on  $\mathbb{R}$  is

$$\mathbb{R} = \{0\} \cup (-\infty, 0) \cup (0, \infty)$$

and the associated stratification on the complexification is

$$\mathbb{C} = \{(0, 0)\} \cup (-\infty, 0) \times \{0\} \cup (0, \infty) \times \{0\} \cup \{x + iy \in \mathbb{C} \mid y > 0\} \cup \{x + iy \in \mathbb{C} \mid y < 0\}.$$

Then  $\text{Sd}(M(\mathcal{A} \otimes \mathbb{C}))$ ,  $M(\mathcal{A} \otimes \mathbb{C})^{\text{op}}$ , and  $(M(\mathcal{A} \otimes \mathbb{C})^{\text{op}})^{\text{op}}$  are given in Figure 31.  $\square$

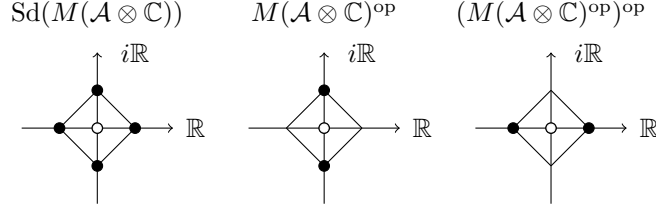


Figure 41: Stellar duals for  $M(\mathcal{A} \otimes \mathbb{C})$

## 7.4 The Barycentric Subdivision of Face Categories

We conclude this paper by proving that the barycentric subdivision of a totally normal cellular stratified space corresponds to the barycentric subdivision of the face category.

Let us first recall the definition of the barycentric subdivision of a small category. We use a definition in Noguchi's papers [Nog11, Nog13]. See also the paper [dH08] by del Hoyo.

**Definition 7.32.** For a small category  $C$ , the *barycentric subdivision*  $\text{Sd}(C)$  is a small category defined by

$$\begin{aligned} \text{Sd}(C)_0 &= \coprod_n \overline{N}_n(C), \\ \text{Sd}(C)(f, g) &= \{\varphi : [m] \rightarrow [n] \mid g \circ \varphi = f\} / \sim \end{aligned}$$

for  $f : [m] \rightarrow C$  and  $g : [n] \rightarrow C$ , where  $\sim$  is the equivalence relation generated by the following relation: for functors  $\varphi, \psi : [m] \rightarrow [n]$  with  $g \circ \varphi = f$  and  $g \circ \psi = f$ ,  $\varphi \sim \psi$  if and only if the morphism  $g(\min\{\varphi(i), \psi(i)\}) \leq g(\max\{\varphi(i), \psi(i)\})$  in  $C(g(\min\{\varphi(i), \psi(i)\}), g(\max\{\varphi(i), \psi(i)\}))$  is an identity morphism in  $C$  for any  $i$  in  $[m]$ .

The description can be simplified for acyclic categories as follows.

**Lemma 7.33.** *Let  $C$  be an acyclic small category. For  $f, g \in \text{Sd}(C)_0$ , the set of morphisms  $\text{Sd}(C)(f, g)$  consists of a single point, if there exists  $\varphi$  with  $f = g \circ \varphi$ , and an empty set otherwise.*

*Therefore  $\text{Sd}(C)$  is a poset.*

*Proof.* Since  $C$  is acyclic,  $C(x, x) = \{1_x\}$  for any objects  $x \in C_0$ . This implies that for  $\varphi, \psi : [m] \rightarrow [n]$  with  $f = g \circ \varphi = g \circ \psi$ ,  $\varphi \sim \psi$  if and only if  $g(\varphi(i)) = g(\psi(i))$  for all  $i \in [m]$ . In other words, all elements in  $\{\varphi : [m] \rightarrow [n] \mid g \circ \varphi = f\}$  are equivalent to each other. Hence  $\text{Sd}(C)(f, g)$  is a single point if the above set is nonempty.  $\square$

In order to compare  $C(\text{Sd}(X))$  and  $\text{Sd}(C(X))$  for a totally normal stellar stratified space  $X$ , we need to understand the cellular stratification on  $\text{Sd}(X)$ . By Corollary 7.1, we know that  $\text{Sd}(X)$  is a totally normal cell complex when  $X$  is a totally normal stellar stratified space. Cells are parametrized by elements in  $\overline{N}(C(X))$ . Let us denote the cell corresponding to  $\mathbf{b} \in \overline{N}(C(X))$  by  $e_{\mathbf{b}}$ . Cell structure maps are given as follows.

