Kyushu J. Math. **67** (2013), 419–427
doi:10.2206/kyushujm.67.419

ON LOCAL COEFFICIENTS OF SIMPLICIAL COALGEBRAS

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(Received 3 September 2012)

Abstract. This paper generalizes the notion of local coefficients and fundamental groups of spaces to simplicial coalgebras. We define a Hopf algebra $\pi_1(C)$ from a simplicial coalgebra C as a generalization of a fundamental group, and show that a module over $\pi_1(C)$ corresponds to a local coefficient of C. As a consequence, the Hoschild cohomology of a Hopf algebra H with a coefficient M coincides with the cohomology of the nerve simplicial coalgebra of H with the local coefficient M_* associated with M.

1. Introduction

A simplicial coalgebra is a simplicial object in the category of coalgebras. It is a model of generalized spaces for homotopy theory. It is well known that differential graded algebras over $\mathbb{Q}$ are an algebraic model for the rational homotopy theory of spaces. On the characteristics other than zero, Goerss suggested that simplicial coalgebras should be regarded as an algebraic model for p-local homotopy theory in [**Goe95**].

This paper extends the notion of local coefficients and fundamental groups in spaces to simplicial coalgebras, where spaces mean simplicial sets. A local coefficient of a space S can be regarded as a comodule over the free simplicial coalgebra generated by S. We can naturally define a local coefficient of a simplicial coalgebra C as a comodule over C. On the other hand, in the case of a space S, a local coefficient of S corresponds to a representation of the fundamental group $\pi_1(S)$ (see [**Hal83**]). In the case of a simplicial coalgebra C, we observe that the Hopf algebra $\pi_1(C)$ gives an algebraic counterpart of the fundamental group of a space: a local coefficient of C can be translated as a module over $\pi_1(C)$.

THEOREM 1.1. *Let C be a connected simplicial coalgebra. There exists a Hopf algebra $\pi_1(C)$ such that the category of modules over $\pi_1(C)$ is equivalent to the category of local coefficients of C.*

We also extend the notion of the cohomology of spaces with local coefficients to simplicial coalgebras. It is obtained from the complex of comodule maps. By the above relation between local coefficients and modules, the Hoschild cohomology of a Hopf algebra H with a coefficient coincides with the cohomology of the nerve NH of H with a local coefficient.

2010 Mathematics Subject Classification: Primary 55N25, 16G30.
Keywords: local coefficients; simplicial coalgebras; Hoschild (co)homology.

THEOREM 1.2. *Let H be a Hopf algebra and M be a module over H. The Hoschild cohomology* $\mathrm{HH}^*(H, M)$ *of H with the coefficient M coincides with the cohomology* $\mathrm{H}^*(NH, M_*)$ *of the nerve NH of H with the local coefficient M_* associated with M.*

The category of spaces is embedded in the category of cocommutative simplicial coalgebras through the free functor. Hence, we can regard a non-cocommutative simplicial coalgebra as a non-(co)commutative geometric object [**Con91**]. In that sense, the Hopf algebra $\pi_1(C)$ of a non-cocommutative simplicial coalgebra C is a model of non-(co)commutative fundamental groups.

This paper is organized as follows. We recall the definition of local coefficients of spaces and the relation with fundamental groups in Section 2. The classical definition of local coefficient is given as a family of modules indexed by simplices [**Hal83**]. We generalize it to simplicial coalgebras as a comodule over simplicial coalgebras in Section 3. The cohomology of simplicial coalgebras with local coefficients is defined as the cohomology generated by a complex of comodule maps. In Section 4, we construct a Hopf algebra $\pi_1(C)$ from a simplicial coalgebra C such that a local coefficient of C corresponds to a module over $\pi_1(C)$. Finally, we prove that the Hoschild cohomology of a Hopf algebra H with a coefficient coincides with the cohomology of the nerve NH of H with a local coefficient.

2. Local coefficients of spaces

We fix a field k throughout this paper. Recall the notion of local coefficients of spaces and the cohomology of spaces with local coefficients. A space (simplicial set) S is a family of sets $\{S_n\}$ indexed by the non-negative integer with the face map $d_j : S_n \longrightarrow S_{n-1}$ and the degeneracy map $s_i : S_n \longrightarrow S_{n+1}$ for $0 \leqq i, j \leqq n$ satisfying the simplicial relations (see [**May92**]). An element of $c \in S_n$ is called an n-simplex of S.

