Location Problems on the Lattice —three facilities case

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1 Introduction

Facility location has been studied in many ways and fields. One approach to this is a game-theoretic approach which uses the Nash equilibrium concept and was originated by the famous Hotelling (1929)'s work.

The simplest form of this type of the location problem is this. There is a line segment with end points 0 and 1. The buyers or consumers are uniformly distributed on the segment. There are facilities which produce a homogeneous product. The facilities seek their location on the segment trying to maximizing their share among all the buyers. In this simplest form there is no price competition among facilities. Also the demand of buyers is completely inelastic, which means one buyer buys one and only one unit of the product of one of the facilities.

In this framework it is well established that there is no pure strategy equilibrium in the case of three facilities. And also known that the number three is exceptional : for two and any number greater than three of facilities there exist pure strategy equilibria(um).

Shaked (1982) found mixed strategy equilibria in the case of three facilities—players in terms of game theory . However, As Serra and ReVelle(1995, p.367) pointed out, the underlying assumption that buyers exist on the continuum of the line segment as well as that facilities can locate at any point of this continuum of the line segment is unrealistic. Just imagine how actually a facility can locate at, say, the point of $\sqrt{2}/2$?

In this paper, I assume more realistic conditions for three facilities case. First, although buyers are still distributed uniformly on the continuum of the line segment, facilities can only locate at finite points on the segment. Second, I proceed to the case under the assumption that both buyers and facilities can only locate on finite points on the segment.

Under such assumptions I found that the problem of the existence of Nash equilibria appears differently than it does in the case of the traditional continuity assumptions.

2 Facilities can only locate at finite points

First, I consider the case where buyers are distributed uniformly while facilities can only locate at finite points. Specifically, I assume that a line segment equipped with a coordinate system is given and that it begins at 0 and ends at $n-1$, where n is an integer greater than 1. A facility can

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locate at a point whose coordinate is an integer only. In other words, facilities can only locate at the lattice points of the line segment. However, it is allowed that more than one facilities can locate at the same point. In such case, it is assumed that buyers are equally divided among the facilites which locate at the same point. For n, there are n lattice points where facilities can locate.

Under the assumptions above, for the case of $n=5$ there exists a pure strategy Nash equilibrium. Take a look at Figure 1. In Figure 1 a dot represents a facility. In this figure, each of the left and right facilities obtains buyers corresponding to the length of 1.5 line segment while the middle facility obtains 1 length of buyers. It is easy to see that no facilities can gain more buyers than now by moving to a different point including one already occupied by another facility. Therefore Figure 1 shows a pure strategy Nash equilibrium.

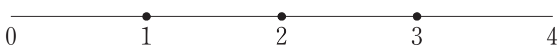
Also it is easy to see that for the case of $n=1$ through 4 there are also equilibria. In the case of $n=1$, the proof is even trivial.

However, the next proposition is true.

Proposition 1: For the case of $n \geq 6$, there are no pure strategy equilibria.

The proof is shown in Appendix. This can be interpreted as such that if n becomes greater, the situation will become analogous to the case of continuum. It might be surprising that even for a rather small number like $n=6$, it becomes analogous to the continuum case.

Figure 1:



3 Both buyers and facilities are located on the lattice. —Case 1

Now I proceed to the case under the assumption that buyers are not uniformly distributed but that they are located only at the lattice points of the line segment. Specifically there is given a line segment with a coordinate system starting at 0 and ending at $n-1$, where n is an integer greater than 1. There are n points whose coordinate are 0, 1, 2, ⋯, $n-1$. At each point there are six buyers. A facility can locate at one of those n points. Also facilities can locate at the same point. Buyers buy at the nearest facility. If there is only one facility for buyers at some point, all six buyers there buy at the facility. If more than one facility are located at the nearest point(s), buyers are divided equally among the facilities: if there are two such facilities, half of buyers, that is to say, three buyers buy at one of the two facilities and the other buyers buy at the other facility. If there are three such facilities, two buyers buy at each facility.