**Lemma 7.34.** *For each  $k$ , fix a homeomorphism  $D^k \cong \Delta^k$ . Let  $X$  be a totally normal stellar stratified space. For  $\mathbf{b} \in \overline{N}_k(C(X))$ , the composition*

$$D^k \cong \Delta^k = B[k] \xrightarrow{B\mathbf{b}} BC(X) = \text{Sd}(X)$$

*defines a cell structure on the cell corresponding to  $\mathbf{b}$ , where we regard  $\mathbf{b}$  as a functor  $\mathbf{b} : [k] \rightarrow C(X)$ .*

*Proof.* The map  $B\mathbf{b} : B[k] \rightarrow BC(X)$  is induced by the map  $\overline{N}\mathbf{b} : \overline{N}([k]) \rightarrow \overline{N}(C(X))$ . As we have seen in the proof of Proposition 7.3, a cell structure map on the cell corresponding to  $\mathbf{b}$  is given by the composition

$$\Delta^k \cong \{\mathbf{b}\} \times \Delta^k \hookrightarrow \prod_k \overline{N}_k(C(X)) \times \Delta^k \longrightarrow \|\overline{N}(C(X))\| \cong BC(X).$$

Since  $\overline{N}_k([k])$  consists of a single point, the above composition can be identified with

$$\Delta^k \cong \overline{N}_k([k]) \times \Delta^k \xrightarrow{\overline{N}(\mathbf{b}) \times 1} \prod_k \overline{N}_k(C(X)) \times \Delta^k \longrightarrow \|\overline{N}(C(X))\| \cong BC(X)$$

and the result follows.  $\square$

For simplicity, we use the standard simplices  $\Delta^k$  as the domains of cells in  $\text{Sd}(X)$ . The cell structure map for  $e_{\mathbf{b}}$  is identified with  $B\mathbf{b}$  by Lemma 7.34.

**Theorem 7.35.** *For any totally normal stellar stratified space  $X$ , we have an isomorphism of categories*

$$\text{Sd}(C(X)) \cong C(\text{Sd}(X)).$$

*Proof.* By definition, objects in  $\text{Sd}(C(X))$  are elements of the nondegenerate nerve of  $C(X)$ . On the other hand, objects in  $C(\text{Sd}(X))$  are in one-to-one correspondence with cells in  $\text{Sd}(X) = BC(X)$ . Under the stratification in Proposition 7.3, we obtain a bijection  $C(\text{Sd}(X))_0 \cong \text{Sd}(C(X))_0$ .

For  $\mathbf{b} \in \overline{N}_k(C(X))$  and  $\mathbf{b}' \in \overline{N}_m(C(X))$ , we have

$$C(\text{Sd}(X); \pi_{\text{Sd}(X)})(e_{\mathbf{b}'}, e_{\mathbf{b}}) = \{f : \Delta^m \rightarrow \Delta^k \mid B\mathbf{b}' = B\mathbf{b} \circ f\}.$$

Since  $B\mathbf{b}'|_{\text{Int}(\Delta^m)}$  is injective,  $f|_{\text{Int}(\Delta^m)}$  is also injective. The condition  $B\mathbf{b}' = B\mathbf{b} \circ f$  implies that  $f|_{\text{Int}(\Delta^m)}$  is a PL map and hence  $f$  is a PL map. Since  $B\mathbf{b}|_{\text{Int}(\Delta^k)}$  is injective, such a PL map is unique if it were to exist. It is given by  $f = B\varphi$  for some poset map  $\varphi : [m] \rightarrow [k]$ .

On the other hand, by Lemma 7.33,  $\text{Sd}(C(X))(\mathbf{b}', \mathbf{b})$  is nonempty (and a single point set) if and only if there exists a poset map  $\varphi : [m] \rightarrow [k]$  with  $\mathbf{b}' = \mathbf{b} \circ \varphi$ . Thus the classifying space functor  $B(-)$  induces an isomorphism of categories

$$B : \text{Sd}(C(X)) \longrightarrow C(\text{Sd}(X)).$$

$\square$

**Remark 7.36.** Note that we obtained an isomorphism of categories instead of an equivalence. Since  $\text{Sd}(C(X))$  is a poset, it implies that  $C(\text{Sd}(X))$  is also a poset. Thus the barycentric subdivision  $\text{Sd}(X)$  of a totally normal stellar stratified space is a regular cell complex.

## A Generalities on Quotient Maps

In our definition of cell structures, we required the cell structure map  $\varphi : D \rightarrow \bar{e}$  of a cell  $e$  to be a quotient map. In order to perform operations on cellular stratified spaces, such as taking products and subspaces, we need to understand basic properties of quotient maps.

## A.1 Definitions

It is well-known that the quotient topology does not behave well with respect to certain operations of topological spaces, such as taking products and subspaces. We need to impose stronger conditions.

**Definition A.1.** A surjective continuous map  $f : X \rightarrow Y$  is called *bi-quotient*, if, for any  $y \in Y$  and any open covering  $\mathcal{U}$  of  $f^{-1}(y)$ , there exists finitely many  $U_1, \dots, U_k \in \mathcal{U}$  such that  $\bigcup_{i=1}^k f(U_i)$  contains a neighborhood of  $y$  in  $Y$ .

Another important class of maps are hereditarily quotient maps.

**Definition A.2.** A surjective continuous map  $f : X \rightarrow Y$  is called *hereditarily quotient* if, for any  $y \in Y$  and any neighborhood  $U$  of  $f^{-1}(y)$ ,  $f(U)$  is a neighborhood of  $y$ .

## A.2 Properties

This quotient topology condition imposes some restrictions on the topology of  $\bar{e}$ , especially when  $e$  is closed. For example,  $\bar{e}$  is metrizable for any closed cell  $e$ . A proof can be found in a book [LW69] by Lundell and Weingram. Their proof can be modified to obtain the following extension of this fact.