Definition 2.1. (Halperin [**Hal83**]) A *local system* L of a space S consists of the following.

1. A family of modules L_c indexed by the simplices c of S.
2. A family of morphisms $d_j^c : L_c \longrightarrow L_{d_j c}$ and $s_i^c : L_c \longrightarrow L_{s_i c}$ satisfying the simplicial relations.

A morphism $L \longrightarrow L'$ between two local coefficients of S is a family of morphisms $L_c \longrightarrow L'_c$ compatible with d_j^c and s_i^c. A local system L of S is said to be a *local coefficient* when d_j^c and s_i^c are isomorphisms for all i, j and c. The category of local coefficients of S, denoted by $\mathcal{L}(S)$, is defined as a full subcategory of the category of local systems.

The cohomology of S with a local coefficient L is defined by the following. A *p-cochain* is a function

$$\alpha : S_p \longrightarrow \prod_{c \in S_p} L_c$$

such that $\alpha(c) \in L_c$ for each $c \in S_p$. Let $C^p(S, L)$ be the module consisting of p-cochains and

$$\partial^j : C^p(S, L) \longrightarrow C^{p+1}(S, L)$$

be the morphism given by $\partial^j(\alpha)(c) = (d_j^c)^{-1}(\alpha(d_j(c))) \in L_c$ for $c \in S_{p+1}$. The pair $(C(S, F), \partial)$ is a cosimplicial module, and it induces a cochain complex $C(S, F)$ with the differential

$$\sum_{j=0}^{p+1}(-1)^j\partial^j : C^p(S, L) \longrightarrow C^{p+1}(S, L).$$

The cohomology of a space S with a local coefficient L of S is defined by the cohomology of the above complex

$$\mathrm{H}^*(S, L) := \mathrm{H}^*(C(S, L)).$$

A local coefficient of a space is closely related with the fundamental group. The fundamental group of a space S is obtained as the quotient group of the free group generated from S_1 by the relation of 2-simplices (see [**May92**]).

Definition 2.2. Let S be a pointed space. The fundamental group $\pi_1(S)$ of S is defined by the following. Let $* \in S_0$ be the base point of S and Σ be the subspace of S defined by

$$\Sigma_n = \{x \in S_n \mid v_j(x) = *, 0 \leqq j \leqq n\}$$

where v_j is the induced map from the injection $\{0\} \longrightarrow \{0, 1, \ldots, n\}$, $0 \mapsto j$. Denote the free group generated from Σ_1 by GS_1. The fundamental group of $\pi_1(S)$ is the quotient group of GS_1 by the normal subgroup generated from $\{d_1^{-1}(\sigma)d_0(\sigma)d_2(\sigma) \mid \sigma \in \Sigma_2\}$.

The following is a well-known fact about local coefficients and fundamental groups [**Hal83**].

THEOREM 2.3. *Let $\mathcal{M}(\pi_1(S))$ be the category of representations of $\pi_1(S)$. Then there exists an equivalence of categories between $\mathcal{L}(S)$ and $\mathcal{M}(\pi_1(S))$.*

Proof. Let L be a local coefficient of S. For $e \in S_1$, the automorphism

$$L_* \xrightarrow{(d_1^e)^{-1}} L_e \xrightarrow{d_0^e} L_*.$$

gives a representation $\pi_1(S) \longrightarrow \mathrm{GL}(L_*)$. Thus, the correspondence $L \mapsto L_*$ induces an equivalence $\mathcal{L}(S) \longrightarrow \mathcal{M}(\pi_1(S))$. □

3. Local coefficients of simplicial coalgebras

Let L be a local system of a space S and let L_n denote the coproduct $\bigoplus_{c\in S_n} L_c$. The maps

$$d_j^L = \bigoplus_{c\in S_n} d_j^c : L_n \longrightarrow L_{n-1}$$

and

$$s_i^L = \bigoplus_{c\in S_n} s_i^c : L_n \longrightarrow L_{n+1}$$

make L a simplicial module. Moreover, L_n has a left comodule structure over the coalgebra kS_n given by $\sigma \mapsto c \otimes \sigma$ if $\sigma \in L_c$. Here, the comultiplication of kS_n is given by $\Delta(\sigma) = \sigma \otimes \sigma$ for all $n \geqq 0$. Therefore, we can regard L as a simplicial left comodule over the simplicial coalgebra kS. A simplicial coalgebra C is a family of coalgebras $\{C_n\}_{n\geqq 0}$ with coalgebra maps d_j and s_i satisfying the simplicial relations as same as a space.

