Demand Revelation for a Risky Public Good under Separable Non-Expected Utility Preferences*

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Abstract : This paper reexamines the demand-revealing property of the Groves-mechanism for a public good whose consumption level is uncertain under non-expected utility preferences. In addition to the no income effect assumption on utility evaluation of the deterministic consumption, we need to impose on the preferences the corresponding separability condition in terms of risk evaluation so that the Groves-mechanism remains to be demand-revealing in the dominant strategies. With this strong separability, the paper shows that the Groves-mechanism is the only class that is demand-revealing in the dominant strategies if the preferences conform to betweenness. Under the separability in terms of the deterministic consumption only, the Groves-mechanism loses a dominant strategy equilibrium at the same time it loses the demand-revealing property. The paper proceeds to introduce the cost uncertainty of a risky public good. In this setting, even with the strong separability condition on the preferences, the Groves-mechanism is no longer demand-revealing.

1. Introduction

The public goods provision mechanism design has been one of the hottest research themes to circumvent the *free-rider problem* under the noncooperative behavior among agents. There have been proposed several mechanisms which are successful in eliciting the private information among agents, such as their demand schedule or their maximum willingness to pay for the public goods. Clarke (1971), Groves (1973) and Groves and Loeb (1975)[1] developed a class of such *demand-revealing* mechanisms, called *Groves-mechanism.* The resource allocation realized through the Groves-mechanism may not be Pareto-optimal, and as shown in Green and Laffont (1977) this is the only class that provides the demand-revelation in a dominant strategy equilibrium for an economy with transferable utility preferences. Under the general preferences, in a Nash equilibrium, the Groves-mechanism remains to be demand-revealing. There developed the efficient mechanisms that can implement the Lindahl allocation for the economy with general transitive preferences by compromising to adopt Nash as an equilibrium concept. Such studies include Groves and Ledyard (1977), Hurwicz (1979), and Tian (1990).

Focusing on the Groves-mechanism within the complete information setting, this paper extends the conventional approach in two respects. Until now, the literature has dealt with the

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1) The first attempt to overcome the "free rider" problem was made by Vickrey (1961) who elicited the marginal schedule of demand (supply) of each consumer (producer) in response to the other agents' reported demand (supply) schedule. The essencially same class of mechanism for public good provision with a little variations was proposed independently by Clarke (1971), Groves (1973) and Groves & Loeb (1975).

case of deterministic public goods. Thus our first extension is to consider a risky public good whose level of services varies depending upon the circumstances, just as the private goods incur a stochastic nature in their consumption process or in its production process. Examples include an outdoor public pool, a commonly used radio transmitter or receiver which are influenced by weather conditions. The other type of example may be a public investment in the scientific research on agricultural product such as grain whose result can be commonly accessed by all agents in the agricultural sector but whose degree of success is normally uncertain.

The second extension is to introduce alternatives to the expected utility hypothesis as the underlying preferences of consumers in the economy. There is a significant behavioral literature following Allais (1953) which demonstrates the lack of empirical validity of the expected utility hypothesis[2]. This prompted the growing research interests to generalize the expected utility preferences (see Fishburn 1988). There follow some recent studies to investigate the economic implications of the non-expected utility preferences in different economic settings. These studies contribute not only in generalizing a number of critical results under the expected utility preferences[3], but also in revealing new implications that have been hidden under the expected utility analysis (such as Karni & Safra (1989), Crawford (1990), Dekel, Safra & Segal (1991), and Chew & Nishimura 1992b).) However, the application studies have mainly concerned with the economy with only private goods.

In this paper dealing with the economy with a public good, under the non-expected utility preferences, we can separate the substitution decision between the private good and the public good from the risk evaluation unlike the case of expected utility preferences. In other words, the private good consumption has an independent influence upon one's evaluation of the risky prospect in addition to creating the income effect under the non-expected utility preferences. This paper shows that on top of the separability between the private good and the public good in the deterministic utility evaluation, we need to have a corresponding separability in terms of risk evaluation to maintain the Groves-mechanism robust. With this strong separability, we demonstrate that the Groves-mechanism is the only class that is demand-revealing in terms of the dominant strategy equilibrium, if we narrow down the set of preferences to the subclass that is both quasi-concave and quasi-convex in probabilities.

Without the strong separability, we no longer have the dominant strategy equilibrium. Moreover, unlike the case of a deterministic public good, even in the sense of Nash equilibrium, the mechanism is not demand-revealing because of the remaining influence of the private good consumption through the risk evaluation part of the preferences.