**Lemma A.3.** Suppose  $\varphi : D \rightarrow \bar{e} \subset X$  is an  $n$ -cell structure with  $\varphi^{-1}(y)$  compact for each  $y \in \bar{e}$ . Then  $\bar{e}$  is metrizable. In particular, it is Hausdorff and paracompact<sup>33</sup>.

*Proof.* For  $y, y' \in \bar{e}$ , define

$$\bar{d}(y, y') = \min \{d(x, x') \mid x \in \varphi^{-1}(y), x' \in \varphi^{-1}(y')\},$$

where  $d$  is the metric on  $D^n$ . By assumption,  $\varphi^{-1}(y)$  and  $\varphi^{-1}(y')$  are compact and  $\bar{d}(y, y')$  is defined. The compactness of  $\varphi^{-1}(y)$  and  $\varphi^{-1}(y')$  also implies that  $\bar{d}$  is a metric on  $\bar{e}$ .

Let us verify that the topology defined by  $\bar{d}$  coincides with the quotient topology by  $\varphi$ . The continuity of  $\varphi$  with respect to the metric topologies on  $D$  and  $\bar{e}$  implies that open subsets in the  $\bar{d}$ -metric topology are open in the quotient topology. Conversely let  $U$  be an open subset of  $\bar{e}$  with respect to the quotient topology. We would like to show that, for each  $y \in U$ , there exists  $\delta > 0$  such that the open disk  $U_\delta(y; \bar{d})$  around  $y$  with radius  $\delta$  with respect to the metric  $\bar{d}$  is contained in  $U$ . Let  $\delta$  be a Lebesgue number of the open covering  $\{\varphi^{-1}(U)\}$  of the compact metric space  $\varphi^{-1}(y)$ . For  $y' \in U_\delta(y; \bar{d})$ , there exist  $x \in \varphi^{-1}(y)$  and  $x' \in \varphi^{-1}(y')$  such that  $d(x, x') < \delta$ . Thus  $x' \in U_\delta(x; d) \subset \varphi^{-1}(U)$  by the definition of Lebesgue number. Or  $y' = \varphi(x') \in U$ . And we have  $U_\delta(y'; \bar{d}) \subset U$ .  $\square$

**Definition A.4.** We say a cell structure  $\varphi : D \rightarrow \bar{e}$  is *relatively compact* if  $\varphi^{-1}(y)$  is compact for each  $y \in \bar{e}$ . We also say that the cell  $e$  is *relatively compact*.

In particular, when  $\varphi : D \rightarrow \bar{e}$  is proper (i.e. closed and each  $\varphi^{-1}(y)$  is compact),  $\bar{e}$  is metrizable. On the other hand, the properness of  $\varphi$  implies that  $\varphi$  is a bi-quotient map.

It is straight forward to verify that a hereditarily quotient map can be restricted freely.

**Lemma A.5.** Any hereditarily quotient map  $f : X \rightarrow Y$  is a quotient map. More generally, for any subspace  $A \subset Y$ , the restriction  $f|_{f^{-1}(A)} : f^{-1}(A) \rightarrow A$  is hereditarily quotient, hence a quotient map.

---

<sup>33</sup>Theorem 41.4 in [Mun00]

*Proof.* Suppose  $f$  is hereditarily quotient. For a subset  $U \subset Y$ , suppose that  $f^{-1}(U)$  is open in  $X$ . For a point  $y \in U$ ,  $f^{-1}(U)$  is a neighborhood of  $f^{-1}(y)$ . Since  $f$  is hereditarily quotient,  $f(f^{-1}(U)) = U$  is a neighborhood of  $y$  in  $Y$ . Thus  $y$  is an interior point of  $U$  and it follows that  $U$  is an open subset of  $Y$ .

Since the definition of hereditarily quotient map is local,  $f|_{f^{-1}(A)}$  is hereditarily quotient for any  $A \subset Y$ .  $\square$

**Remark A.6.** See also Arhangel'skii's paper [Arh63] for hereditarily quotient maps.

**Lemma A.7.** *Any bi-quotient map is hereditarily quotient. In particular, it is a quotient map.*

*Proof.* By definition.  $\square$

Michael [Mic68] proved that bi-quotient maps are abundant.

**Lemma A.8.** *Any one of the following conditions implies that a map  $f : X \rightarrow Y$  is bi-quotient:*

1.  $f$  is open.
2.  $f$  is proper.
3.  $f$  is hereditarily quotient and the boundary  $\partial f^{-1}(y)$  of each fiber is compact.

*Proof.* Proposition 3.2 in [Mic68].  $\square$

Recall that a product of quotient maps may not be a quotient map. There exist a space  $X$  and a quotient map  $f : Y \rightarrow Z$  such that the product  $1_X \times f : X \times Y \rightarrow X \times Z$  is not a quotient map. The following fact is a well-known result of J.H.C. Whitehead [Whi48].

**Lemma A.9.** *For a locally compact Hausdorff space  $X$ ,  $1_X \times f$  is a quotient map for any quotient map  $f$ .*

Unfortunately the domain of cell structures may not be locally compact.