Definition 3.1. Let C be a simplicial coalgebra. A simplicial module M is said to be a simplicial left C-comodule if M_n is a left C_n-comodule for all $n \geqq 0$ and the comodule structure maps are compatible with the face and degeneracy maps of C and M.

In order to define the local coefficients of a simplicial coalgebra, we need the following notion of cotensor product. Aguiar formulated a definition of internal categories using coalgebras (comonoids), comodules and cotensor products in a general monoidal category [**Agu97**].

Definition 3.2. Let C_1, C_2, C_3 be coalgebras and M be a (C_1, C_2)-comodule and N be a (C_2, C_3)-comodule. The cotensor product $M \Box_{C_2} N$ is a (C_1, C_3)-comodule defined as the following equalizer

$$M \Box_{C_2} N \longrightarrow M \otimes N \underset{\mu_M \otimes 1}{\overset{1 \otimes \mu_N}{\rightrightarrows}} M \otimes C_2 \otimes N$$

where μ_M is the right C_2-comodule structure map and μ_N is the left C_2-comodule structure map.

Definition 3.3. Let C be a simplicial coalgebra and L be a simplicial left C-comodule. We can regard C_n as a (C_n, C_n)-comodule through the comultiplication Δ of C_n. Moreover,

$$(1 \circ d_j) \circ \Delta : C_n \longrightarrow C_n \otimes C_{n-1}$$

gives a (C_n, C_{n-1})-comodule structure on C_n for $0 \leqq j \leqq n$. When we regard C_n as the above bicomodule, denote $C_n \Box_{C_{n-1}} L_{n-1}$ by $C_n \Box_j L_{n-1}$. Similarly, $C_n \Box^i L_{n+1}$ denotes $C_n \Box_{C_{n+1}} L_{n+1}$ induced by the right (C_{n+1})-comodule structure

$$(1 \circ s_i) \circ \Delta : C_n \longrightarrow C_n \otimes C_{n+1}$$

on C_n for $0 \leqq i \leqq n$. Let μ_n be the left C_n-comodule structure map of L_n and d_*^L, s_*^L be the face and degeneracy maps of L_n, respectively. The maps

$$(1 \otimes d_j^L) \circ \mu_n : L_n \longrightarrow C_n \otimes L_{n-1}$$

and

$$(1 \otimes s_i^L) \circ \mu_n : L_n \longrightarrow C_n \otimes L_{n+1}$$

induce the canonical maps $\tilde{d}_j : L_n \longrightarrow C_n \Box_j L_{n-1}$ and $\tilde{s}_i : L_n \longrightarrow C_n \Box^i L_{n+1}$. A simplicial left C-comodule L is said to be a *local coefficient* if both $\tilde{d}_j$ and $\tilde{s}_i$ are isomorphisms as left C_n-comodule maps for all i, j and n.

The category of simplicial left C-comodules is denoted by Com_C and the category of local coefficients of C, denoted by $\mathcal{L}(C)$, is defined as a full subcategory of Com_C.

Example 3.4. Let L be a local coefficient of a space S, then L is a local coefficient of the simplicial coalgebra kS generated by S.

We can also define the cohomology of a simplicial coalgebra with a local coefficient. We use the following Sweedler's notation. Let Δ be the comultiplication of a coalgebra C, then denote $\Delta(c)$ by $\sum c_{(1)} \otimes c_{(2)} \in C \otimes C$ for $c \in C$.

Definition 3.5. Let L be a local coefficient of a simplicial coalgebra C. Let $\mathrm{C}^p(C, L)$ be the module of left C_p-comodule maps from C_p to L_p. Define a map

$$\partial^j : \mathrm{C}^p(C, L) \longrightarrow \mathrm{C}^{p+1}(C, L)$$

by $\partial^j(\alpha)(c) = \sum \tilde{d}_j^{-1}(c_{(1)} \otimes \alpha(d_j c_{(2)}))$ for $c \in C_{p+1}$.