The objective of facilities is to obtain maximum buyers. Facilities decide their location at the same time. I seek pure strategy Nash equilibria of this one shot game. I assert that in the case of $n=5$ there is no pure strategy equilibrium in contrast with the case discussed in the previous section, although for $n=1$ through 4 there exist pure strategy equilibria.

Again take a look at Figure 1 as an example. Now under the assumption in this section, Six buyers are situated at each of 0, 1, 2, 3, 4. The left facility obtains 12 buyers, the central one 6, and the right one 12. But if the central facility moves to the point of 1 where the left one is located, it will obtain 8 buyers while the left facility will obtain 8 and the right one 14. This means the central facility can be better off by changing its location and that the case of Figure 1 is not a Nash equilibrium. In the case of other than Figure 1 example, it is easy to see that there is no pure strategy Nash equilibrium.

Proposition 2: For the case of $n \geq 5$, there are no pure strategy equilibria.

The proof is shown in Appendix.

4 Both buyers and facilities are located on the lattice. —Case 2

In this section, I change the assumptions slightly differently than in the case of the previous section. Now facilities can only locate at even points while buyers are situated at each integer point as in the case of the previous section. That is, facilities can locate at one of the points 0, 2, 4, ⋯, $2(n-1)$, n is an integer greater than 1. These changed assumptions reflect the fact that residential areas and commercial ones are usually distinct. However, the distinction is not thorough since at even points facilities and buyers can coexist.

Other than this change the assumptions are the same as before: Six buyers are situated at each point of integers. Buyers buy at the nearest facility. If more than one facility are located at the nearest point(s), buyers are divided equally among the facilities.

Proposition 3: Under these assumptions there are no pure strategy equilibria for the case of $n \geq 5$.

The proof is shown in Appendix.

5 Conclusions

Some researchers have begun to approach the location problem from the standpoint of discrete models as Daniel Serra and Charles ReVelle (1995). Although the model presented here is the simplest of all the location problem models, I showed that continuous models and discrete ones are quite different from each other. Since discrete models are more realistic, I think that more attention ought to be paid to than it is.

Appendix

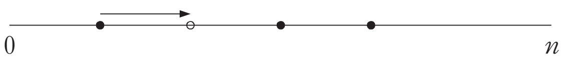
Before we proceed to the proof of any of propositions above, we notice that if there is an unoccupied point between two facilities, such locations do not consist of a Nash equilibrium.

Because clearly by moving to the unoccupied point one of the two will be better off (See Figure 2). Therefore we only need to consider three(in more detail eight) cases.

Also, note that any combination of the locations of the facilities is essentially equivalent to that symmetric with respect to the midpoint of the line segment in terms of buyers the facilities obtain. So here is a remark : it is suffice to prove the propositions only in the case where all the three facilities locate at points left to the midpoint or the two locate at left points and only the third at right to the midpoint.

Proof of Proposition 1

Figure 2 :



The first case is the one where three facilities locate side by side.

The second case is the one where two facilities locate at the same point and the third facility locate at the point immediately left or right to the two. This case breaks down into the left case and the right one.

The third case is the one where three facilities locate at the same point.

Now we consider the first case. We divide it into two cases, that is, that of $n=2m$ (n is even, $m\geq 3$) and $n=2m+1$ (n is odd, $m\geq 2$).

In the case of $n=2m$:

Take a look at Figure 3. Here $k\geq 0$, $k+2\leq m+1$, $k=0, 1, 2, \cdots$ (note the remark just above). By moving to the same point as located by the right facility, the central facility obtains $\frac{2m-(k+1)}{2}$. The central facility obtains 1 at the original point.

$$\frac{2m-(k+1)}{2}-1=\frac{2m-k-3}{2}\geq\frac{2m-(m-1)-3}{2}\ (\because k+2\leq m+1)=\frac{m-2}{2}>0,$$

for $m\geq 3$, $\therefore \frac{2m-(k+1)}{2}>1$.