The paper proceeds to introduce another source of uncertainty in the model, that is, the uncertain production cost of public good[4]. We show that, even with the strong separability in the risk evaluation, the Groves-mechanism is no longer demand-revealing under non-expected utility preferences.

2) See e. g., MacCrimmon & Larsson (1979) and Machina (1982, 83).

3) Such studies include Chew, Karni, & Safra (1987), Chew, Epstein, & Zilcha (1988), Machina (1989), and Chew & Nishimura (1992a).

4) The case of uncertain production cost can be found in Nitzan and Romano (1990) on the private provision of the public good.

In section 2, we present a sketch of non-expected utility preferences. The basic model and the notations are stated in section 3. Section 4 provides the first result followed by section 5 which deals with the case of uncertain cost. A concluding remark appears in section 6.

2. Smooth Non-Expected Utility

Consider a finite dimensional outcome space J. D_J is a set of cumulative probability distribution functions defined over J. Let Ω denote a set of preference orderings $\gtrsim$ which satisfy the completeness, transitivity, and continuity in the convergence in probabilities (see Billingsley 1968). It is known that there exists a real-valued utility function U mapping from D_J to $\mathbb{R}$ which represents the preference orderings in Ω.

In the subsequent analysis, we restrict our attentions to $\mathfrak{U}_H$, a subset of real-valued utility functionals which are smooth in terms of *Hadamard differentiability* (we call *Hadamard smooth*) defined below. Let $\hat{\Pi}$ denote the set of all path, $\{H_{(\cdot)} \subset D_J | H_{(\cdot)}: [0,1] \rightarrow D_J\}$. We say that the path is smooth if $\frac{\partial}{\partial r} H_r(x)|_{0+}$ exists and we denote the set of all smooth paths by $\Pi \subset \hat{\Pi}$, i.e., $\Pi = \{H_{(\cdot)} \subset \hat{\Pi} | \forall x \in J, \frac{\partial}{\partial r} H_r(x)|_{0+} \text{ exists.}\}$.

Definition 2.1 : A functional $U : D_J \rightarrow \mathbb{R}$ is *Hadamard differentiable* at $P \in D_J$ if there is a bounded continuous function $u(\cdot\,; P) : J \rightarrow \mathbb{R}$ such that for any smooth path $H_{(\cdot)} \in \Pi$ and $H_0 = P$,

$$\frac{\partial}{\partial r} U(H_r)|_{0+} = \int_J u(x; P) \cdot d\left[\frac{\partial}{\partial r} H_r(x)|_{0+}\right]. \tag{2.1}$$

where the function $u(\cdot\,;\cdot)$ is called *Hadamard derivative.*

Hadamard smoothness can be regarded as a local linear approximation along the smooth path just as we are accustomed to such a local approximation for real-valued function on $\mathbb{R}^m$. When we consider local behavior of utility functionals such as risk attitude, the Hadamard derivatives play a similar role to von-Neumann-Morgenstern utility. Especially, when u is increasing with respect to $x \in J$ component-wise, preference orderings are consistent with the first-order stochastic dominance. In fact, if we rewrite the condition (2.1) as (2.2),

$$U(H_r) - U(P) = r\int_J u(x; P) \cdot d\left[\frac{\partial}{\partial r} H_r(x)|_{0+}\right] + o(r), \tag{2.2}$$

then we can immediately tell that in the case of the expected utility functional, the Hadamard derivative $u(\cdot\,; P)$ does not depend on P and corresponds to von-Neumann-Morgenstern utility.

Every non-expected utility preference representation retaining transitivity can be characterized by the way of abandoning the independence axiom[5] (see Chew & Epstein (1989)). A subset of non-expected utility preferences of particular interest is a class which conforms to the property called *betweenness.*

Definition 2.2 : The preference relations belonging to Ω is said to satisfy *betweenness* if for any

5) The preference ordering $\succ$ (strict preference) is said to satisfy the *independence axiom* if for any P, Q, R $\in$ S, if $P \succ Q$, then for any $\beta \in (0, 1)$, $\beta P + (1-\beta)R \succ \beta Q + (1-\beta)R$.

P, Q $\in D_J$, with $P \lesssim Q$, then for any $\beta \in (0, 1)$, $P \lesssim \beta P + (1-\beta) Q \lesssim Q$.