**Example A.10.**  $D^2 - \{(1, 0)\}$  is locally compact, while  $\text{Int}(D^2) \cup \{(1, 0)\}$  is not locally compact.  $\square$

The domain  $D$  of an  $n$ -cell structure  $\varphi : D \rightarrow \bar{e}$  is often a stratified subspace of  $D^n$  under a normal cell decomposition of  $D^n$ . In other words,  $D$  is obtained from  $D^n$  by removing cells. In the Example A.10,  $D^2$  is regarded as a cell complex  $D^2 = e^0 \cup e^1 \cup e^2$ .  $D^2 \setminus e^0$  is locally compact, while  $D^2 \setminus e^1$  is not. More generally we have the following criterion of locally compact subspaces in a CW complex.

**Proposition A.11.** *Let  $X$  be a locally finite CW complex and  $A$  be a subcomplex, then  $X \setminus A$  is locally compact.*

This is an immediate corollary to the following fact, which can be found, for example, in Chapter XI of Dugundji's book [Dug78] as Theorem 6.5.

**Lemma A.12.** *Let  $X$  be a locally compact Hausdorff space. A subspace  $A \subset X$  is locally compact if and only if there exist closed subsets  $F_1, F_2 \subset X$  with  $A = F_2 \setminus F_1$ .*

*Proof of Proposition A.11.* Since  $X$  is locally finite, it is locally compact. The CW condition implies that  $A$  is closed in  $X$ .  $\square$

Let us go back to the discussion on products of quotient maps. The main motivation of Michael for introducing bi-quotient maps is that they behave well with respect to products.

**Proposition A.13.** For any family of bi-quotient maps  $\{f_i : X_i \rightarrow Y_i\}_{i \in I}$ , the product

$$\prod_{i \in I} f_i : \prod_{i \in I} X_i \longrightarrow \prod_{i \in I} Y_i$$

is a bi-quotient map.

*Proof.* Theorem 1.2 in [Mic68]. □

The following property is also useful when we study cell structures.

**Lemma A.14.** Let  $f : X \rightarrow Y$  be a quotient map. Suppose that  $Y$  is first countable and Hausdorff and that, for each  $y \in Y$ ,  $\partial f^{-1}(y)$  is Lindelöf. Then  $f$  is bi-quotient.

*Proof.* Proposition 3.3(d) in [Mic68]. □

**Corollary A.15.** Let  $\varphi : D \rightarrow \bar{e}$  be a relatively compact cell. Then  $\varphi$  is bi-quotient.

*Proof.* By Lemma A.3,  $\bar{e}$  is first countable and Hausdorff. By assumption each fiber  $\varphi^{-1}(y)$  is compact and so is the boundary  $\partial \varphi^{-1}(y)$ . The result follows from Lemma A.14. □

## B Simplicial Topology

In this second appendix, we recall basic definitions and theorems in PL (Piecewise Linear) topology and simplicial homotopy theory used in this paper. Our references are

- the book [RS72] of Rourke and Sanderson for PL topology, and
- the book [GJ09] by Goerss and Jardine for simplicial homotopy theory.

### B.1 Simplicial Complexes, Simplicial Sets, and Simplicial Spaces

Let us fix notation and terminology for simplicial complexes first. Good references are Dwyer's monograph [DH01] and Friedman's survey article [Fri12].

**Definition B.1.** For a set  $V$ , the power set of  $V$  is denoted by  $2^V$ .

**Definition B.2.** Let  $V$  be a set. An *abstract simplicial complex* on  $V$  is a family of subsets  $K \subset 2^V$  satisfying the following condition:

- $\sigma \in K$  and  $\tau \subset \sigma$  imply  $\tau \in K$ .

$K$  is called *finite* if  $V$  is a finite set.

**Definition B.3.** An *ordered simplicial complex*  $K$  is an abstract simplicial complex whose vertex set  $P$  is partially ordered in such a way that the induced ordering on each simplex is a total order.

An  $n$ -simplex  $\sigma \in K$  with vertices  $v_0 < \dots < v_n$  is denoted by  $\sigma = [v_0, \dots, v_n]$ .

There are several ways to define the geometric realization of an abstract simplicial complex.



**Definition B.4.** For an abstract simplicial complex  $K$  with vertex set  $V$ , define a space  $\|K\|$  by

$$\|K\| = \left\{ f \in \text{Map}^f(V, \mathbb{R}) \left| \sum_{v \in \sigma} f(v) = 1, f(v) \geq 0, \sigma \in K \right. \right\},$$

where  $\text{Map}^f(V, \mathbb{R})$  is the set of maps from  $V$  to  $\mathbb{R}$  whose values are 0 except for a finite number of elements. It is equipped with the compact-open topology. The space  $\|K\|$  is called the *geometric realization* of  $K$ .

**Lemma B.5.** Suppose the vertex set  $V$  of an abstract simplicial complex  $K$  is finite. Choose an embedding

$$i : V \hookrightarrow \mathbb{R}^N$$

for a sufficiently large  $N$  so that the  $i(V)$  is affinely independent. Then we have a homeomorphism

$$\|K\| \cong \left\{ \sum_{v \in V} a_v i(v) \left| \sum_{v \in \sigma} a_v = 1, a_v \geq 0, \sigma \in K \right. \right\}.$$

**Example B.6.** Consider  $2^V \setminus \{\emptyset\}$  for  $V = \{0, \dots, n\}$ . This is an abstract simplicial complex. Then we have a homeomorphism

$$\|2^V \setminus \{\emptyset\}\| \cong \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \left| \sum_i t_i = 1, t_i \geq 0 \right. \right\} = \Delta^n.$$

$\Delta^n$  is a convex polytope having  $(n+1)$  codimension 1 faces. Each codimension 1 face can be realized as the image of the map

$$d^i : \Delta^{n-1} \longrightarrow \Delta^n$$

defined by

$$d^i(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}).$$

We also have maps

$$s^i : \Delta^n \longrightarrow \Delta^{n-1}$$

defined by

$$s^i(t_0, \dots, t_n) = (t_0, \dots, t_i + t_{i+1}, t_{i+2}, \dots, t_n).$$

□

For an ordered simplicial complex  $K$ , we may forget the ordering and apply the above construction. However, there is another construction.