This means the central facility can obtain more buyers by moving to another point, which means in turn this case does not consist of a Nash equilibrium.

Next, we examine the case of $n=2m+1$ (n is odd, $m\geq 2$). Take a look at Figure 4 (which shows a case of $k+2=m+1$ for understanding). Here $k\geq 0$, $k+2\leq m+1$, $k=0, 1, 2, \cdots$. By moving to the same point as located by the right facility, the central facility obtains $\frac{2m+1-(k+1)}{2}$. The central facility obtains 1 at the original point.

$$\frac{2m+1-(k+1)}{2}-1=\frac{2m-k-2}{2}\geq\frac{2m-(m-1)-2}{2}\ (\because k+2\leq m+1)=\frac{m-1}{2}>0,$$

for $m \geq 2$, $\therefore \dfrac{2m+1-(k+1)}{2} > 1$.

This means the central facility can obtain more buyers by moving to antother point, which means in turn this case does not consist of a Nash equilibrium.

We now proceed to one of the second case where two facilities locate at the same point and the third one locates immediately right to the two.

First let us consider the case of $n=2m$, $m \geq 3$. Take a look at Figure 5 (where $0 \leq k \leq m-1$). If one of the facilities at the same point moves to the point immediately right to the third point, the buyers it gets will change from $\dfrac{2k+1}{4}$ to $2m-\dfrac{2k+3}{2}$.

$$\left(2m-\frac{2k+3}{2}\right)-\frac{2k+1}{4}=\frac{8m-6k-7}{4} \geq \frac{8m-6(m-1)-7}{4}\ (\because k \leq m-1)$$
$$=\frac{2m-1}{4}>0, \text{ for } m \geq 3, \ \therefore 2m-\frac{2k+3}{2}>\frac{2k+1}{4}.$$

This means this facility can obtain more buyers by moving to another point, which means in turn this case does not consist of a Nash equilibrium.

In the case of $n=2m+1$, $m \geq 2$, the proof goes likewise except that $0 \leq k \leq m$:

$$\left[(2m+1)-\frac{2k+3}{2}\right]-\frac{2k+1}{4}=\frac{8m-6k-3}{4} \geq \frac{2m-3}{4}\ (\because k \leq m)>0. \ \therefore (2m+1)-\frac{2k+3}{2}>\frac{2k+1}{4}.$$

Next we proceed to the other of the second case where two facilities locate at the same point and the third one locates immediately left to the two.

First let us consider the case of $n=2m$, $m \geq 3$. Take a look at Figure 6 (where $k \geq 0$, $k+1 \leq m$). If one of the facilities at the same point moves to the point immediately right to the third point, the buyers it gets will change from $\dfrac{2m-\frac{2k+1}{2}}{2}$ to $2m-\dfrac{2k+3}{2}$.

$$\left(2m-\frac{2k+3}{2}\right)-\frac{2m-\frac{2k+1}{2}}{2}=\frac{4m-2k-5}{4} \geq \frac{4m-2(m-1)-5}{4}(\because k \leq m-1)$$
$$=\frac{2m-3}{4}>0 \quad \therefore 2m-\frac{2k+3}{2}>\frac{2m-\frac{2k+1}{2}}{2}.$$

Now in the case of $n=2m+1$, $m \geq 2$. We have to compare $(2m+1)-\dfrac{2k+3}{2}$ to $\dfrac{(2m+1)-\frac{2k+1}{2}}{2}$. It easy to see

$$(2m+1)-\frac{2k+3}{2} > \frac{(2m+1)-\frac{2k+1}{2}}{2}, \text{ for } m \geq 3.$$

Finally we proceed to the third case. Take a look at Figure 7 (where $n=2m$).

It is easy to see that any of the three can be better off by moving to another point either in the case of $n=2m$ or $n=2m+1$.