LEMMA 3.6. *The map $\partial^j(\alpha)$ is a left C_{p+1}-comodule map making the following diagram commutative.*

$$\begin{array}{ccc} C_{p+1} & \xrightarrow{\partial^j(\alpha)} & L_{p+1} \\ {\scriptstyle d_j}\downarrow & & \downarrow{\scriptstyle d_j^L} \\ C_p & \xrightarrow[\alpha]{} & L_p \end{array}$$

Proof. By Definition 3.5, $\partial^j(\alpha) = \tilde{d}_j^{-1} \circ (1 \otimes (\alpha \circ d_j)) \circ \Delta$, hence this is a left C_{p+1}-comodule map. On the other hand,

$$\begin{array}{ccccccccc} C_{p+1} & \xrightarrow{\Delta} & C_{p+1} \otimes C_{p+1} & \xrightarrow{1\otimes d_j} & C_{p+1} \otimes C_p & \xrightarrow{1\otimes\alpha} & C_{p+1}\Box_j L_p & \xleftarrow[\cong]{\tilde{d}_j} & L_{p+1} \\ {\scriptstyle d_j}\downarrow & & \downarrow{\scriptstyle d_j\otimes d_j} & & \downarrow{\scriptstyle d_j\otimes 1} & & \downarrow{\scriptstyle d_j\Box 1} & & \downarrow{\scriptstyle d_j^L} \\ C_p & \xrightarrow[\Delta]{} & C_p \otimes C_p & = & C_p \otimes C_p & \xrightarrow[1\otimes\alpha]{} & C_p\Box_{C_p} L_p & \xrightarrow[\varepsilon\Box 1]{\cong} & L_p \end{array}$$

is a commutative diagram and α is the composition of the under horizontal maps. □

LEMMA 3.7. *Let L be a local coefficient of a simplicial coalgebra C. For $i < j \leqq p$, both $C_{p+1}\Box_j(C_p\Box_i L_{p-1})$ and $C_{p+1}\Box_i(C_p\Box_{j-1} L_{p-1})$ are isomorphic to the following equalizer*

$$C_{p+1} \otimes L_{p-1} \underset{1\otimes\mu}{\overset{(1\otimes d_i d_j)\Delta\otimes 1}{\rightrightarrows}} C_{p+1} \otimes C_{p-1} \otimes L_{p-1}.$$

Proof. By the simplicial relations $d_i d_j = d_{j-1} d_i$ for $i < j$. □

PROPOSITION 3.8. *The map ∂ is satisfies the cosimplicial relations.*

Proof. Lemma 3.7 implies that $\partial^j \circ \partial^i = \partial^i \circ \partial^{j-1} : \mathrm{C}^{p-1}(C, L) \longrightarrow \mathrm{C}^{p+1}(C, L)$ for $i < j \leqq p$. We can similarly show that the other cases of cosimplicial relations hold. □

Definition 3.9. Let L be a local coefficient of a simplicial coalgebra C. The cohomology $\mathrm{H}^*(C, L)$ of C with L is defined as the cohomology of the cosimplicial module $\mathrm{C}(C, L)$.

Example 3.10. The following are some basic examples of the cohomology of simplicial coalgebras with local coefficients.

1. Let L be a local coefficient of a space S, then $\mathrm{H}^*(kS, L) = \mathrm{H}^*(S, L)$.
2. Let C be a simplicial coalgebra and regard it as a local coefficient of C itself through the comultiplication. Since $\mathrm{Com}_k(C, C) = \mathrm{Hom}_k(C, k)$, the cohomology $\mathrm{H}^*(C, C)$ is the cohomology $\mathrm{H}^*(C, k)$ where C is the simplicial module forgetting the coalgebra structure of C.