**Definition B.7.** For an ordered simplicial complex  $K$  with vertex set  $V$  Let  $K_n$  be the set of  $n$ -simplices in  $K$ . Each element  $\sigma$  in  $K_n$  can be written as  $\sigma = (v_0, \dots, v_n)$  with  $v_0 < \dots < v_n$ . Under such an expression, define

$$d_i(\sigma) = (v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n).$$

Define

$$\|K\| = \left( \coprod_n K_n \times \Delta^n \right) / \sim,$$

where the relation  $\sim$  is generated by

$$(\sigma, d^i(\mathbf{t})) \sim (d_i(\sigma), \mathbf{t}).$$

This is called the *geometric realization* of  $K$ .

**Lemma B.8.** For a finite ordered simplicial complex, the above two constructions of the geometric realization coincide.

The above construction can be extended to simplicial sets and simplicial spaces.

**Definition B.9.** A *simplicial set*  $X$  consists of

- a sequence of sets  $X_0, X_1, \dots$ ,
- a family of maps  $d_i : X_n \rightarrow X_{n-1}$  for  $0 \leq i \leq n$ ,
- a family of maps  $s_i : X_n \rightarrow X_{n+1}$  for  $0 \leq i \leq n$

satisfying the following relations

$$\begin{cases} d_i \circ d_j = d_{j-1} \circ d_i, & i < j \\ d_i \circ s_j = s_{j-1} \circ d_i, & i < j \\ d_j \circ s_j = 1 = d_{j+1} \circ s_j \\ d_i \circ s_j = s_j \circ d_{i-1}, & i > j + 1 \\ s_i \circ s_j = s_{j+1} \circ s_i, & i \leq j. \end{cases}$$

The maps  $d_i$ 's and  $s_j$ 's are called *face operators* and *degeneracy operators*, respectively.

When each  $X_n$  is a topological space and maps  $d_i, s_j$  are continuous,  $X$  is called a *simplicial space*.

**Remark B.10.** It is well known that defining a simplicial set  $X$  is equivalent to defining a functor

$$X : \Delta^{\text{op}} \longrightarrow \mathbf{Sets},$$

where  $\Delta$  is the full subcategory of the category of posets consisting of  $[n] = \{0 < 1 < \dots < n\}$  for  $n = 0, 1, 2, \dots$

**Example B.11.** For a topological space  $X$ , define

$$S_n(X) = \text{Map}(\Delta^n, X).$$

The operators  $d^i$  and  $s^i$  on  $\Delta^n$  induce

$$\begin{aligned} d_i & : S_n(X) \longrightarrow S_{n-1}(X) \\ s_i & : S_{n-1}(X) \longrightarrow S_n(X). \end{aligned}$$

When each  $S_n(X)$  is equipped with the compact-open topology, these maps are continuous and we obtain a simplicial space  $S(X)$ . This is called the *singular simplicial space*.

Usually  $S_n(X)$ 's are merely regarded as sets and  $S(X)$  is regarded as a simplicial set, in which case we denote it by  $S_n^\delta(X)$ .  $\square$

**Example B.12.** Let  $K$  be an ordered simplicial complex on the vertex set  $V$ . Define

$$s(K)_n = \left\{ \left[ \underbrace{v_0, \dots, v_0}_{i_0}, \underbrace{v_1, \dots, v_1}_{i_1}, \dots, \underbrace{v_k, \dots, v_k}_{i_k} \right] \mid \sigma = [v_0, \dots, v_k] \in K, \sum_{j=0}^k i_j = n \right\}.$$

Then the collection  $s(K) = \{s(K)_n\}$  becomes a simplicial set. This is called the *simplicial set generated by  $K$* .  $\square$

**Definition B.13.** The *geometric realization* of a simplicial space  $X$  is defined by

$$|X| = \left( \prod_n X_n \times \Delta^n \right) / \sim$$

where the relation  $\sim$  is generated by

$$\begin{aligned} (x, d^i(\mathbf{t})) &\sim (d_i(x), \mathbf{t}), \\ (x, s^i(\mathbf{t})) &\sim (s_i(x), \mathbf{t}). \end{aligned}$$

The map induced by the evaluation maps

$$\Delta^n \times S_n(X) \longrightarrow X$$

is denoted by

$$\text{ev} : |S(X)| \longrightarrow X.$$

Note that the geometric realization of an ordered simplicial complex is defined only by using face operators.

