An alternative proof of uniqueness of the equilibrium of Rubinstein's bargaining game of alternating offers

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In this article we give an alternative proof of uniqueness of the equilibrium of Rubinstein (1982)'s bargaining game of alternating offers. In our proof, we use backward induction to obtain a subgame perfect equilibrium since the game has the one deviation property.

1. The description of the game.

Two players offer the way of splitting of unity in turn. The game begins in period 1. In every odd-numbered period n player 1 offers a division $(s_n, 1-s_n)$. Player 2 will accept this offer or reject it. If player 2 accepts, then the game ends and the payoffs are s_n for player 1 and $1-s_n$ for player 2. If player 2 rejects, then the game will continue to the next even-numbered period. In every even-numbered period m player 2 offers a division $(s_m, 1-s_m)$. Player 1 will accept this offer or reject it. If player 1 accepts, then the game ends and the payoffs are s_m for player 1 and $1-s_m$ for player 2. If player 1 rejects, then the game will continue to the next odd-numbered period.

2. The description of the strategies of the game

We represent the strategies of the game as follows.

Expression 1 $(s_1, 1-s_1), (s_2, 1-s_2), \cdots, (s_n, 1-s_n), \cdots$

and

$$(*, 1-t_1), (t_2, *) (*, 1-t_3) \cdots$$

The player whose turn it is to make an offer makes an offer $(s_n, 1-s_n)$.

$(*, 1-t_n)$ or $(t_n, *)$ represents the minimum or the infimum share of the player whose turn it is to accept or reject the other player's offer. Whether it represents the minimum or the infimum depends on players' strategies:

If $s_n \geq t_n$ (a minimum case) or $s_n > t_n$ (an infimum case), player 1 will accept s_n; otherwise she will reject it. If $1-s_n \geq 1-t_n$ (a minimum case) or $1-s_n > 1-t_n$ (an infimum case), player 2 will accept $1-s_n$; otherwise he will reject it.

3. **Assumption 1:** Let discount factors be $0 < \delta_1, \delta_2 < 1$ for players 1 and 2 respectively. At any period n, players prefer more payoffs in terms of their values discounted to period n. This is an ordinary assumption.

4. We first consider a finite game which ends at period n $(n \geq 2)$ exogenously given payoffs $(s, 1-s)$, where $0 \leq s \leq 1$.

Proposition 1: Assume that the game ends at period n $(n \geq 2)$ exogenously given payoffs $(s, 1-s)$, where $0 \leq s \leq 1$. If we base our reasoning on backward induction[1)], there is an equilibrium such that the game ends in period $n-1$ and it is unique for such equilibriums. So that if period $n-1$ is reached, it will end there. The offer is $(1-\delta_2(1-s), \delta_2(1-s))$ if player 1 makes her offer in period $n-1$ and it is $(\delta_1 s, 1-\delta_1 s)$ if player 2 makes his offer in period $n-1$.

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1) In order to use backward induction to obtain a subgame perfect equilibrium, the game must have the one deviation property. A game which has only finite stages has this property. (see Fudenberg and Tirole 1991, pp.108-110).

Proof:

We first treat the case where player 1 makes her offer in period $n-1$. Let player 1's offer in period $n-1$ be $(u, 1-u)$. If player 2 accepts, then player 2's payoff will be $1-u$; if he rejects, then the game will continue to the next final period and end there with a payoff $1-s$ for him. Player 2 accepts an offer $(u, 1-u)$ by Assumption 1 if

$$1-u > \delta_2(1-s)$$

Player 2 is indifferent between accepting and rejecting if

$$1-u = \delta_2(1-s)$$

So that player 2 will accept $\delta_2(1-s)+\varepsilon$ in period $n-1$ if $\varepsilon > 0$ and is indifferent between accepting and rejecting if $\varepsilon = 0$. If player 2 accepts, the payoff for player 1 in period $n-1$ is $u = 1-\delta_2(1-s)-\varepsilon$. If the game continues to the next final period, then player 1 will get a payoff s. So that she compares the payoffs $1-\delta_2(1-s)-\varepsilon$ and $\delta_1 s$. Actually, for sufficiently small $\varepsilon > 0$,

Expression 2 $1-\delta_2(1-s)-\varepsilon > \delta_1 s$

For, a linear function of s

$$1-\delta_2-\varepsilon+(\delta_2-\delta_1)s$$

assumes a minimum value of $1-\delta_2-\varepsilon$ (if $\delta_2 \geq \delta_1$) or $1-\delta_1-\varepsilon$ (if $\delta_2 < \delta_1$) and either value is positive for sufficiently small $\varepsilon > 0$ in the range of $0 \leq s \leq 1$. So that in the range of $0 \leq s \leq 1$, the function always

$$1-\delta_2-\varepsilon+(\delta_2-\delta_1)s > 0.$$

This implies Expression 2.