3. Let C be a coalgebra and we regard it as the simplicial coalgebra where all face and degeneracy maps are the identity map 1_C. A local coefficient L of C is a simplicial C-comodule whose d_j^L and s_i^L are all isomorphisms. Moreover, we have $\partial^j = \partial^k : \mathrm{C}^p(C, L) \longrightarrow \mathrm{C}^{p+1}(C, L)$ for any $0 \leqq j, k \leqq p+1$ since

$$d_0^L = s_0^{-1} = d_1^L = s_1^{-1} = d_2^L = \cdots = d_{p+1}^L : L_{p+1} \longrightarrow L_p.$$

Thus, the differential is zero if p is even and is an isomorphism if p is odd. Then $\mathrm{H}^0(C, L) = \mathrm{Com}_C(C, L_0)$ and $\mathrm{H}^n(C, L) = 0$ for $n \neq 0$. □

4. Fundamental Hopf algebra

Theorem 2.3 implies that a local coefficient of a space S corresponds to a module over the Hopf algebra $k\pi_1(S)$. Hence, we expect that a local coefficient of a simplicial coalgebra C corresponds to a module over some Hopf algebra associated with C. We construct such a Hopf algebra using a slightly modifying method of the construction of universal Hopf algebra for a coalgebra [**Tak71, Por08**].

Definition 4.1. A connected simplicial coalgebra is a simplicial coalgebra C with $C_0 = k$.

Definition 4.2. Let C be a connected simplicial coalgebra. A Hopf algebra $\pi_1(C)$ is defined as follows: let C_p^{op} be the opposite coalgebra of C_p and define V as the coproduct of the following form

$$V_p := C_p \oplus C_p^{\mathrm{op}} \oplus C_p \oplus C_p^{\mathrm{op}} \oplus \cdots \quad (\text{for } p \geqq 0).$$

We obtain the shift map $S_p : V_p \longrightarrow V_p^{\mathrm{op}}$ given by $S_p(c_1, c_2, \ldots, c_n) = (0, c_1, c_2, \ldots, c_n)$. The tensor algebra TV_p has a unique bialgebra structure such that the inclusion $V_p \hookrightarrow TV_p$ is a coalgebra map. The map S_p can be extended uniquely to a bialgebra map $\widetilde{S}_p : TV_p \longrightarrow (TV_p)^{\mathrm{op}}$. Let I be the two-sided ideal in TV_1 generated by all elements of the form $\sum c_{(1)} \otimes S_1(c_{(2)}) - \varepsilon(c)$ and $\sum S_1(c_{(1)}) \otimes c_{(2)} - \varepsilon(c)$ and $\sum d_0(\sigma_{(1)}) \otimes d_2(\sigma_{(2)}) - d_1(\sigma)$ for $c \in V_1$ and $\sigma \in V_2$, where ε is the counit map of C_1. The ideal I is a coideal of TV since S_1 and d_i $(i = 0, 1, 2)$ are all coalgebra maps. Moreover, we have $\widetilde{S}_1(I) \subset I$. Hence, $\pi_1(C) = TV/I$ is Hopf algebra with the antipode map $S : \pi_1(C) \longrightarrow \pi_1(C)^{\mathrm{op}}$ induced from $\widetilde{S}_1$. The Hopf algebra $\pi_1(C)$ is called the fundamental Hopf algebra of C.

By the above definition, the fundamental Hopf algebra $\pi_1(k\Sigma)$ of the simplicial coalgebra $k\Sigma$ generated by the subspace Σ of a pointed space S in Definition 2.2 is the Hopf algebra $k\pi_1(S)$ generated by the fundamental group $\pi_1(S)$.

Definition 4.3. Let B be a bialgebra. Define a connected simplicial coalgebra NB as $NB_n = B^{\otimes n}$ with the face map $d_j : NB_n \longrightarrow NB_{n-1}$ given by

$$d_j(b_1 \otimes \cdots \otimes b_n) = \begin{cases} \varepsilon(b_1) b_2 \otimes \cdots \otimes b_n, & j = 0, \\ b_1 \otimes \cdots \otimes b_j b_{j+1} \otimes \cdots \otimes b_n, & 0 < j < n, \\ b_1 \otimes \cdots \otimes b_{n-1} \varepsilon(b_n), & j = n, \end{cases}$$

and the degeneracy map $s_i : NB_n \longrightarrow NB_{n+1}$ given by

$$s_i(b_1 \otimes \cdots \otimes b_n) = b_1 \otimes \cdots b_i \otimes 1 \otimes b_{i+1} \otimes \cdots \otimes b_n$$

for $0 \leqq i \leqq n$, where 1 is the unit and ε is the counit map of B. We call NB the nerve of B.