**Definition B.14.** A  $\Delta$ -set  $X$  consists of

- a sequence of sets  $X_0, X_1, \dots$ , and
- a family of maps  $d_i : X_n \rightarrow X_{n-1}$  for  $0 \leq i \leq n$ ,

satisfying the following relations

$$d_i \circ d_j = d_{j-1} \circ d_i,$$

for  $i < j$ .

When each  $X_n$  is equipped with a topology under which  $d_i$ 's are continuous,  $X$  is called a  $\Delta$ -space.

**Remark B.15.** A  $\Delta$ -set  $X$  can be identified with a functor

$$X : \Delta_{\text{inj}}^{\text{op}} \longrightarrow \mathbf{Sets},$$

where  $\Delta_{\text{inj}}$  is the subcategory of  $\Delta$  consisting of injective maps. In particular, any simplicial set can be regarded as a  $\Delta$ -set.

**Definition B.16.** The *geometric realization* of a  $\Delta$ -space  $X$  is defined by

$$\|X\| = \left( \prod_n X_n \times \Delta^n \right) / \sim,$$

where the relation  $\sim$  is generated by

$$(x, d^i(\mathbf{t})) \sim (d_i(x), \mathbf{t}).$$

**Remark B.17.** Note that any simplicial space  $X$  can be regarded as a  $\Delta$ -space. However the geometric realization of  $X$  as a  $\Delta$ -space,  $\|X\|$ , is much larger than that of  $X$  as a simplicial space.  $\|X\|$  is often called the fat realization.

In order to study the homotopy type of simplicial complexes, and more generally, regular cell complexes, the notion of regular neighborhood is useful. Let us recall the definition.

**Definition B.18.** Let  $K$  be a cell complex. For  $x \in K$ , define

$$\text{St}(x; K) = \bigcup_{x \in \bar{e}} e.$$

This is called the *open star* around  $x$  in  $K$ . For a subset  $A \subset K$ , define

$$\text{St}(A; K) = \bigcup_{x \in A} \text{St}(x; K).$$

When  $K$  is a simplicial complex and  $A$  is a subcomplex,  $\text{St}(A; K)$  is called the *regular neighborhood* of  $A$  in  $K$ .

The regular neighborhood of a subcomplex is often defined in terms of vertices.

**Lemma B.19.** *Let  $A$  be a subcomplex of a simplicial complex  $K$ . Then*

$$\text{St}(A; K) = \bigcup_{v \in \text{sk}_0(A)} \text{St}(v; K).$$

## B.2 Locally Cone-like Spaces

Let us first define the operation  $*$  on subsets of a Euclidean space.

**Definition B.20.** For subspaces  $P, Q \subset \mathbb{R}^n$ , define

$$P * Q = \{(1-t)p + tq \mid p \in P, q \in Q, 0 \leq t \leq 1\}.$$

This is called the *convex sum* or *join* of  $P$  and  $Q$ .

When  $P$  is a single point  $v$ ,  $v * Q$  is called the *cone* on  $Q$  with vertex  $v$ .

**Remark B.21.** When  $P$  and  $Q$  are closed and “in general position”,  $P * Q$  agrees with the join operation.

**Definition B.22.** We say a subspace  $P \subset \mathbb{R}^n$  is *locally cone-like*, if, for any  $a \in P$ , there exists a compact subset  $L \subset P$  such that the cone  $a * L$  is a neighborhood of  $a$ .

**Remark B.23.** Locally cone-like spaces are called polyhedra in the Rourke-Sanderson book [RS72].

**Example B.24.** The half space  $\mathbb{R}_+^n = \{\mathbf{x} \in \mathbb{R}^n \mid x_n \geq 0\}$  is locally cone-like. □

**Example B.25.** A vector (or affine) subspace of  $\mathbb{R}^n$  is locally cone-like. □

**Example B.26.** A convex polytope in  $\mathbb{R}^n$  is locally cone-like. □

**Lemma B.27.** *The class of locally cone-like spaces is closed under the following operations:*

- *finite intersections,*
- *finite products, and*
- *locally finite unions.*

**Corollary B.28.** *Euclidean polyhedral complexes are locally cone-like.*

The following theorem characterizes compact locally cone-like spaces.

**Theorem B.29.** *Any compact locally cone-like space can be expressed as a union of finite number of simplices.*

### B.3 PL Maps Between Polyhedral Complexes

In PL topology, we study triangulated spaces. Following Rourke and Sanderson, let us consider locally cone-like spaces. We also consider polyhedral complexes in a Euclidean space.

**Definition B.30.** Let  $P$  and  $Q$  be locally cone-like spaces. A map

$$f : P \longrightarrow Q$$

is said to be *piecewise-linear (PL)* if, for each  $a \in P$ , there exists a cone neighborhood  $N = a * L$  such that

$$f(\lambda a + \mu x) = \lambda f(a) + \mu f(x)$$

for  $x \in L$  and  $\lambda, \mu \geq 0$  and  $\lambda + \mu = 1$ .

**Lemma B.31.** *PL maps are closed under the following operations:*

- *products,*
- *compositions,*
- *the cone construction.*

Another important property of PL maps is the extendability.