The preferences are both quasi-concave and quasi-convex at the same time on the domain D_J. In terms of utility functionals, this property requires the utility value of the probability mixture of lotteries P and Q to be in-between the utility value of P and Q. Among the existing non-expected utility functionals proposed, the class of betweenness-conforming preferences contains weighted utility (Chew & MacCrimmon (1979), Chew (1983)), implicit weighted utility[6] (Dekel (1986), Chew (1989)) and skewed symmetric utility (Fishburn (1983)).

Betweenness has a strong normative appeal and received a various degree of empirical support (see e.g., Chew & Waller (1986)). Further the betweenness-conforming preferences have the advantage of simple representational utility functionals. The corresponding Hadamard derivative, u(• ; P) of (2.1) or (2.2) for betweenness-conforming utility U at P, depends only upon the value of P, U(P) rather than P itself for every $P \in D_J$.

3. Risky Public Good and Groves-Mechanism

3—a. Risky Public Good

The services from a public good are subject to uncertainty after it is produced. Specifically we consider a risky public good as a two-outcome simple lottery $F_X \equiv p_n\delta_X + (1-p_n)\delta_{X_d}$, where δ_X is a unit probability assigned at point x which is the level of public good in the normal state with probability of occurrence p_n, and x_d is the fixed service level in the disaster state. Let X be a subset of $\mathbb{R}$ which defines the possible values of the realized service level from the public good. D_X is a set of cumulative probability distribution function defined over X, whose element is $F_x \in D_X$ and x, $x_d \in X$. Let $\mathscr{K}$ denote a set of available public good projects such that $\mathscr{K} \subseteq D_X$, whose element is $K \in \mathscr{K}$. Obviously $\mathscr{K}$ is compact. For an example, when $\mathscr{K} = \{F_X, \delta_0\}$, then available projects are either a given risky public good F_x or none.

3—b. Groves-mechanism

We consider a simple economy with n agents where a risky public good and a unique private good y are consumed. A bundle of private goods is regarded as a composite good. The private good can be used as an input in producing the public good. Let e^i denote the endowment of consumer i, which consists of only private good.

The Groves-mechanism is a public good allocation mechanism which accommodates two feasibility constraints. One is the "information decentralization" which requires the mechanism to allow for the existence of strictly private information. Consequently the mechanism involves a communication between agents and the government through a set of "messages". The other

6) For P, Q $\in$ S, $\forall\beta \in (0, 1)$, the implicit weighted utility functional $v(\beta P + (1-\beta)Q)$ is given implicitly in the following expression in terms of its value "a",

$$a = \frac{\beta w(P, a)\, v(P) + (1-\beta)\, w(Q, a)\, v(Q)}{\beta w(P, a) + (1-\beta)\, w(Q, a)}.$$

w takes only a positive value. The product vw is real-valued and vanishes only when v = 0. v may take a value either $-\infty$ or ∞ but not both. Note that the above expression reduces to the expected utility representation whenever w is constant. When $w(\cdot, a) = w(\cdot)$, it becomes weighted utility.

component of the mechanism is the "outcome rule". The outcome rule consists of a rule which determines the level of public good to supply and the corresponding tax level in terms of a private good. This outcome rule must give enough incentive to each consumer to reveal his true individual valuation of the public good, provided that the government can effectively and costlessly communicate with every consumer. This is the second feasibility constraint which is the "incentive compatibility". We shall state these rules formally in definition 3.1 below.

Definition 3.1: Let m^i be a message function, $m^i: \mathscr{K} \rightarrow \mathbb{R}$, submitted by the ith agent to the government. M^i is a set of possible real-valued functions m^i, $m^i \in M^i$, and $M \equiv M^1 \times \cdots\cdots \times M^n$. Let μ and ξ denote the outcome rule $\mu: M \rightarrow \mathscr{K}$ and the tax rule applied to consumes i, $\xi^i: M \rightarrow \mathbb{R}$ respectively. $\xi = \{\xi^1, \cdots\cdots, \xi^n\}$. Denote $T(K)$ as the total efficient cost of producing a particular public good K. Then the Groves-mechanism is characterized by $<M, \mu, \xi>$ such that,

$$M^i = \{m^i \mid m^i: \mathscr{K} \rightarrow \mathbb{R},\ m^i \text{ is differentiable.}\},\ M = M^1 \times \cdots\cdots \times M^n$$

$$\mu(m^1, \cdots\cdots, m^n) = K^* = \operatorname{argmax} \sum_{i=1}^{n} m^i(K) - T(K).$$

$$\xi^i(m^1, \cdots\cdots, m^n) = T[\mu(m^1, \cdots\cdots, m^n)] - \sum_{j \neq i} m^j[\mu(m^1, \cdots\cdots, m^n)] + h^i(m^{-i}).$$

where h^i indicates an arbitrary transfer function, and m^{-i} denotes $(m^1, \cdots\cdots, m^{i-1}, m^{i+1}, \cdots\cdots, m^n)$.