LEMMA 4.4. *Let H be a Hopf algebra, then there exists an isomorphism $\pi_1(NH) \cong H$.*

Proof. Since $NH_1 = H$, the multiplication of H induces a homomorphism $\alpha : \pi_1(NH) \longrightarrow H$ given by $\alpha(h_1 \otimes h_2 \otimes \cdots \otimes h_n) = \prod_{i=1}^{n} h_i$. On the other hand, there exists the canonical inclusion $\beta : H \longrightarrow \pi_1(NH)$. We can see $\alpha \circ \beta = 1_H$. Conversely, for an element $a \otimes b \in NH_2 = H \otimes H$, this also can be regarded as an element of $\pi_1(NH)$. We have

$$ab = d_1(a \otimes b) = d_2 \otimes d_0(\Delta(a \otimes b)) = \sum \varepsilon(a_{(1)})b_{(1)} \otimes a_{(2)}\varepsilon(b_{(2)}) = a \otimes b$$

in $\pi_1(NH)$. Hence, any element represented $h_1 \otimes \cdots \otimes h_n \in H^{\otimes n}$ is equal to $\prod_{i=0}^{n} h_i \in H$ in $\pi_1(NH)$. Thus, $\beta \circ \alpha = 1_{\pi_1(NH)}$. □

THEOREM 4.5. *Let $\mathcal{M}(\pi_1(C))$ be the category of left modules over $\pi_1(C)$ for a connected simplicial coalgebra C. Then, there exists an equivalence of categories between $\mathcal{L}(C)$ and $\mathcal{M}(\pi_1(C))$.*

Proof. A functor $\mathcal{L}(C) \longrightarrow \mathcal{M}(\pi_1(C))$ is defined by the following. Let L be a local coefficient of C. Since C is a connected simplicial coalgebra, L_0 is a k-module and

$$\tilde{d}_j = (1 \otimes d_j^L) : L_1 \longrightarrow C_1 \otimes L_0$$

is an isomorphism for $j = 0, 1$. The maps

$$d_0 \circ \tilde{d}_1^{-1} : C_1 \otimes L_0 \longrightarrow L_0$$

and

$$d_1 \circ \tilde{d}_0^{-1} : C_1^{\mathrm{op}} \otimes L_0 \longrightarrow L_0$$

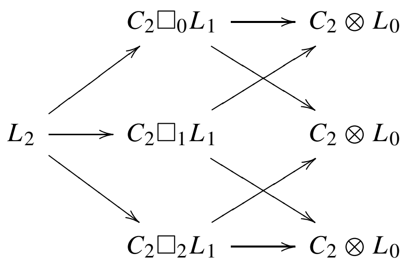
induce a map $V \otimes L_0 \longrightarrow L_0$. Furthermore, it can be extended to a module structure map

$$\nu : TV \otimes L_0 \longrightarrow L_0.$$

We can see

$$\nu((1 \otimes S)\Delta(c) \otimes x) = \nu((S \otimes 1)\Delta(c) \otimes x) = \varepsilon(c)x$$

for $c \in V$ and $x \in L_0$ since $d_i \circ \tilde{d}_i(c \otimes x) = \varepsilon(c)x$ for any $c \in C_1$, $x \in L_0$ and $i = 0, 1$. The simplicial relations of L make the following diagram commutative



where the above maps are isomorphisms $\tilde{d}_*$ induced by the definition of local coefficients. Consider the following map

$$\nu_i : C_2 \otimes L_0 \xrightarrow{1\otimes\tilde{d}_1^{-1}} C_2\square_i L_1 \xrightarrow{1\otimes d_0^L} C_2 \otimes L_0$$

for $i = 0, 1, 2$. For $\sigma \in C_2$ and $x \in L_0$, we have

$$\nu((d_0 \otimes d_2)\Delta(\sigma) \otimes x) = (\varepsilon \otimes 1)(\nu_2(\nu_0(\sigma \otimes x))) = (\varepsilon \otimes 1)(\nu_1(\sigma \otimes x)) = \nu(d_1(\sigma) \otimes x).$$