Hence if player 1 offers $(1-\delta_2(1-s)-\varepsilon, \delta_2(1-s)+\varepsilon)$ in period $n-1$, player 2 accepts it if $\varepsilon > 0$ and is indifferent between accepting and rejecting if $\varepsilon = 0$. Player 1 can be better off by making $\varepsilon > 0$ a little smaller, so that as long as $\varepsilon > 0$, it cannot be an equilibrium in period $n-1$ given payoffs $(s, 1-s)$ in period n. If there is any possibility that player 2 rejects when $\varepsilon = 0$, player 1 can be better off by making ε positive; then, however, again by making $\varepsilon > 0$ a little smaller, player 1 can be better off, so that there is no equilibrium. Hence the strategies that player 1 offers $(1-\delta_2(1-s), \delta_2(1-s))$ and player 2 accepts if his payoff is greater than or equal to $\delta_2(1-s)$ are the only equilibrium and the outcome is that player 1 offers $(1-\delta_2(1-s), \delta_2(1-s))$ and player 2 accepts it.

Next we consider the case where player 2 makes his offer. Let player 2's offer in period $n-1$ be $(u, 1-u)$. If player 1 accepts, then player 1's payoff will be u ; if he rejects, then the game will continue to the next final period and end there with a payoff s for him. Player 1 accepts the offer $(u, 1-u)$ by Assumption 1 if

$$u > \delta_1 s$$

Player 1 is indifferent between accepting and rejecting if

$$u = \delta_1 s$$

So that player 1 will accept $\delta_1 s+\varepsilon$ in period $n-1$ if $\varepsilon > 0$ and is indifferent between accepting and rejecting if $\varepsilon = 0$. If player 1 accepts, the payoff for player 2 in period $n-1$ is $1-u = 1-\delta_1 s-\varepsilon$. If the game continues to the next final period, then player 2 will get a payoff $1-s$. So that she compares the payoffs $1-\delta_1 s-\varepsilon$ and $\delta_2(1-s)$. Actually, for sufficiently small $\varepsilon > 0$,

Expression 3 $1-\delta_1 s-\varepsilon > \delta_2(1-s)$

For, clearly Expression 3 is equivalet to Expression 2.

By exactly the same argument as in the case where player 1 makes an offer, the only equilibrium is that player 2 offers $(\delta_1 s, 1-\delta_1 s)$ in period $n-1$ and player 1 accepts it if her payoff is greater than or equal to $\delta_1 s$ and the outcome is that player 2 offers $(\delta_1 s, 1-\delta_1 s)$ in period $n-1$ and player 1 accepts it.

□

5. Next we consider a case where the game potentially continues infinitely.

Proposition 2: Any subgame perfect equilibrium is, if one does exist, such that player 1 makes an offer in period 1 and player 2 accepts it and the game ends there.

Proof:

First we show that the case where no agreement is reached is not a subgame perfect equilibrium. For, no agreement means the strategies that either player continues infinitely to reject the other player' offer in turn. However, if player 1 changes her strategy and offers $\delta_2 < 1 - s_1 < 1$ for player 2's share in period 1, then player 2 should accept since he cannot hope more if the game continues to later periods. If player 2 accepts the offer, player 1 will get s_1 and it is greater than zero which will be obtained in the case of no agreement.

So if a subgame perfect equilibrium does ever exist, it must be such that in some period n, the player whose turn it is to make an offer makes some offer and the other player will accept it and the game will end there. Now assume $n \geq 2$. We again work with backward induction[2]. If $n \geq 2$, by Proposition 1 the game ends at period $n - 1$. However, if so, again by Proposition 1 the game ends at period $n - 2$. By a repetitive argument of this kind we are led to the conclusion that the game ends at period 1, which is a contradiction.

□

6. In view of Proposition 2, we learn that any subgame beginning in period n, if it is reached, will end there according to the equilibrium strategies. Formally:

Proposition 3: Any subgame perfect equilibrium is, if one does exist, such that if period n is reached, the player whose turn it is to make an offer makes some offer and the other player will accept it and the game will end there.