All the information about the available public projects $\mathscr{K}$ is commonly known among all agents in the economy. The message function m^i will be interpreted as the agent i's valuation of a given public good K in terms of a private good. Based upon this communication through $(m^1, \cdots\cdots, m^n)$, the government determines $K^* \in \mathscr{K}$ so that it maximizes the total reported consumer surplus according to the outcome rule μ. Corresponding to that level, the government applies the tax rule $\{\xi^i\}$ which tells consumer i to pay the total reported consumer surplus of the other consumers as if he were not in this economy $T(K^*) - \sum_{j \neq i} m^j(K^*)$, and the possible transfer h^i that is independent of his message m^i. From now on, we assume $h^i = 0$ for all i without losing generality.

To sum up, this tax rule is equipped with a built-in incentive for each agent to transmit his information correctly to the government by excluding the direct influence of that agent's message from his own tax rate. Together with the Nash behavioral assumption, this tax rule leads agents to regard the tax rate as given. In the economy with no income effect for the case of deterministic public good, the Groves-mechanism induces $\{m^i\}$ to reflect each agent's maximum willingness to pay for the given public good in terms of a dominant strategy.

4. Demand Revelation without Cost Uncertainty

4—a. Consumption Prospect and Separability

Consider a commodity space J which is a closed convex subset of $\mathbb{R}^2_+ \cup \{0\}$. Let D_J be a set of cumulative probability functions whose element is denoted by $H_{x,y}$. $\delta_{x,y} \in D_J$ indicates unit probability mass concentrating on the point (x, y) in $\mathbb{R}^2$. We represent by $F_{x,y} \in D_J$ a consumption prospect where an agent consumes a risky public good $F_x \in D_X$ and a deterministic private good y, such that $F_{x,y} \equiv p_n \delta_{x,y} + (1-p_n)\delta_{xd,y}$, $0 < p_n < 1$.

Let U^i be the ith participant's utility functional such as $U^i: D_J \rightarrow \mathbb{R}$. The functional U is assumed to be bounded and Hadamard smooth. Let $u(x, y; H)$ denote Hadamard derivative of U evaluated at a particular decision point $H \in D_J$. "u" is real-valued, non-negative function which is continuous and non-decreasing in both x and y.

In the case of deterministic public good, we need to have the separability between the private good and the public good in the utility function so that the Groves-mechanism is demand-revealing in the dominant strategies. In the case of risky public good, however, just the separability for the deterministic utility is not enough to induce the utility functional U for the whole risky prospect $F_{x,y}$ to be quasi-linear between y and F_x. For an example, in the case of weighted utility $WU(F_{x,y})$ which has a form,

$$WU(F_{x,y}) = \frac{\int w(x,y)\, v(x,y) dF_{x,y}}{\int w(x,y)\, dF_{x,y}}$$

the utility for the deterministic consumption corresponds to $v(x,y)$. Even if this $v(x,y)$ is quasi-linear between x and y, we can easily see that the whole utility value U of $F_{x,y}$ is not separable between y and F_x.

Observing that Hadamard derivative plays the role analogous to von-Neumann-Morgenstern utility, we define the following two conditions on the Hadamard derivative, which are *weak separability* and *strong separability*.

Definition 4.1 (Weak Separability): Hadamard derivative "u" is said to be *weakly separable* between the private good and the public good if,

$$u(x, y; F_{x,y}) = y + u(x; F_{x,y}), \tag{4.1}$$

for every $F_{x,y} \in D_J$.

Definition 4.2 (Strong Separability): Hadamard derivative "u" is said to be *strongly separable* between the private good and the public good if,

$$u(x, y; F_{x,y}) = y + u(x; F_x), \tag{4.2}$$

for every $F_{x,y} \in D_J$ and $F_x \in D_X$, where F_x represents the part of uncertainty regarding only with the public good.

In what follows, we utilize the expression such as $U(F_x, y)$ for $U(F_{x,y})$ to simplify the notation. Further, we assume the Hadamard derivatives are increasing with respect to x.