This completes the proof of Proposition 1. □

Figure 3 :

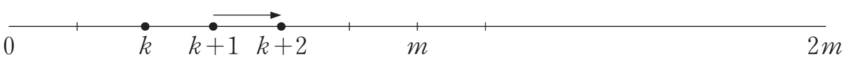


Figure 4 :

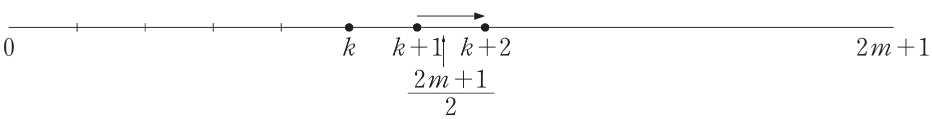


Figure 5 :

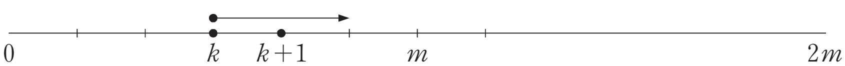


Figure 6 :

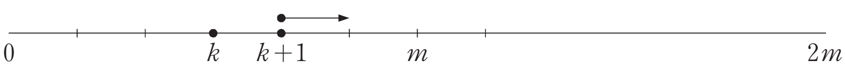
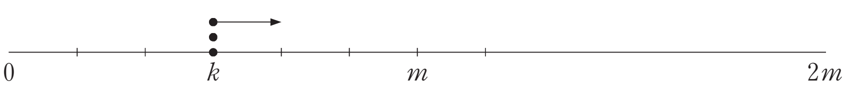


Figure 7 :



Proof of Proposition 2

To prove Proposition 2, we divide the proof into the same three cases as in the proof of Proposition 1. We also use Figure 3 through 7 now paying attention to the fact that at each point there are 6 buyers.

In the case of Figure 3, the central facility obtains 6 buyers. In this case, it is suffice to consider in the case of $k+1 \leq m$(See the remark above.) By moving to the same point as the right facility, it obtains $3[2m-(k+1)]+2$. To compare 6 to this,

$$3[2m-(k+1)]+2-6=6m-3k-7 \geq 6m-3(m-1)-7 (\because k \leq m-1)$$
$$=3m-4>0. (\because m \geq 2 \text{ by assumption}) \quad \therefore 3[2m-(k+1)]+2>6.$$

This means the central facility can obtain more buyers by moving to another point, which means in turn this case does not consist of a Nash equilibrium.

In the case of Figure 4, we compare 6 to $3[(2m+1)-(k+1)]+2$. Since $3[(2m+1)-(k+1)]+2 > 3[2m-(k+1)]+2$ and the condition $k+1 \leq m$ still must hold, it is clear that $3[(2m+1)-(k+1)]+2>6$ by the calculation just above.

As for the second and the third cases, the proof goes exactly as in that of Proposition 1. So I omit the rest of the proof and conclude it. □

Proof of Proposition 3

Take a look at Figure 8. We prove only this case does not consist of a Nash equilibrium. In the other cases the proof goes likewise.

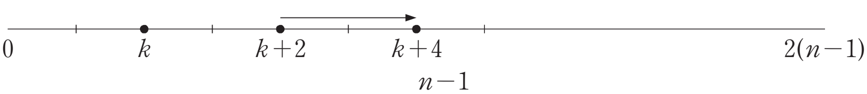
Buyers the central facility obtains change from 12 to $3[2(n-1)-(k+2)]+2$. To compare both,

$3[2(n-1)-(k+2)]+2-12=6n-3k-22 \geq 6n-3(n-3)-22 (\because k+2 \leq n-1)=3n-13>0 (\because n \geq 5$ by assumption) $\therefore 3[2(n-1)-(k+2)]+2>12$.

This means the case of Figure 8 does not consist of a Nash equilibrium.

I conclude the proof. □

Figure 8 :



Refferences

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