Proof:

This proposition is almost obvious in view of Proposition 2. To be precise, any subgame of the game is itself a game of alternating offers. So that Proposition 2 applies to this subgame.

□

Proposition 4: If the game has an equilibrium where players' offer in each period is

$$(s_1, 1 - s_1), (s_2, 1 - s_2), \cdots, (s_n, 1 - s_n), \cdots,$$

then

s_{2k-1} and s_{2k+1} $(k = 1, 2 \cdots)$ has a relation such that

$$s_{2k+1} = \frac{s_{2k-1} + \delta_2 - 1}{\delta_1 \delta_2}$$

Proof:

We work with backward induction. If period $2k + 1$ is reached, the game will end there by Proposition 3 with payoffs $(s_{2k+1}, 1 - s_{2k+1})$. By Proposition 1 it must be that player 2 offers in period $2k$ $(\delta_1 s_{2k+1}, 1 - \delta_1 s_{2k+1})$ and that the game ends there. Hence again by Proposition 1 it must be that player 1 offers in period $2k - 1$

$$(1 - \delta_2 (1 - \delta_1 s_{2k+1}), \delta_2 (1 - \delta_1 s_{2k+1}))$$

and player 2 accepts it and the game ends in period $2k - 1$.

This implies

Expression 4 $s_{2k-1} = 1 - \delta_2 (1 - \delta_1 s_{2k+1})$

By solving this expression for s_{2k+1}, we get

$$s_{2k+1} = \frac{s_{2k-1} + \delta_2 - 1}{\delta_1 \delta_2}$$

□

7. The next proposition and its proof is our main aim.

Proposition5: If there ever exists a subgame perfect equilibrium, it is unique and starts in period 1 with player 1's offer $(s, 1 - s)$ where

$$s = \frac{1 - \delta_2}{1 - \delta_1 \delta_2}.$$

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2) This potentially infinitely continuing game has also the one deviation property since payoffs are bounded(see Fudenberg and Tirole 1991, the same place as above).

Player 2 accepts this offer and the game ends in period 1.

Proof:

By Proposition 4, we know that if the game has an equilibrium where players' offer in each period is

$$(t_1, 1-t_1), (t_2, 1-t_2), \cdots, (t_m, 1-t_m), \cdots,$$

then

$s_n \equiv t_{2n-1}$ and $s_{n+1} \equiv t_{2n+1}$ (n =1, 2,⋯) has a relation such that

Expression 5 $s_{n+1} = \dfrac{s_n + \delta_2 - 1}{\delta_1 \delta_2}$

Expression 5 inductively defines a sequence$\{s_n\}$ (n = 1, 2, ⋯).

By putting $s_n = s_{n+1} = s$ in Expression 5 we obtain the equation

$$s = \frac{s + \delta_2 - 1}{\delta_1 \delta_2}$$

Solving this we obtain

$$s = \frac{1 - \delta_2}{1 - \delta_1 \delta_2}$$

Next we show that s_1 cannot assume any value other than s.

Assume, to the contrary, that

Expression 6 $s_1 > \dfrac{1 - \delta_2}{1 - \delta_1 \delta_2}$

Expression 6 implies

$$s_1 < \frac{s_1 + \delta_2 - 1}{\delta_1 \delta_2}$$

Hence in view of Expression 5

$$s_1 < s_2$$

Also we see

$$\frac{1 - \delta_2}{1 - \delta_1 \delta_2} < s_1 < s_2$$

Now if we assume

$$s_n > \frac{1 - \delta_2}{1 - \delta_1 \delta_2}$$

then by the same argument we obtain

$$\frac{1 - \delta_2}{1 - \delta_1 \delta_2} < s_n < s_{n+1}$$

So that by mathematical induction we learn

Expression 7 $\dfrac{1 - \delta_2}{1 - \delta_1 \delta_2} < s_1 < s_2 < s_3 < \cdots < s_n < \cdots$

So that the sequence $\{s_n\}$ is monotonically increasing. This sequence is not bounded above. For, to the contrary, if it were bounded above, then it would converge to a certain limit. Let this limit be a. By taking the limit of both sides of Expression 5, we obtain

$$a = \frac{a + \delta_2 - 1}{\delta_1 \delta_2}$$

which implies

$$a = \frac{1 - \delta_2}{1 - \delta_1 \delta_2}$$

However, it could not be possible since by Expression 7

$$a = \frac{1 - \delta_2}{1 - \delta_1 \delta_2} < \lim_{n \to \infty} s_n = a.$$

So that the sequence $\{s_n\}$ is not bounded above. Hence there is a number N such that $s_N > 1$. However it could not be possible since $0 \le s_n \le 1$ for $n = 1, 2, \cdots$. A contradiction. So that s_1 cannot assume any value grater than s.