4—b. Demand-Revelation Property

The choice of equilibrium concept in the Groves-mechanism is naturally Nash. If the preferences are quasi-concave in probability mixture, a Nash equilibrium exists in mixed strategies[7]. Consequently we assume that the utility preferences are quasi-concave in probabil-

7) This has been observed in Crawford (1990) for the case where pure strategy sets are finite. This result for more general settings of continuous pure strategy sets follows, for example, from modifying the classical result of Glicksberg (1952) based on the observation that quasi-concavity implies that the best response correspondences are convex-valued. See Dekel, Safra, and Segal (1992) for the case of the two-person game.

ities. When we say the Groves-mechanism is demand-revealing, we mean that the mechanism induces each individual to report one's reservation price which is defined in the usual manner.

Definition 4.3 : The message $v^i \in M^i$ such that $v^i: \mathscr{X} \to \mathbb{R}$ is said to be *truth-telling* if v^i satisfies,

$$(4.3) \qquad U^i(F_x, e^i - v^i(F_x)) = U^i(\delta_o, e^i).$$

It follows that $v^i(\delta_o) = 0$.

Definition 4.4 : The Groves-mechanism is called *demand-revealing* if the n-tuple truth-telling strategies $(v^1, \cdots\cdots, v^n) \in M$ is Nash equilibrium strategies of the Groves-mechanism $<\mu, \xi, m>$, that is, for every player i and for every $m^i \in M^i$, v^i satisfies,

$$(4.4) \qquad U(\mu(v^i, v^{-i}), e^i - \xi(v^i, v^{-i})) \geq U(\mu(m^i, v^{-i}), e^i - \xi(m^i, v^{-i})),$$

where $v^{-i} = (v^1, \cdots, v^{i-1}, v^{i+1}, \cdots, v^n)$.

We will first state the robust result with preferences whose Hadamard derivatives are strongly separable.

Theorem 1 : Let U represent preference relations $\gtrsim \in \Omega$. U is assumed to be Hadamard smooth and its Hadamard derivatives are $u(\cdot,\cdot;\cdot)$. The Groves mechanism $<\mu, \xi, m>$ of definition 3.1 is demand-revealing in a dominant strategy equilibrium if and only if the Hadamard derivatives $u(\cdot,\cdot;\cdot)$ are strongly separable.

Proof : Consider a representative individual i. We will omit the superscript i whenever there is no fear of confusion. Let $z(\cdot) = \sum_{j \neq i} m^j(\cdot)$. If the strategy of sending $v(\cdot)$ is optimal, due to the condition (4.4), it satisfies the following inequality for any $m \in M^i$ and any z,

$$(4.5) \qquad U(F_x^*, e - T(F_x^*) + z(F_x^*)) \equiv U(F_{x(v,z), e-T+z})$$
$$> U(\hat{F}_x, e - T(\hat{F}_x) + z(\hat{F}_x)) \equiv U(F_{x(m,z), e-T+z}),$$

where $F_x^*, \hat{F}_x \in D_X$, and $F_x^* \equiv F_{x(v,z)}$ maximizes $v(F_x) + z(F_x) - T(F_x)$, and $\hat{F}_x \equiv F_{x(m,z)}$ maximizes $m(F_x) + z(F_x) - T(F_x)$. We consider the payoff $F_{x(v,z), e-T+z} \in D_J$ as a path parameterized by v. Then we differentiate $U(F_{x(v,z), e-T+z})$ in the sense of Hadamard,

$$(4.6) \qquad \frac{\partial}{\partial v} U(F_{x(v,z), e-T+z}) = \lim_{\epsilon \to 0} \int_J u(\cdot, \cdot; F^*_{x, e-T+z}) d\left\{\frac{1}{\epsilon}[(F_{x(v+\epsilon,z), e-T+z}) - F_{x(v,z), e-T+z}]\right\}.$$

With the strong separability, we have,

$$(4.7) \qquad \frac{\partial}{\partial v} U(F_{x(v,z), e-T+z}) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_X u(\cdot; F_x^*)[p_n \delta_{x(v+\epsilon,z)} + (1-p_n)\delta_{x_d}] - T(\hat{F}_x) + z(\hat{F}_x)$$
$$- \left\{\int_X u(\cdot; F_x^*) d[p_n \delta_{x(v,z)} + (1-p_n)\delta_{x_d}] - T(F_x^*) + z(F_x^*)\right\},$$

where $\hat{F}_x$ denotes $F_{x(v+\epsilon,z)}$. With enough continuity on U, we can find $v(F_{x(v+\epsilon,z)})$ for small ϵ such that,

$$(4.8) \qquad U[F_{x(v+\epsilon,z)}, e - v(F_{x(v+\epsilon,z)})] = U[F_{x(v,z)}, e - v(F_{x(v,z)})] = U(\delta_0, e) \equiv 0.$$