**Lemma B.32.** *Let  $P$  be a convex polytope and*

$$f : \text{Int}P \longrightarrow \mathbb{R}^n$$

*be a PL map. Then it has a PL extension*

$$\tilde{f} : P \longrightarrow \mathbb{R}^n.$$

**Theorem B.33.** *Let  $K$  and  $L$  be Euclidean polyhedral complexes. For any PL map*

$$f : K \longrightarrow L,$$

*there exist simplicial subdivisions  $K'$  and  $L'$  of  $K$  and  $L$ , respectively, such that the induced map*

$$f : K' \longrightarrow L'$$

*is simplicial.*

*Proof.* By Theorem 2.14 in [RS72]. □

## C Topological Categories

In this third appendix, we recall basics of topological categories. Our references are Segal's paper [Seg68] and the article by Dwyer in [DH01].

## C.1 Topological Acyclic Categories

**Definition C.1.** A *topological quiver*  $Q$  is a diagram of spaces of the form

$$s, t : Q_1 \longrightarrow Q_0.$$

For a topological quiver  $Q$ , define

$$N_n(Q) = \{(u_n, \dots, u_1) \mid s(u_n) = t(u_{n-1}), \dots, s(u_2) = t(u_1)\}.$$

An element of  $N_n(Q)$  is called an *n-chain* of  $Q$ .

**Definition C.2.** A *topological category*  $C$  is a topological quiver equipped with two more maps

$$\begin{aligned} i & : C_0 \longrightarrow C_1 \\ \circ & : N_2(C) \longrightarrow C_1 \end{aligned}$$

making the following diagrams commutative

1.

$$\begin{array}{ccc} N_3(C) & \xrightarrow{\circ \times 1} & N_2(C) \\ \downarrow 1 \times \circ & & \downarrow \circ \\ N_2(C) & \xrightarrow{\circ} & C_1, \end{array}$$

2.

$$\begin{array}{ccccc} C_1 & \xrightarrow{i \times 1} & N_2(C) & \xleftarrow{1 \times i} & C_1 \\ & \searrow & \downarrow \circ & \swarrow & \\ & & C_1 & & \end{array}$$

Elements of  $C_0$  are called *objects*. An element  $u \in C_1$  with  $s(u) = x$  and  $t(u) = y$  is called a *morphism* from  $x$  to  $y$  and is denoted by  $u : x \rightarrow y$ . The subspace of morphisms from  $x$  to  $y$  is denoted by  $C(x, y)$ , i.e.  $C(x, y) = s^{-1}(x) \cap t^{-1}(y)$ . For  $x \in C_0$ ,  $i(x)$  is called the *identity morphism* on  $x$  and is denoted by  $1_x : x \rightarrow x$ .

When  $C_0$  has the discrete topology,  $C$  is called a *top-enriched category*.

**Definition C.3.** An *acyclic topological category* is a top-enriched category  $C$  in which, for any pair of distinct objects  $x, y \in C_0$ , either  $C(x, y)$  or  $C(y, x)$  is empty and, for any object  $x \in C_0$ ,  $C(x, x)$  consists of the identity morphism.

**Remark C.4.** When the topology of  $C_0$  is not discrete, we need to assume that  $C_1$  decomposes into a disjoint union of  $i(C_0)$  and its complement.

The following fact is well known.

**Lemma C.5.** For an acyclic topological category  $C$ , define a relation  $\leq$  on  $C_0$  as follows:

$$x \leq y \text{ if and only if } C(x, y) \neq \emptyset.$$

Then the relation  $\leq$  is a partial order on  $C_0$ .

**Definition C.6.** For an acyclic topological category  $C$ , the poset  $(C_0, \leq)$  is called the *underlying poset* of  $C$  and is denoted by  $P(C)$ . The canonical functor from  $C$  to  $P(C)$  is denoted by

$$\pi_C : C \longrightarrow P(C).$$

In this paper, we are concerned with cellular structures.

**Definition C.7.** A topological category  $C$  is said to be *cellular* if both  $C_0$  and  $C_1$  have structures of cellular stratified spaces and the structure maps  $s, t, i, \circ$  are morphisms of cellular stratified spaces.

**Remark C.8.** In the above definition, we assume that each  $C(y, z) \times C(x, y)$  is a cellular stratified space under the product stratification. See Proposition 6.14 and discussions in §6.2 for conditions under which the bi-quotient assumption in Lemma C.17 is satisfied.

When  $C_0$  is not discrete it is not easy to define cellularity, since the pullback of stratified spaces over a stratified space may not be a stratified space in general.

Functors between topological categories are always assumed to be continuous.

**Definition C.9.** A *continuous functor*  $f$  from a topological category  $C$  to another  $D$  is a pair  $f = (f_0, f_1)$  of continuous maps

$$\begin{aligned} f_0 & : C_0 \longrightarrow D_0 \\ f_1 & : C_1 \longrightarrow D_1 \end{aligned}$$

that are compatible with all structure maps of topological category.

Topological categories defined in Definition C.2 are usually called small topological categories, in the sense that collections of objects and morphisms form sets. In this paper the only category which is not small is the category **Spaces** of topological spaces and continuous maps. The continuity of a functor to **Spaces** is defined as follows.

**Definition C.10.** Let  $C$  be a (small) top-enriched category. A functor  $f : C \rightarrow \mathbf{Spaces}$  is said to be *continuous* if, for each pair  $x, y \in C_0$  of objects, the adjoint

$$C(x, y) \times f(x) \longrightarrow f(y)$$

to the map  $f(x, y) : C(x, y) \rightarrow \text{Map}(f(x), f(y))$  is continuous.