Thus, ν induces a module structure map

$$\tilde{\nu} : \pi_1(C) \otimes L_0 \longrightarrow L_0.$$

Define $F : \mathcal{L}(C) \longrightarrow \mathcal{M}(\pi_1(C))$ by $F(L) = L_0$ with the above module structure. The inverse functor is given by the following. Let M be a left $\pi_1(C)$-module. A simplicial left C-comodule M_* is defined by $M_n = C_n \otimes M$ with the left C_n-comodule structure induced by the comultiplication of C_n. The degeneracy map $s_i^M : C_n \otimes M \longrightarrow C_{n+1} \otimes M$ is $s_i \otimes 1$ for $0 \leqq i \leqq n$ and the face map $d_j^M : C_n \otimes M \longrightarrow C_{n-1} \otimes M$ is given by $d_j \otimes 1$ when $j \neq 0$. In the case of $j = 0$, d_0^M is given as the following composition

$$C_n \otimes M \xrightarrow{\Delta\otimes 1} C_n \otimes C_n \otimes M \xrightarrow{1\otimes e\otimes 1} C_n \otimes C_1 \otimes M \xrightarrow{d_0\otimes\nu} C_{n-1} \otimes M$$

where $e : C_n \longrightarrow C_1$ is the map induced by the inclusion $\{0, 1\} \hookrightarrow \{0, 1, \dots, n\}$. They satisfy the simplicial relations since

$$\nu((d_0 \otimes d_2)\Delta(\sigma) \otimes m) = \nu(d_1(\sigma) \otimes m)$$

for $\sigma \in C_2$ and $m \in M$. Hence, M_* is a local coefficient of C by the comodule structure. We can obtain a functor $G : \mathcal{M}(\pi_1(C)) \longrightarrow \mathcal{L}(C)$ by $G(M) = M_*$. Obviously, $FG(M) = M$ for $M \in \mathcal{M}(\pi_1(C))$. Conversely, $GF(L)$ is a simplicial module such that $GF(L)_n = C_n \otimes L_0$ for $L \in \mathcal{L}(C)$. We define a map $f_n : L_n \longrightarrow C_n \otimes L_0$ by $(1 \otimes v) \circ \mu$ where $v : L_n \longrightarrow L_0$ is the map induced by the inclusion $\{0\} \hookrightarrow \{0, \dots, n\}$. We have

$$L_n \cong C_n \Box L_{n-1} \cong C_n \Box C_{n-1} \Box L_{n-2} \cong C_n \Box L_{n-2} \cdots ,$$

then f_n is an isomorphism for all $n \geqq 0$. Moreover, a family of maps $\{f_n\}$ is a simplicial comodule map $L \longrightarrow GF(L)$ by the definition of the face and degeneracy maps of $GF(L)$. □

The above theorem and Lemma 4.4 imply that for a Hopf algebra H, a local coefficient of NH coincides with a module over H. Therefore, the Hoschild cohomology of H with a coefficient M can be regarded as the cohomology of the simplicial coalgebra NH with the local coefficient M_* associated with M.

THEOREM 4.6. *Let H be a Hopf algebra, M be a left module over H and M_* be the local coefficient of NH associated with M by Theorem 4.5 and Lemma 4.4. Then, the Hoschild cohomology* $\mathrm{HH}^*(H, M)$ *of H with the coefficient M coincides with the cohomology* $\mathrm{H}^*(NH, M_*)$ *of the simplicial coalgebra NH with the local coefficient M_*.*

Proof. The Hoschild cohomology $\mathrm{HH}^*(H, M)$ is the cohomology of the Hoschild complex $\mathrm{HC}^n(H, M) = \mathrm{Hom}_k(H^{\otimes n}, M)$. On the other hand, $C^n(NH, M_*)$ is the module of comodule maps $\mathrm{Com}_{NH_n}(NH_n, M_n)$. Since M_n is given by $NH_n \otimes M$, we have

$$\mathrm{Com}_{NH_n}(NH_n, M_n) = \mathrm{Hom}_k(NH_n, M) = \mathrm{Hom}_k(H^{\otimes n}, M).$$

The differentials of both sides of cochain groups also coincide with each other, and therefore the theorem follows. □

Acknowledgements. The author would like to thank Dai Tamaki and Katsuhiko Kuribayashi for helpful discussions.

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