Similarly we can consider the path $F_{x(v,z), e-v(F_{x(v,z)})}$ that gives the payoff 0. Then we have,

$$(4,9) \qquad 0 = \frac{\partial}{\partial v} U(F_{x(v,z), e-v}) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{\int_X u(\cdot; F_x^*) d[p_n \delta_{x(v+\epsilon,z)} + (1-p_n)\delta_{x_d}] - v(F_{x(v+\epsilon,z)})\right.$$
$$\left. - [\int_X u(\cdot; F_x^*) d[p_n \delta_{x(v,z)} + (1-p_n)\delta_{x_d}] - v(F_x^*)]\right\}.$$

Combining (4.7) and (4.9), we have,

(4.10) $\frac{\partial}{\partial v} U(F_{x(v,z),e-T+z}) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\Big\{[-T(\hat{F}_x) + z(\hat{F}_x) + v(\hat{F}_x)] - [-T(F_x^*) + z(F_x^*) + v(F_x^*)]\Big\}.$

From the definition of F_x^* through the Groves-mechanism itself, the inside of the large bracket { } in (4.10) has negative sign for positive ϵ. Then for any z, sending the truth-telling message v yields the local maximum of the payoff.

Suppose that u is weakly separable. Then in RHS of the equation (4.7), we have $u(\cdot; F_{x(v,z),e-T+z})$ in the place for $u(\cdot; F_x^*)$. And in the equation (4.9), the difference between two reservation prices $[v(F_{x(v+\epsilon,z)}) - v(F_x^*)]$ corresponds to the term $\int_X u(\cdot; F_{x(v,z),e-v})\, d\{[p_n\delta_{x(v+\epsilon,z)} + (1-p_n)\delta_{x_d}] - [p_n\delta_{x(v,z)} + (1-p_n)\delta_{x_d}]\}$. Since in general $u(\cdot; F_{x(v,z),e-T+z})$ is different from $u(\cdot; F_{x(v,z),e-v})$, there is no corresponding term with the difference of the reservation prices in equation (4.7). Thus the truth-telling strategy can not bring about a dominant strategy equilibrium. ■.

Green and Laffont (1977) show that the Groves-mechanism is the only class that is demand-revealing in terms of the dominant strategies with transferable utilities. For the risky public good, with strongly separable utilities, we need an additional condition for preferences, which is betweenness.

Theorem 2: Let U represent preference relations $\succsim \in \Omega$, which has the strongly separable Hadamard derivatives. The public good provision mechanism that is demand-revealing in terms of dominant strategies is the Groves-mechanism, if the preferences conform to betweenness.

Proof: As characterized in Green and Laffont (1977), the revelation mechanism is the Groves-mechanism if and only if it satisfies the following two conditions;

i) $\xi(m^1, m^{-1})$ is independent of m^1 at $F_X \in D_J$, in other words,

(4.11) $\xi(m^1, m^{-1}) = \xi(w^1, m^{-1})$ when $\mu(m^1, m^{-1}) = \mu(w^1, m^{-1}) = F_x$.

ii) $-\xi(m^1, m^{-1}) - [-\xi(w^1, m^{-1})] = [-T(F_x^*) + z(F_x^*)] - [-T(P_x) + z(P_x)]$,

where F_x^*, $P_X \in D_J$, and F_x^* maximizes $m^1(F_X) + z(F_X) - T(F_X)$ and P_X maximizes $w^1(F_X) + z(F_X) - T(F_X)$.

If i) fails to hold, there exists m^1, m^{-1}, and w^1 such that $\mu(m^1, m^{-1}) = \mu(w^1, m^{-1}) = F_X$ and $-\xi(m^1, m^{-1}) \geq -\xi(w^1, m^{-1})$. Suppose $w^1 = v^1$, then $F_{x,e-\xi(m^1,m^{-1})}$ dominates $F_{x,e-\xi(w^1,m^{-1})}$ in terms of the first-order stochastic dominance, such that $U(F_X, e - \xi(m^1, m^{-1})) \geq U(F_X, e - \xi(w^1, m^{-1}))$ which implies w^1 is not a dominant strategy.