Likewise, if we assume

$$s_1 < \frac{1 - \delta_2}{1 - \delta_1 \delta_2}$$

then we will get a monotonically decreasing sequence $\{s_n\}$

$$\frac{1 - \delta_2}{1 - \delta_1 \delta_2} > s_1 > s_2 > s_3 > \cdots > s_n > \cdots$$

The argument goes parallel to that above. This sequence $\{s_n\}$ is not bounded below. Hence there is a number M such that $s_M < 0$. However, it could not be possible since $0 \le s_n \le 1$ for $n = 1, 2, \cdots$. So that s_1 cannot assume any value smaller than s.

Hence we have shown that s_1 cannot assume any value other than s.

□

8. We have not yet shown that strategies starting in

period 1 with player 1's offer $(s, 1-s)$ such that

$$s = \frac{1-\delta_2}{1-\delta_1\delta_2}$$

are in fact subgame perfect equilibrium strategies. However, this is an easy matter, so that we will not show it here.

9. Discussion

Ordinary proofs of uniqueness consider the infimums and the supremums of the payoffs of players in any perfect equilibrium of any subgame of this game. In any period, players cannot possibly hope that they get more than their supremums and can certainly expect to get at least their infimums. Then the proofs go on to show that the infimums and the supremums are actually equal. See Fudenberg and Tirole (1991), Osborne and Rubinstein (1994), or Okada (1996). All these proofs exploit the stationarity of the game: the idea was originally suggested by Shaked and Sutton (1984)[3).

The proof of Gibbons (1992) also makes use of this stationarity. Because of a characteristic of this book as an introductory textbook, it uses an informal argument of backward induction. When it comes to the proof of uniqueness, this book also uses the infimum-supremum argument. For simplicity, it treats the case $\delta_1 = \delta_2 = \delta$ and shows that if s_3 is player 1's payoff in period 3, and s_1 that in period 1,

$$s_1 = 1 - \delta(1 - \delta s_3)$$

This is exactly what is obtained when we put $\delta_1 = \delta_2 = \delta$ and $k = 1$ in Expression 4 in this article. However we think there is something to be desired in its proof of uniqueness. It says that player 1's payoff $s_1 = 1 - \delta\ (1 - \delta s_3)$ attains its maximum when s_3 attains its maximum. This is certainly true; however, there is no guarantee that all payoffs of player 1 in period 1 are represented as $s_1 = 1 - \delta\ (1 - \delta s_3)$ using player 1's payoff in period 3.

Gibbons (1992)'s presentation of the game has an advantage that it is informal and, therefore, readable. While hinted by the way of the proof of Gibbons (1992), our proof here is the result of efforts to make it rigorous and clarify some ambiguity still remaining in it.

At first glance our proof of Proposition 5 might seem to be too technical. However, if we represent the sequence $\{s_n\}$ in a graph, a very familiar picture will appear. See Figure 1. This is something which reminds us of a discussion of a fix point of a contraction mapping in mathematics.

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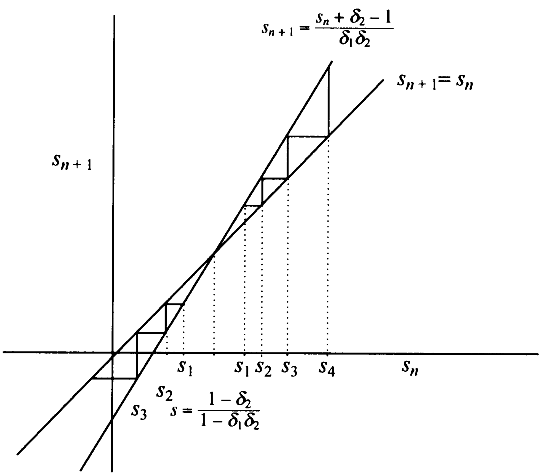
Figure 1



3) Fudenberg and Tirole(1991) gives an alternative proof by using iterated conditional dominance. Kreps(1990) also presents a more simplified case of iterated conditional dominance.