For a continuous functor  $f : C \rightarrow \mathbf{Spaces}$ , define  $\text{tot}(f) = \coprod_{x \in C_0} f(x)$ . The canonical projection onto  $C_0$  is denoted by  $\pi_f : \text{tot}(f) \rightarrow C_0$ . Define  $C_1 \square_{C_0} \text{tot}(f)$  by the following pullback diagram

$$\begin{array}{ccc} C_1 \square_{C_0} \text{tot}(f) & \xrightarrow{s \square 1} & \text{tot}(f) \\ \downarrow & & \downarrow \pi_f \\ C_1 & \xrightarrow{s} & C_0. \end{array}$$

Then the *colimit* of  $f$ ,  $\text{colim}_C f$ , is defined by the following coequalizer diagram

$$C_1 \square_{C_0} \text{tot}(f) \begin{array}{c} \xrightarrow{s \square 1} \\ \xrightarrow{\mu_f} \end{array} \text{tot}(f) \xrightarrow{p_f} \text{colim}_C f,$$

where  $\mu_f$  is defined on each component by the adjoint  $C(x, y) \times f(x) \longrightarrow f(y)$  to  $f(x, y) : C(x, y) \rightarrow \text{Map}(f(x), f(y))$ .

## C.2 Nerves and Classifying Spaces

One of the most fundamental facts is the collection  $N(X) = \{N_n(X)\}_{n \geq 0}$  defined in Definition C.1 forms a simplicial space.

**Lemma C.11.** *For a topological category  $X$ , we have*

$$N_n(X) = \text{Funct}([n], X),$$

where the poset  $[n] = \{0, \dots, n\}$  is regarded as a topological category (with discrete topology). Thus  $N(X)$  defines a functor

$$N(X) : \Delta^{\text{op}} \longrightarrow \mathbf{Spaces}.$$

In other words,  $N(X)$  is a simplicial space for any topological category  $X$ .

**Definition C.12.** The simplicial space  $N(X)$  is called the *nerve* of  $X$ . The source and the target maps on  $X$  can be extended to

$$s, t : N_k(X) \longrightarrow X_0$$

by  $s(\mathbf{f}) = f(0)$  and  $t(\mathbf{f}) = f(k)$ , respectively. These are also called the *source and target maps*.

**Definition C.13.** The geometric realization of  $N(X)$  is called the *classifying space* of  $X$  and is denoted by  $BX$ .

When we form the geometric realization of a simplicial space, nondegenerate chains are essential.

**Definition C.14.** For a topological category  $X$ , define

$$\bar{N}_n(X) = N_n(X) \setminus \bigcup_i s_i(N_{n-1}(X)).$$

Elements of  $\bar{N}_n(X)$  are called *nondegenerate  $n$ -chains*.

**Lemma C.15.** *When  $P$  is a poset regarded as a small category,  $\bar{N}(P) = \{\bar{N}_n(P)\}$  is an ordered simplicial complex and we have*

$$BP = \|\bar{N}(P)\|.$$

More generally, we have the following description.

**Lemma C.16.** *When  $X$  is a topological acyclic category, the simplicial structure on  $N(X)$  can be restricted to give a structure of  $\Delta$ -space on  $\bar{N}(X)$ . Furthermore the composition*

$$\|\bar{N}(X)\| \longrightarrow \|N(X)\| \longrightarrow |N(X)| = BX$$

is a homeomorphism.

When an acyclic topological category  $C$  is cellular in the sense of Definition C.7, the classifying space  $BC$  has a canonical cell decomposition.

**Lemma C.17.** *Let  $C$  be an acyclic topological category in which each morphism space  $C(x, y)$  is equipped with a structure of cellular stratified space whose cell structures are bi-quotient. Then the classifying space  $BC$  has a structure of cellular stratified space.*

*In particular when all morphism spaces are cell complexes, the classifying space  $BC$  has a structure of cell complex.*



*Proof.* Under the identification  $BC \cong \|\overline{N}(C)\|$ , any point in  $BC$  can be represented by a pair  $(x, t) \in \overline{N}_k(C) \times \text{Int}(\Delta^k)$  uniquely. In other words, we have a decomposition

$$BC = \prod_{k=0}^{\infty} \overline{N}_k(C) \times \text{Int}(\Delta^k)$$

as sets. The bi-quotient assumption on cell structures on  $C(x, y)$  implies that  $\overline{N}_k(C)$  has a structure of cellular stratified space<sup>34</sup>. Thus the above decomposition and the cellular stratification on each  $\overline{N}_k(C)$  define a stratification on  $BC$ .

For each cell  $\varphi : D \rightarrow \bar{e} \subset \overline{N}_k(C)$ , let us denote the cell in  $BC$  corresponding to  $e \times \text{Int}(\Delta^k)$  by  $e \times e^k$ . The composition

$$D \times \Delta^k \xrightarrow{\varphi \times 1_{\Delta^k}} \overline{N}_k(C) \times \Delta^k \longrightarrow \|\overline{N}(C)\| = BC$$

defines a cell structure on  $e \times e^k$ . □

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<sup>34</sup>See Definition 6.13 for product cellular stratifications

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