If ii) fails, there exists $\epsilon > 0$ such that,

(4.12) $-\xi(m^1, m^{-1}) - [-\xi(w^1, m^{-1})] = [-T(F_x^*) + z(F_x^*)] - [-T(P_X) + z(P_X)] + \epsilon.$

Let $\hat{w}^1(F_x^*) = -z(F_x^*) + T(F_x^*)$, $\hat{w}^1(P_X) = -z(P_X) + T(P_X) + \delta$, $0 < \delta < \epsilon$, and $\hat{w}^1(F_X) = -c$ for $F_X \neq F_x^*$ or P_X where $c > \max_{F_X \in \mathcal{X}} [z(F_X) - T(F_X)]$. Since P_X maximizes $\hat{w}^1(F_X) + z(F_X) - T(F_X)$, from (4.11) we have $\xi(\hat{w}^1, m^{-1}) = \xi(w^1, m^{-1})$. It follows that,

(4.13)
$$-\xi(m^1, m^{-1}) - (-\xi(\hat{w}^1, m^{-1})) = [-T(F_x^*) + z(F_x^*)] - [T(P_X) + z(P_X)] + \epsilon$$
$$= -\hat{w}^1(F_x^*) + \hat{w}^1(P_X) + \epsilon - \delta.$$

Suppose that $\hat{w}^1(F_X)$ is the reservation price solving the following, (4.14) $$U(F_X, e - \hat{w}^i(F_X)) = U(\delta_0, e).$$

Let $F^\gamma = \gamma F^*_{X, e-\xi(m^i, m^{-i})} + (1-\gamma) P_{X, e-\xi(\hat{w}^i, m^{-i})}$ for $\gamma \in [0,1]$ and take Hadamard differential of the following sort.

(4.15)
$$\begin{aligned}\frac{\partial}{\partial\gamma}U(F^\gamma)|_{\gamma=0} &= \int_J u(\cdot,\cdot;P_{X,e-T+z})\,d[F^*_{X,e-T+z} - P_{X,e-T+z}]\\ &= \int_X u(\cdot;P_X)\,d[F^*_X - P_X] - \xi(m^i, m^{-i}) - (-\xi(\hat{w}^i, m^{-i}))\\ &= \left\{\int_X u(\cdot;P_X)\,d[F^*_X - P_X] - \hat{w}^i(F^*_X) + \hat{w}^i(P_X)\right\} + \epsilon - \delta.\end{aligned}$$

From the definition of the reservation prices, the inside of the large bracket { } in the RHS of (4.15) is zero. Then $\frac{\partial}{\partial\gamma}U(F^\gamma)|_{\gamma=0}$ has positive sign, which implies that $F^*_{X, e-\xi(m^i, m^{-i})}$ is preferred to $P_{X, e-\xi(\hat{w}^i, m^{-i})}$, due to betweenness. This in turn means that sending message m enables an agent to achieve higher utility than sending the true message $\hat{w}$. ■.

5. Demand Revelation with Cost Uncertainty

This section introduces another source of uncertainty. We consider a discrete public good in the sense that there exists the minimum required cost to produce a given level of public good. Here we assume that minimum cost is random. The decision to evaluate a given public project takes place before the cost uncertainty resolves. The studies of discrete public goods have been growing in the voluntary contribution literature, and the case of uncertain cost has been dealt by Nitzan and Romano (1990). Though having a discrete public good contributes positively to the voluntary contribution mechanism, it shades the robustness of the Groves-mechanism when we have cost uncertainty.

For the case of discrete public good, $\mathscr{X}$ is a two-element set, such that $\{F_X, \delta_0\}$. Thus the outcome rule μ and ξ can be interpreted as,

$$M = \prod_{i=1}^{n} M^i, \text{ where } M^i = \{m^i \in M^i | m^i : \mathscr{X} \to \mathbb{R}\},$$

(5.1)
$$\mu(m^1, \cdots\cdots, m^n) = \begin{cases} F_X & \text{if } \sum_{i=1}^{n} m^i \geq T,\\ \delta_0 & \text{otherwise,}\end{cases}$$

$$\xi^i(m^1,\cdots\cdots,m^n) = T[\mu(m^1,\cdots\cdots,m^n)] - \sum_{j\neq i} m^j[\mu(m^1,\cdots\cdots,m^n)],$$

where T is a random variable following the cumulative probability distribution G(•) over $[\underline{c}, \bar{c}]$ with the density function g(•).

In the case of surplus, we assume that the surplus belongs to the production sector. Within this framework, we will show that the Groves-mechanism is not demand-revealing for the agents with Hadamard smooth non-expected utility preferences even under the strong separability condition on their Hadamard derivatives.

Theorem 3 : Consider the Groves-mechanism $<\mu, \xi, m>$ defined in (5.1) for a risky public good F_X with uncertain cost T. The truth-telling messages constitute a dominant strategy equilibrium if and only if the agents posses the expected utility preferences.

Proof : The sufficient part is known. Suppose that U represents a preference relation $\gtrsim \in \Omega$, which is Hadamard smooth with strongly separable Hamadard derivatives u. For a given $z = \sum_{j \neq 1} m^j(\cdot)$, when a representative agent i sends a message m, his payoff prospect is given by H(m, z) $\in D_J$ such that,

$$H(m, z) \equiv \int_{\underline{c}}^{m+z} F_{x, e-T+z} dg(T) + \int_{m+z}^{\bar{c}} \delta_{0, e} dg(T). \tag{5.2}$$

If the strategy of sending m^1 is optimal in terms of dominant strategy, m^1 satisfies the following first-order condition for any z,

$$0 = \frac{\partial}{\partial v} U(H(m, z)) = \int_J u(\cdot, \cdot, H(m, z)) d[\frac{\partial}{\partial m} H(m, z)] \tag{5.3}$$

$$= \int_J u(\cdot, \cdot; H(m, z)) d[g(m+z)(F_{x, e-m} - \delta_{0, e})].$$

With the strong separability, we have,

$$\frac{\partial}{\partial m} U(H(m, z)) = \int_X u(\cdot; H'(m, z)) d[p_n \delta_x + (1 - p_n) \delta_{x_d} - \delta_0] - m = 0. \tag{5.4}$$

where H'(m, z) is $\int_{\underline{c}}^{m+z} F_x dg(T) + \int_{m+z}^{\bar{c}} \delta_0 \, dg(T)$. We note that the condition (5.4) depends on z through H'(m, z). Therefore, m can not be a dominant strategy.

From the definition of the reservation price, v is implicitly given by,

$$U(F_{x, e-v}) = U(\delta_{0, e}). \tag{5.5}$$

Maintaining the same utility level, we have the following relation,

$$\frac{dv}{dx} = \frac{\partial U(F_{x, e-v}) / \partial x}{\partial U(F_{x, e-v}) / \partial v} = \frac{\int_J u(\cdot, \cdot; F_{x, e-v}) d[\frac{\partial}{\partial x} F_{x, e-v}]}{\int_J u(\cdot, \cdot; F_{x, e-v}) d[\frac{\partial}{\partial v} F_{x, e-v}]} \tag{5.6}$$

$$= - \int_X u'(x; F_x) \, dF_x.$$

When we move x and m to maintain the equation (5.4), it is easy to see that m does not correspond to v in (5.6), since $u(\cdot; F_x)$ is different from $u(\cdot; H'(v, z))$ unless U is expected utility. ■.

6. Conclusion

The Groves-mechanism for a deterministic public good is demand-revealing in terms of dominant strategies under separable (transferable) preferences. The mechanism achieves demand-revealing property through its tax rule which prevents an individual's tax rate from having a direct effect of his own message to the government. This paper reexamines the Groves-mechanism for a risky public good under non-expected utility preferences within the complete information settings in two parts. In the first part, the uncertain aspect of the risky public good exists in its random service level. In the second part, there is an additional uncertainty in the cost.

Under non-expected utility preferences, we can separate the substitution decision between private goods and public goods from the evaluation of the risk unlike the case of expected utility. In the first part, we show that the Groves-mechanism is demand-revealing in terms of dominant strategies if preferences achieve separability with respect not only to the substitution decision but also to the risk evaluation, which we call the strong separability. Further, we show that the Groves-mechanism is the only class that is demand-revealing in dominant strategies if we narrow the set of preferences down to the subclass that is weighted linear in probabilities.

Without the separability with respect to the risk evaluation attitude, we no longer have a dominant strategy equilibrium. In the Nash equilibrium, unlike the case of deterministic public good, the paper demonstrates a break-down of the demand-revealing property of the Groves-mechanism. This result stems from the structure of the non-expected utility preferences that lets the private good consumption recreate a link between one's message and his valuation of the tax rate independently through the risk evaluation.

In the second part with cost uncertainty, we observe that the demand-revealing property of the Groves-mechanism breaks down even under the strong separability. A decision under the non-expected utility preferences depends upon the specific risky prospect at which the decision is evaluated. In this case, the cost uncertainty recaptures the link between the individual's message and one's evaluation of the tax rate since the individual's message level determines the probability distribution of the consumption plan of a risky public good.

We conjecture that our result of the Groves-mechanism can be extended to the other class of incentive compatible mechanisms. The rigorous examination on this issue will be a future work.

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